Unambiguous discrimination between linearly dependent states with multiple copies

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A set of quantum states can be unambiguously discriminated if and only if they are linearly independent. However, for a linearly dependent set, if C copies of the state are available, then the resulting C particle states may form a linearly independent set, and be amenable to unambiguous discrimination. We obtain one necessary and one sufficient condition for the possibility of unambiguous discrimination among N states given that C copies are available and that the single copies span a D-dimensional space. These conditions are found to be identical for qubits. We then examine in detail the linearly dependent trine ensemble. The set of $C>1$ copies of each state is a set of linearly independent lifted trine states. The maximum unambiguous discrimination probability is evaluated for all $C>1$ with equal *a priori* probabilities.

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I. INTRODUCTION

Much of the fascination with the information-theoretic properties of quantum systems derives from collective phenomena and processes. On one hand, the information contained in entangled quantum systems is of a collective native, sometimes nonlocal nature, and is central to many intriguing applications of quantum information, such as teleportation and quantum computing. On the other, there are collective operations, such as collective measurements on several quantum systems. Generally speaking, collective measurements on a set of systems can yield more, or better, information than one can obtain by carrying out separate measurements on the individual subsystems, even if these are not entangled. The use of collective measurements is crucial for attaining the true classical capacity of a quantum channel [1], since capacities attained with receivers performing collective measurements on increasingly large strings of signal states are superadditive.

A further illustration of the superiority of collective over individual measurements is the ‘‘non-locality without entanglement’’ discovered by Bennett *et al.* [2]. This refers to the fact that one can construct a set of orthogonal product states that can be perfectly distinguished only by a collective measurement.

In this paper, we provide a further demonstration of the increased knowledge that can be attained using collective rather than individual measurements, relating to unambiguous state discrimination [3]. Such measurements can reveal, with zero probability of error, the state of a quantum system, even if the possible states are nonorthogonal. Perfect discrimination among nonorthogonal states is impossible, and the price we pay is the nonzero probability of inconclusive results.

It has been established that unambiguous discrimination is possible only for linearly independent states [4]. However, suppose that the possible states form a linearly dependent set, but we have $C>1$ copies of the actual state at our disposal. Unambiguous discrimination is impossible using separate measurements on the individual copies. If, however, the possible C particle states form a linearly independent set, then unambiguous discrimination will be possible by carrying out a collective measurement on all C copies.

In Sec. II, we derive one necessary and one sufficient condition for N states to be amenable to unambiguous discrimination, given that C copies of the state are available and that the possible single-copy states span a finite, D-dimensional space. For a qubit $(D=2)$, these conditions are identical. In Sec. III, we work out in detail a specific example, that of multiple copies of the so-called trine states. The trine set is linearly dependent, although the set comprised of multiple copies of these states is linearly independent for $C\geqslant 2$. Indeed, these states are the lifted trine states recently discussed in a different but related context by Shor [5]. We obtain the maximum discrimination probability for these multitrine states with equal *a priori* probabilities, and find that it has some curious, unexpected features.

II. BOUNDS ON THE MAXIMUM NUMBER OF DISTINGUISHABLE STATES

Consider the following scenario: a quantum system is prepared in one of the N pure states $|\psi_j\rangle$, where $j=1,...,N$. These states are nonorthogonal, and we would like to determine which state has been prepared. If we are unwilling to tolerate errors, then we should adopt an unambiguous discrimination strategy. Such a measurement will have $N+1$ outcomes: N of these correspond to the possible states and a further outcome gives inconclusive results. It has been established that the zero-error constraint leads to a nonzero probability of inconclusive results for nonorthogonal states [4].

Suppose that the $|\psi_j\rangle$ span a D-dimensional Hilbert space $\mathcal{H}$. Clearly, $D\leqslant N$. If $D=N$, then the states are linearly independent. If, on the other hand, $D<N$, then they are linearly dependent. Whether or not the set is linearly independent is crucial, since it is only for linearly independent sets that unambiguous discrimination is possible [4].

If, however, instead of having just one copy of the state, we have $C>1$ copies, that is, one of the states $|\psi_j\rangle^{\otimes C}$, then there is the possibility that, even if $\{|\psi_j\rangle\}$ is a linearly dependent set, $\{|\psi_j\rangle^{\otimes C}\}$ may be linearly independent, making unambiguous discrimination possible. It is of interest to determine the conditions under which this is so. Here, we will obtain two general results relating to the number of states that can be unambiguously discriminated, given that the single copies span a D-dimensional space and that C copies of the state are available. First, we will show that the number of states that can be unambiguously discriminated satisfies the inequality

$$N \leqslant \binom{C+D-1}{C}. \tag{2.1}$$

To see why, let us denote by $\mathcal{H}_{\mathrm{sym}}$ the symmetric subspace of $\mathcal{H}^{\otimes C}$. The states $|\psi_j\rangle^{\otimes C}$ are invariant under any permutation of the states of the single copies, and thus lie in $\mathcal{H}_{\mathrm{sym}}$. Denoting by D_{sym} the dimension of $\mathcal{H}_{\mathrm{sym}}$, it can be shown that [6]

$$D_{\mathrm{sym}} = \binom{C+D-1}{C}. \tag{2.2}$$

The $|\psi_j\rangle^{\otimes C}$ will be linearly dependent if N is greater than the dimension of $\mathcal{H}_{\mathrm{sym}}$. This, together with Eq. (2.2), leads to inequality (2.1), which is a necessary condition for unambiguous discrimination among N states spanning a D-dimensional space given C copies of the state.

This bound holds for all pure states. It is tight, in the sense that for all C,D, there exists a set of N pure states $\{|\psi_j\rangle\}$ such that the equality in (2.1) is satisfied and the set $\{|\psi_j\rangle^{\otimes C}\}$ is linearly independent. To prove this, we make use of the fact that $\mathcal{H}_{\mathrm{sym}}$ is the subspace of $\mathcal{H}^{\otimes C}$ spanned by the states $|\psi\rangle^{\otimes C}$, for all $|\psi\rangle \in \mathcal{H}$. The set of states $\{|\psi\rangle^{\otimes C}\}$ is linearly dependent. However, every linearly dependent set spanning a vector space $\mathcal{V}$ contains a linearly independent subset that is a basis for $\mathcal{V}$ [7]. Let $\{|\psi_j\rangle^{\otimes C}\}$ be such a subset of $\{|\psi\rangle^{\otimes C}\}$ for $\mathcal{V} = \mathcal{H}_{\mathrm{sym}}$. These states are linearly independent and satisfy the equality in (2.1) since $N = D_{\mathrm{sym}}$.

We now show that any N distinct pure states can be unambiguously discriminated if

$$N \leqslant C+D-1. \tag{2.3}$$

Here, the elements of the set $\{|\psi_j\rangle\}$ are considered distinct if and only if $|\langle \psi_{j'}|\psi_j\rangle| < 1 \, \forall j \neq j'$. It will suffice to show that if $N = C+D-1$ then the states $|\psi_j\rangle^{\otimes C}$ are linearly independent. To see why, we simply note that, if this can be shown, then our more general claim will be true as a consequence of the fact that any subset of a linearly independent set is also linearly independent.

To prove the sufficiency of (2.3), we assume that $N = C + D - 1$ and again make use of the fact that any linearly dependent set contains a linearly independent spanning subset. The set $\{|\psi_j\rangle\}$ then has a subset of D linearly independent states, which we shall denote by S^1_{LI}. Without loss of generality, we can relabel all states according to the index j in such a way that $|\psi_j\rangle \in S^1_{\mathrm{LI}}$ for $j = 1,...,D$.

Let us now consider the sets $S^r_{\mathrm{LI}} = \{|\psi_j\rangle^{\otimes r} | j = 1,...,D+r-1\}$, for $r = 1,...,C$. Notice that S^1_{LI} accords with our previous definition and that $S^C_{\mathrm{LI}} = \{|\psi_j\rangle^{\otimes C}\}$. We will use induction to prove that $\{|\psi_j\rangle^{\otimes C}\}$ is linearly independent. The set S^1_{LI} is linearly independent by definition, and we will show that if S^{r-1}_{LI} is linearly independent then so is S^r_{LI}. To do this, we shall require the following lemma.

Lemma. Let $\{|\chi_k\rangle\} \in \mathcal{H}$ and $\{|\phi_k\rangle\} \in \mathcal{H}'$ be sets of distinct, normalized state vectors which have equal cardinality. Consider any normalized states $|\chi\rangle \in \mathcal{H}$ and $|\phi\rangle \in \mathcal{H}'$ such that $|\chi\rangle$ is distinct from all elements of $\{|\chi_k\rangle\}$. If the set $\{|\phi_k\rangle\}$ is linearly independent, then so is the set $\{|\phi_k\rangle \otimes |\chi_k\rangle\} \cup (|\phi\rangle \otimes |\chi\rangle)$.

A proof of this is given in the Appendix. The linear independence of S^{r-1}_{LI} implies that of S^r_{LI} if we make the identifications

$$S^{r-1}_{\mathrm{LI}} = \{|\phi_k\rangle\} \tag{2.4}$$

$$\{|\psi_j\rangle | j = 1,...,D+r-2\} = \{|\chi_k\rangle\}, \tag{2.5}$$

$$|\psi_{D+r-1}\rangle^{\otimes r-1} = |\phi\rangle, \tag{2.6}$$

$$|\psi_{D+r-1}\rangle = |\chi\rangle \tag{2.7}$$

for $r = 2,...,C$. Thus, the set $S^C_{\mathrm{LI}} = \{|\psi_j\rangle^{\otimes C}\}$ is linearly independent, and this completes the proof. We have shown that Eq. (2.3) is a sufficient condition for unambiguous discrimination among N states spanning a D-dimensional space given C copies of the state.

Like the necessary condition in Eq. (2.1), this bound is the tightest we can obtain using N, C, and D alone, in the sense that for all values of these parameters that do not satisfy Eq. (2.3) there exists a set of states $\{|\psi_j\rangle^{\otimes C}\}$ that is linearly dependent. To prove this, suppose that for $j = 1,...,D$, the $|\psi_j\rangle$ are linearly independent and that for $j = D+1,...,N$, $|\psi_j\rangle = a_j|\psi_{D-1}\rangle + b_j|\psi_D\rangle$, for some complex coefficients a_j, b_j. If Eq. (2.3) is not satisfied, then $N \geqslant C + D$ and the subspace spanned by $|\psi_{D-1}\rangle$ and $|\psi_D\rangle$ contains at least $C+2$ states. We will now see that the set of states $\{|\psi_j\rangle^{\otimes C} | j = D-1, D, \ldots, N\}$ is linearly dependent. For $j = D-1,...,N$, the $|\psi_j\rangle$ all lie in the same two-dimensional subspace, so that the corresponding C-fold copies $|\psi_j\rangle^{\otimes C}$ lie in the symmetric subspace of C qubits, which, from Eq. (2.2), is $C+1$ dimensional. It follows that, if there are at least $C+2$ of these states, they must be linearly dependent. This implies that the entire set of C-fold copies $\{|\psi_j\rangle^{\otimes C} | j = 1,...,N\}$ is linearly dependent.

The necessary and sufficient conditions (2.1) and (2.3) for the linear independence of C copies of N states with single-copy Hilbert space dimension D are thus the most complete statements that can be made about the possibility of unambiguous discrimination given only these three parameters. These two bounds are also, in general, different from each other, which implies that for a particular set of states additional, more detailed information about the set may be useful.

However, this is not the case for $D = 2$. For the case of qubits, these bounds are identical, and equal to $C+1$. Thus, the necessary and sufficient condition for the possibility of unambiguous discrimination among N pure, distinct states of a qubit, given C copies of the state, is that

$$N \leqslant C+1. \tag{2.8}$$

The generality of this result is quite remarkable, since it is completely independent of the actual states involved. These will, however, have a strong bearing on the maximum probability of success.

III. DISCRIMINATION AMONG MULTITRINE STATES

A. Trine and lifted trine states

Having discussed in the preceding section the conditions under which unambiguous discrimination among multiple copies of linearly dependent states is possible, let us examine in detail one particular example, that of the so-called trine ensemble. Consider a qubit whose two-dimensional Hilbert space is denoted by $\mathcal{H}_2$. Let $\{|x\rangle,|y\rangle\}$ be an orthonormal basis for $\mathcal{H}_2$. Then the following states, if they have equal *a priori* probabilities equal to $\frac{1}{3}$, form the trine ensemble:

$$|t_1\rangle=|y\rangle, \tag{3.1}$$

$$|t_2\rangle=\frac{1}{2}(-|y\rangle+\sqrt{3}|x\rangle), \tag{3.2}$$

$$|t_3\rangle=\frac{-1}{2}(|y\rangle+\sqrt{3}|x\rangle. \tag{3.3}$$

These states are clearly linearly dependent, and so cannot be unambiguously discriminated at the level of one copy. Given only a single copy, we must tolerate a nonzero error probability in any attempt to distinguish among these states. The minimum error probability is equal to $\frac{1}{3}$ [8]. The optimum such measurement has recently been carried out in the laboratory, where the trine ensemble was implemented as a set of nonorthogonal optical polarization states [9]. Applications of the trine ensemble and optimal measurements to quantum key distribution are discussed in [10].

The trine ensemble may be regarded as a special case of a more general ensemble of states having the same threefold rotational symmetry, but also having a component in a third direction, which exists in a larger, three-dimensional Hilbert space $\mathcal{H}_3\supset\mathcal{H}_2$. Let this third dimension be spanned by the vector $|z\rangle$ orthogonal to both $|x\rangle$ and $|y\rangle$. This generalized trine ensemble may be written as

$$|T_j(\lambda)\rangle=\lambda|z\rangle+\sqrt{1-\lambda^2}|t_j\rangle, \tag{3.4}$$

for some real parameter $\lambda\in[0,1]$ known as the *lift* parameter. When $\lambda=0$, the $|T_j(\lambda)\rangle$ are just the coplanar trine states. If, however, $0<\lambda<1$, then the states are lifted out of the plane and, for $\lambda<1$, are linearly independent. These are known as *lifted trine states* [5]. As the lift parameter is increased, the states become increasingly distinct until $\lambda=1/\sqrt{3}$, at which point they are orthogonal. Increasing λ further serves to draw the three states closer to the $|z\rangle$ axis until $\lambda=1$, at which point $|T_j(\lambda)\rangle=|z\rangle$.

In this section, we show that the set of C-fold copies of the trine ensemble, which we refer to as a multitrine ensemble, may be represented as a lifted trine ensemble. We will use this, together with the fact that the maximum discrimination probability for lifted trine states can be derived exactly, to determine the maximum discrimination probability for multiple copies of the trine states.

To show that the states $|t_j\rangle^{\otimes C}$ are lifted trine states, we will make use of the fact that the states $|\tau_j(\lambda)\rangle=|T_j(\lambda)\rangle\otimes|t_j\rangle$, for $\lambda\in[0,1)$, are also lifted trine states, with a different, nonzero lift parameter. To see this, let us define the three orthogonal states

$$|X\rangle=\sqrt{2/(1+\lambda^2)}(\lambda|z\rangle\otimes|x\rangle-\frac{\sqrt{1-\lambda^2}}{2}(|x\rangle\otimes|y\rangle+|y\rangle\otimes|x\rangle), \tag{3.5}$$

$$|Y\rangle=\sqrt{2/(1+\lambda^2)}(\lambda|z\rangle\otimes|y\rangle-\frac{\sqrt{1-\lambda^2}}{2}(|x\rangle\otimes|x\rangle+|y\rangle\otimes|y\rangle), \tag{3.6}$$

$$|Z\rangle=\frac{1}{\sqrt{2}}(|x\rangle\otimes|x\rangle+|y\rangle\otimes|y\rangle). \tag{3.7}$$

Then the $|\tau_j(\lambda)\rangle$ may be written as

$$|\tau_1(\lambda)\rangle=L|Z\rangle+\sqrt{1-L^2}|Y\rangle, \tag{3.8}$$

$$|\tau_2(\lambda)\rangle=L|Z\rangle+\frac{\sqrt{1-L^2}}{2}(-|Y\rangle+\sqrt{3}|X\rangle), \tag{3.9}$$

$$|\tau_3(\lambda)\rangle=L|Z\rangle-\frac{\sqrt{1-L^2}}{2}(|Y\rangle+\sqrt{3}|X\rangle), \tag{3.10}$$

where the parameter L is

$$L=\sqrt{(1-\lambda^2)/2}. \tag{3.11}$$

Comparison of the $|\tau_j(\lambda)\rangle$ with the $|T_j(\lambda)\rangle$ shows that they are indeed lifted trine states, with lift parameter L, given by Eq. (3.11). Also, for any $\lambda\in[0,1)$, $L>0$ and the $|\tau_j(\lambda)\rangle$ are linearly independent.

We can now use simple induction to show that the states $|t_j\rangle^{\otimes C}$ are lifted trine states. In the above argument, if we let $|T_j(\lambda)\rangle=|t_j\rangle$, i.e., take $\lambda=0$, then we find that $|\tau_j(\lambda)\rangle=|t_j\rangle^{\otimes 2}$, and that these are lifted trine states with lift parameter $1/\sqrt{2}$. For the inductive step, we can say that, if $|t_j\rangle^{\otimes C-1}$ is a set of lifted trine states with lift parameter L_{C-1}, then so is the set $|t_j\rangle^{\otimes C}$, with some lift parameter L_C. It follows from Eq. (3.11) that these lift parameters for successive values of C obey the recurrence relation

$$L_C=\sqrt{(1-L_{C-1}^2)/2}, \tag{3.12}$$

with the boundary condition $L_1=0$. The solution is

$$L_C=\left\{\frac{1}{3}\left[1-\left(\frac{-1}{2}\right)^{C-1}\right]\right\}^{1/2}. \tag{3.13}$$

Thus, the states $|t_j\rangle^{\otimes C}$ are lifted trine states with lift parameter given by Eq. (3.13).

B. Discrimination among lifted trine states

To determine the maximum discrimination probability for the lifted trine ensemble, we make use of the theorem in [11] which gives the maximum discrimination probability for equally probable, linearly independent symmetrical states.

A set of N linearly independent symmetric states can be expressed as

$$|\psi_j\rangle=\sum_{k=0}^{N-1} c_k e^{2\pi ijk/N}|u_k\rangle, \tag{3.14}$$

where $\Sigma_{k=0}^{N-1}|c_k|^2=1$, $c_k\neq 0$, and $\langle u_{k'}|u_k\rangle=\delta_{k'k}$. The maximum discrimination probability is

$$P_{\max}=N\min_k|c_k|^2. \tag{3.15}$$

For the lifted trine states, we define the following orthogonal states:

$$|u_0\rangle=|z\rangle, \tag{3.16}$$

$$|u_1\rangle=\frac{e^{5\pi i/6}}{\sqrt{2}}(|x\rangle+i|y\rangle), \tag{3.17}$$

$$|u_2\rangle=\frac{e^{-5\pi i/6}}{\sqrt{2}}(|x\rangle-i|y\rangle). \tag{3.18}$$

In terms of these states, one can easily verify that the lifted trine states have the form

$$|T_1(\lambda)\rangle=\lambda|u_0\rangle+\sqrt{(1-\lambda^2)/2}(e^{2\pi i/3}|u_1\rangle+e^{4\pi i/3}|u_2\rangle), \tag{3.19}$$

$$|T_2(\lambda)\rangle=\lambda|u_0\rangle+\sqrt{(1-\lambda^2)/2}(e^{4\pi i/3}|u_1\rangle+e^{8\pi i/3}|u_2\rangle), \tag{3.20}$$

$$|T_3(\lambda)\rangle=\lambda|u_0\rangle+\sqrt{(1-\lambda^2)/2}(|u_1\rangle+|u_2\rangle). \tag{3.21}$$

We can verify that these expressions are of the form (3.14) if we take the coefficients c_k to be

$$c_0=\lambda, \tag{3.22}$$

$$c_1=c_2=\sqrt{(1-\lambda^2)/2}. \tag{3.23}$$

Making use of these expressions and employing Eq. (3.15), we find that the maximum discrimination probability for the lifted trine states is

$$P_{\max}=3\min\left(\lambda^2,\frac{1-\lambda^2}{2}\right). \tag{3.24}$$

The behavior of $P_{\max}$ as a function of λ is illustrated in Fig. 1. For $0\leqslant\lambda\leqslant 1/\sqrt{3}$, $P_{\max}=2\lambda^2$, which increases monotonically to 1 until the orthogonality point $\lambda=1/\sqrt{3}$. For $1/\sqrt{3}\leqslant\lambda\leqslant 1$, $P_{\max}=(3/2)(1-\lambda^2)$, which decreases monotonically, reaching zero when $\lambda=1$, at which point all three states are identical.

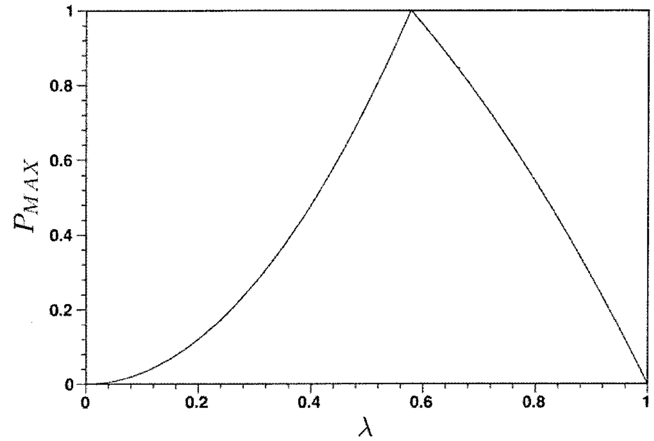

FIG. 1. Maximum probability $P_{\max}$ of unambiguous discrimination among lifted trine states as a function of the lift parameter λ. For $\lambda=0,1$, the states are linearly dependent and so unambiguous discrimination is impossible. However, at $\lambda=1/\sqrt{3}\approx 0.557$, the states are orthogonal and can be discriminated with unit probability.

We have shown how to calculate the maximum discrimination probability for lifted trine states. We will now see how these results can be used to obtain the maximum discrimination probability for multiple copies of the trine states.

C. Discrimination among multitrine states

We are now in a position to calculate the maximum discrimination probability for the states $|t_j\rangle^{\otimes C}$. It follows from Eq. (2.8) that the necessary and sufficient condition for unambiguous discrimination is that $C\geqslant 2$. Making use of Eqs. (3.13) and (3.24), we see that this is given by

$$\begin{aligned}P_{\max}(|t_j\rangle^{\otimes C})&=3\min\left(L_C^2,\frac{1-L_C^2}{2}\right)=3\min(L_C^2,L_{C+1}^2)\\&=\min\left(1-\left(\frac{-1}{2}\right)^{C-1},1-\left(\frac{-1}{2}\right)^{C}\right).\end{aligned} \tag{3.25}$$

It is quite straightforward to show that the smaller of these two terms is determined solely by whether C is even or odd, and we find

$$P_{\max}(|t_j\rangle^{\otimes C})=\begin{cases}1-2^{-C}, & \text{even } C\\ 1-2^{-(C-1)}, & \text{odd } C\end{cases}. \tag{3.26}$$

Some interesting observations can be made about this result. First, the minimum probability of inconclusive results, given by $1-P_{\max}(|t_j\rangle^{\otimes C})$, decreases exponentially with increasing C, with even and odd cases considered separately. However, Eq. (3.26) has one peculiar, unexpected feature, which is that $P_{\max}(|t_j\rangle^{\otimes C})=P_{\max}(|t_j\rangle^{\otimes C+1})$ for even C. That is, adding another copy to an even number of copies does not increase the maximum discrimination probability. This behavior provides an interesting exception to the trend observed in state estimation/discrimination that the more copies we have of the state, the better we can determine it [3].

One further curious feature of the maximum discrimination probability in Eq. (3.26) is that it can be attained by carrying out collective measurements only on pairs of copies of the state. Suppose that C is even: if it is not, we simply discard one of the copies, in view of the above results. We divide the set of copies into $C/2$ pairs, and carry out an optimal discrimination measurement on each pair. The probability of success for one pair is $P_{\max}(|t_j\rangle^{\otimes 2})$. The success probability for all C copies by this method is simply the probability that not all of the $C/2$ pairwise measurements give inconclusive results, which is $1-[1-P_{\max}(|t_j\rangle^{\otimes 2})]^{C/2}$. Into this we insert $P_{\max}(|t_j\rangle^{\otimes 2})=3/4$, which is the special case of Eq. (3.26) for $C=2$, and obtain the general maximum discrimination probability for all C. The ability to do optimum discrimination for this ensemble with only pairwise measurements is clearly convenient from a practical perspective.

IV. DISCUSSION

It is impossible to discriminate unambiguously among a set of linearly dependent states. If, however, we have access to more than one copy belonging to such a set, then the compound states may be linearly independent, and thus amenable to unambiguous discrimination. This is the issue we addressed in this paper.

It is natural to search for any general limitations on the extent to which this is achievable. The most natural parameters to consider are D, the dimension of the Hilbert space of a single copy, C, the number of copies, and N, the number of states. We derived one necessary and one sufficient condition, respectively Eqs. (2.1) and (2.3), for N states to be amenable to unambiguous discrimination for fixed C and D. These conditions were shown to be tight and, for $D=2$, identical. Combining them solves the problem completely for qubits.

We then worked out in detail the specific example of unambiguous discrimination among $C \geqslant 2$ copies of the trine states. We showed how such multitrine states can be interpreted as lifted trine states, for which the maximum unambiguous discrimination probability can be calculated exactly. We also found that if C is even then adding a further copy, strangely, fails to increase the maximum discrimination probability. Also, we described how the optimum discrimination measurement for arbitrary $C \geqslant 2$ can be carried out by performing optimum discrimination measurements only on pairs of copies.

We conclude with an observation regarding the related subject of probabilistic cloning. It was established by Duan and Guo [12] that a set of quantum states can be probabilistically copied exactly if and only if they are linearly independent. This result is rigorously correct for $1 \to M$ cloning. If, however, $1 < C < M$ copies of the state are initially available, then sometimes $C \to M$ cloning will be possible for linearly dependent sets. A sufficient condition is that the $|\psi_j\rangle^{\otimes C}$ are linearly independent. This may be accomplished, for example, by carrying out an unambiguous discrimination measurement to determine the state and then manufacturing M copies of the state.

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APPENDIX: PROOF OF LEMMA

Proof. We prove this by contradiction. If the set $\{|\phi_k\rangle \otimes |\chi_k\rangle\} \cup (|\phi\rangle \otimes |\chi\rangle)$ is linearly dependent, then there exist coefficients b and b_k such that

$$b|\phi\rangle \otimes |\chi\rangle + \sum_k b_k |\phi_k\rangle \otimes |\chi_k\rangle = 0, \tag{A1}$$

where not all of the coefficients in $\{b, b_k\}$ are zero. In fact, we can show that at least two of the b_k are nonzero. If only one of the b_k were nonzero, then, depending on whether or not $b=0$, the corresponding $|\chi_k\rangle$ will be equal to either $|\chi\rangle$ (up to a phase) or the zero vector. The latter possibility contradicts the premises of the lemma (normalization). The former does also, since it would imply that, for the nonzero $b_k, |\chi_k\rangle$ is not distinct from $|\chi\rangle$.

The set $\{|\phi_k\rangle\}$ is linearly independent, so there exists a set of reciprocal states $\{|\tilde{\phi}_k\rangle\} \in \mathcal{H}'$ such that $\langle \tilde{\phi}_{k'}|\phi_k\rangle = \langle \tilde{\phi}_k|\phi_k\rangle \delta_{kk'}$ and $\langle \tilde{\phi}_k|\phi_k\rangle \neq 0 \forall k$. Acting on Eq. (A1) throughout with $\langle \tilde{\phi}_k| \otimes 1$ gives

$$b\langle \tilde{\phi}_k|\rangle|\chi\rangle + b_k \langle \tilde{\phi}_k|\phi_k\rangle|\chi_k\rangle = 0 \quad \forall\ k. \tag{A2}$$

The fact that at least two of the b_k are nonzero implies that the corresponding $|\chi_k\rangle$ will be indistinct, contradicting the premise. This completes the proof.

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