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Designing optimum completely positive maps for quantum teleportation

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We study a general teleportation scheme with an arbitrary state of the pair of particles (2 and 3) shared by Alice and Bob, and arbitrary measurements on the input particle 1 and one of the members (2) of the pair on Alice's side. We find an efficient iterative algorithm for identifying optimum operations on Bob's side. In particular, we find that simple unitary transformations on his side are not always optimal even if particles 2 and 3 are perfectly entangled. We describe the most interesting protocols in the language of extremal completely positive maps.

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Of many potential applications of quantum information processing, the quantum teleportation is probably the most appealing example. Several experiments [1] have been done since the first proposal in [2], confirming thus that teleportation—a popular subject of science-fiction literature—is indeed feasible, at least if one deals with simple quantum objects.

Ideal teleportation requires a source of maximally entangled particles and a very delicate measurement on the sender's side. These are not always easy to do with realistic experimental devices. For example, it has been shown in [3] that a never-failing Bell-state measurement is impossible with linear elements and detectors only. More severe limitations might arise if the teleported object gets more complicated.

However, even with restricted resources there is still the possibility to optimize the teleporting scheme. Optimization over the joint measurement on particles 1 and 2 *and* the operation on the particle 3 for the given state of the shared pair (particles 2 and 3) has been done in [4]. Here we put a more severe restriction on the resources and ask the following question: What is the optimum operation on particle 3 (output of teleportation) for the given resources of the shared entangled pairs of particles (2 and 3), *and* of the measurements performed jointly on particle 2 and the input particle 1? Several attempts have been done in this direction. In [5] the authors analyzed a teleportation with realistic linear elements and maximally entangled shared state. Optimization of protocols with arbitrary shared entangled states has been pursued in [6] and [7], but the optimization was done over unitary transformations only.

The main goal of this paper is to consider the most general case. For the given resources of the sender (type of the measurement on particles 1 and 2) and resources on the quantum channel shared by two parties (the state of particles 2 and 3), we will optimize the teleportation protocol by finding the optimum operation that should be applied by the receiver on particle 3. In contrast to the naive picture suggesting that every interaction of a quantum system with environment leads to a "loss of information," we here find that such an interaction on receiver's side might enhance the fidelity of teleportation even if the pair of particles constituting the quantum channel are perfectly entangled.

Let us consider the teleportation of an unknown state $\overline{\rho_1}$ of particle 1 between two parties called Alice and Bob. Let us assume that before the teleportation starts they share two particles in an arbitrary state τ_{23}. Alice then performs a measurement on particles 1 and 2 and sends the outcome by a classical channel to Bob. Based on the classical communication he receives, he performs a transformation on particle 3. The optimum transformation is such that the final state of particle 3 gets as close to the input state $\overline{\rho_1}$ as possible on the average.

The measurement performed by Alice will be described by a positive operator-valued measure (POVM) $\{\Pi_{12}^j\}$, $\Sigma_j \Pi_{12}^j = 1$. Its elements generate probabilities of all possible outcomes of Alice's measurement. Consider the situation where one such particular outcome, say $j=a$, has been registered with probability p_a. The state of the third particle conditioned on this result reads

$$\rho_3^a = \frac{1}{p_a} \mathrm{Tr}_{12}\{\overline{\rho_1}\tau_{23}\Pi_{12}^a\} = \frac{1}{p_a}\mathrm{Tr}_1\{\overline{\rho_1}O_{13}^a\}, \tag{1}$$

where $O_{13}^a = \mathrm{Tr}_2\{\tau_{23}\Pi_{12}^a\}$. Now Bob applies a ($a$ dependent) transformation on this state. The most general transformation is a trace preserving completely positive (CP) map of the form [8]

$$\Phi^a(\rho_3^a) = \sum_k A_{k3}^a \rho_3^a A_{k3}^{a\dagger}, \quad \sum_k A_{k3}^{a\dagger} A_{k3}^a = 1. \tag{2}$$

The optimality of the given set of transformations will be judged by the fidelity of the corresponding teleported state averaged over all possible outcomes a and over a distribution of input states $\overline{\rho_1}$; the averaging over input states will be denoted $\langle \cdots \rangle$. To simplify further considerations let us assume that the input states are pure [9]. In that case the fidelity of teleportation can be defined as $F = \mathrm{Tr}\{\overline{\rho_3}\rho_3\}$, where $\overline{\rho_3}$ is the input state in the Hilbert space of particle 3 and ρ_3 is the teleported state. Using Eqs. (1) and (2) the average fidelity becomes

$$\langle F \rangle = \sum_a \mathrm{Tr}_{13}\left\{ \langle \overline{\rho_3}\,\overline{\rho_1} \rangle \sum_k A_{k3}^a O_{13}^a A_{k3}^{a\dagger} \right\}. \tag{3}$$

This expression is to be maximized over the set of all possible operations applied to particle 3.

Although the following optimization can be carried out for arbitrary dimensional Hilbert spaces, we will illustrate the idea on the simple example of spin-1/2 particles. The generalization to more dimensions is straightforward.

First one has to choose an appropriate distribution of input states. Obviously, the choice depends on the prior knowledge one has about the state to be teleported. We will assume a complete ignorance of Alice and Bob about the incoming state, which is the usual situation described by the isotropic distribution of input states. More general situations can be handled analogously, see the reference after Eq. (4). Further, the input density matrix will be decomposed in some basis of Hermitian generators. In the case of spin 1/2, the convenient choice is the basis of Pauli spin matrices. After substituting the decomposition into Eq. (3), and integrating over the whole surface of Poincare sphere, we get

$$\langle F\rangle=\frac{1}{2}+\frac{1}{12}\sum_a\sum_k \mathrm{Tr}\{\vec{\sigma}\cdot A_k^a\vec{O}^a A_k^{a\dagger}\}, \tag{4}$$

where $\vec{O}^a=\mathrm{Tr}_1\{\vec{\sigma}_1 O_{13}^a\}$, $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$, and where we have now dropped the unnecessary subscript of particle 3 [10].

Since CP maps corresponding to different registrations a are independent, each term on the right-hand side of Eq. (4) can be maximized independently. Omitting therefore notation a and using the constraint $\Sigma_k A_k^\dagger A_k=1$, the expression to be maximized is

$$\sum_k \mathrm{Tr}\{\vec{\sigma}A_k\vec{O}A_k^\dagger - A_k\Lambda A_k^\dagger\}=\text{maximum}. \tag{5}$$

Variation of this expression with respect to $A_k^\dagger$ gives the extremal equation in the form

$$\sum_i \sigma_i A_k O_i = A_k\Lambda, \tag{6}$$

where Λ is the (Hermitian) Lagrange operator. It can be determined from Eq. (6) as follows:

$$\Lambda=(\vec{X}\cdot\vec{O}+\vec{O}\cdot\vec{X})/2, \tag{7}$$

where we have introduced Hermitian operators $\vec{X}=\Sigma_k A_k^\dagger\vec{\sigma}A_k$ that provide another representation of the CP map $\{A_k\}$. Equation (6) can be brought to the form suitable to iterations. Multiplying it by $A_k^\dagger\sigma_j$ from the left and summing over k we obtain

$$\vec{X}\Lambda=\vec{O}-i|\vec{X}\times\vec{O}|, \tag{8}$$

a formula suitable to iterative solving is obtained by adding $\vec{0}=\vec{X}-\vec{X}$ to the left-hand side of Eq. (8) and rearranging

$$\vec{X}=\vec{X}+\vec{O}-(i|\vec{X}\times\vec{O}|+\vec{X}\Lambda+\mathrm{H.c.}). \tag{9}$$

The iterative algorithm for finding optimum CP maps based on Eqs. (7) and (9) is the main formal result of the present article. Starting from some "unbiased" CP map, for example, $\vec{X}=\vec{0}$ (means that particle 3 is always brought to the maximally mixed state), the equations can be successively iterated until the stationary point is attained. In this way we get the operators $\vec{X}$ corresponding to the optimum transformation of particle 3.

Notice that the average fidelity of teleportation bears a very simple form when expressed in terms of $\vec{X}$,

$$\langle F\rangle=\frac{1}{2}+\frac{1}{12}\sum_a \mathrm{Tr}\{\vec{X}^a\cdot\vec{O}^a\}. \tag{10}$$

Notice also that $\langle F\rangle$ is a linear functional of $\vec{X}$. This means that all its maxima lie on the boundary of the set of physically allowed operators $\vec{X}$ that is determined by the constraint of complete positiveness of the corresponding transformations. So there is a clear connection between optimum teleportation protocols and extremal CP maps. The topology of CP maps is a well studied field related to many problems in quantum information processing. We will use some of the recently derived results on CP maps for the discussion of some cases of special interest.

But before we come to this point let us first demonstrate the usefulness of our iterative optimizing algorithm on an interesting example involving spin-1/2 systems. We will consider the realistic situation with a perfect source of shared particles but with an imperfect measurement. The imperfect measurement will be drawn from this one-parametric family of POVMs,

$$\begin{aligned}
\Pi_{12}^a&=\frac{\sin^2\theta}{2}|--\rangle\langle--|+\frac{1}{2}|\phi^a\rangle\langle\phi^a|\\
|\phi^a\rangle&=\cos\theta|+-\rangle-|-+\rangle,\\
\Pi_{12}^b&=\frac{\sin^2\theta}{2}|++\rangle\langle++|+\frac{1}{2}|\phi^b\rangle\langle\phi^b|\\
|\phi^b\rangle&=\cos\theta|-+\rangle+|+-\rangle,\\
\Pi_{12}^c&=\frac{\sin^2\theta}{2}|+-\rangle\langle+-|+\frac{1}{2}|\phi^c\rangle\langle\phi^c|\\
|\phi^c\rangle&=\cos\theta|--\rangle+|++\rangle,\\
\Pi_{12}^d&=\frac{\sin^2\theta}{2}|-+\rangle\langle-+|+\frac{1}{2}|\phi^d\rangle\langle\phi^d|\\
|\phi^d\rangle&=\cos\theta|++\rangle-|--\rangle.
\end{aligned} \tag{11}$$

Here $|+\rangle$ and $|-\rangle$ are two orthogonal states, for example, states spin up and spin down in the z direction and $\theta\in[0,\pi/2]$. One boundary point $\theta=0$ corresponds to the perfect Bell measurement. Alice gets no information about the incoming state, but the teleportation with fidelity one is possible. The other boundary point $\theta=\pi/2$ corresponds to a projective measurement on the first particle and no measure-

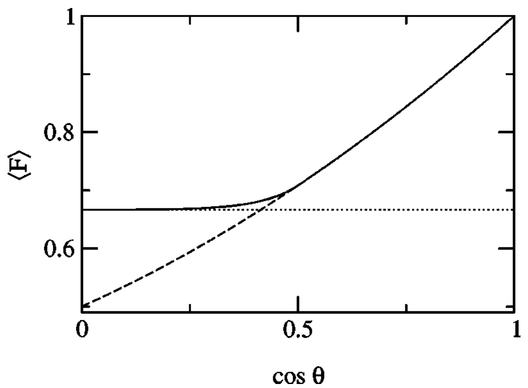

FIG. 1. Fidelities of optimum CP maps for Alice's measurements (11). Solid line shows the performances of optimum CP maps; dashed line shows the performances of optimum unitary operations. Dotted horizontal line shows the boundary between classical and quantum teleportation protocols.

ment on the second one. In this case $\Pi^a = \Pi^d$ and $\Pi^b = \Pi^c$, and we have only two distinct outcomes $\Pi^a + \Pi^d = |-\rangle\langle -|_1 \otimes \hat{1}_2$ and $\Pi^b + \Pi^c = |+\rangle\langle +|_1 \otimes \hat{1}_2$. Alice gets maximum amount of information about the input state, but the "teleported" state bears no relation to the input state—the quantum resources are wasted.

For intermediate values of θ, less information is extracted about the input state and an imperfect quantum teleportation is possible. As we have mentioned above, we will take the particles 2 and 3 in a maximally entangled state, for instance, let them be in the singlet state $\tau_{23} = \frac{1}{4}(1 - \sigma_{1x}\sigma_{2x} - \sigma_{1y}\sigma_{2y} - \sigma_{1z}\sigma_{2z})$. Accordingly, the operators $\vec{O}^a$ have the form,

$$\vec{O}^a = 4R\vec{\sigma} + 4\vec{r}, \tag{12}$$

$$R = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix}, \quad \vec{r} = \begin{pmatrix} 0 \\ 0 \\ -\sin^2\theta \end{pmatrix}. \tag{13}$$

The operators $\vec{O}$ generated by the remaining three POVM elements Π^b, Π^c, and Π^d differ from Eq. (13) only in signs of its elements, and hence, their contribution to the average fidelity (10) is the same.

Fidelities (10) of the optimum Bob's transformations that were found by our iterative algorithm (7) and (9) for the above Alice's measurements are shown in Fig. 1 (solid line). As could have been anticipated, the fidelity continuously changes from the classical limit $\langle F \rangle = 2/3$ to the maximum value of $\langle F \rangle = 1$. It is interesting to note that the optimum CP maps for $\cos\theta \geqslant 1/2$ are actually unitary operations. However, for $\cos\theta < 1/2$ unitary operations are not optimum, see Fig. 1 (dashed line). This becomes a trivial statement if $\cos\theta = 0$. In that case Alice performs no measurement on particle 2, and therefore the state of particle 3 remains a complete mixture for any outcome a she obtains. So the fidelity of the "teleported" state is 1/2 if one is allowed to perform unitary transformations only. In contrast, if one adopts more general transformations, Bob can construct the input state with fidelity 2/3 on the basis of the outcomes of Alice's measurement (which is here optimal state estimation), thus attaining the classical limit.

The most interesting situations correspond to Alice's measurements that allow nonclassical teleportation only if followed by a *nonunitary* operation on Bob's particle. In our case this happens for $0 < \cos\theta \leqslant \sqrt{2} - 1$ (see later) and the optimum CP map turns out to be a kind of decoherence process [11].

Let us emphasize again that the above example of teleporting spin 1/2 system has been chosen for the sake of simplicity only. If needed, our iterative algorithm could be straightforwardly generalized to larger Hilbert spaces. The generalization consists of replacing Pauli spin matrices by the appropriate basis of Hermitian operators in that space and replacing the integration over the surface of the Poincare sphere by the integration over the surface of generalized N-dimensional sphere. Nevertheless, the relative simplicity of the set of CP maps operating on a 2-dimensional space allows one to get further insight into the optimum teleportation protocols and obtain analytical results. We will therefore stick to the teleportation of spin 1/2 particles in the following.

First let us express the operation acting on Bob's particle using operators $\vec{X}$,

$$\Phi(1/2 + \vec{w}\cdot\vec{\sigma}/2) = 1/2 + (\vec{t} + T\vec{w})\cdot\vec{\sigma}/2 \tag{14}$$

where $\vec{t}$ and T are defined as follows: $\vec{X} = T\vec{\sigma} + \vec{t}$, and $\vec{w}$ is the Bloch vector defining the state of the Bob's particle before he applies the operation Φ. To each operation on the Bob's particle there corresponds a simple transformation of the Poincare sphere: The unit sphere is mapped onto an ellipsoid, the lengths of its axes being the eigenvalues of T, which is translated by $\vec{t}$ from the origin. Of course the ellipsoid has to lie within the unit Poincare sphere (positivity). However, not all such ellipsoids define *completely* positive maps that are the most general maps in quantum mechanics. Recently, it has been shown [12] how to parameterize the set of *extremal* CP maps comprising the boundary of the convex set of all CP maps. This set contains all Bob's optimum transformations. Matrix T can be always brought to the diagonal form by a unitary transformation. When in diagonal form, extremal CP maps can be parametrized by two angles u and v,

$$T = \begin{pmatrix} \cos u & 0 & 0 \\ 0 & \cos v & 0 \\ 0 & 0 & \cos u \cos v \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} 0 \\ 0 \\ \sin u \sin v \end{pmatrix},$$

with $u \in [0,2\pi)$, $v \in [0,\pi)$. Now let us show how the above example can be elegantly solved using this trigonometric parametrization of extremal CP maps. Due to the form of matrix R [Eq. (13)] it can be shown that the optimum CP map is degenerated, $u = 2\pi - v$, so there is only one free parameter left.

Substituting the trigonometric parametrization with $u = 2\pi - v$ and the POVM element (13) into Eq. (10) and maximizing the fidelity with respect to v, one easily finds the analytical expression for the optimum fidelity:

$$\langle F_{\mathrm{opt}}\rangle=\begin{cases}\dfrac{\cos^4\theta-4\cos^2\theta+2}{3-6\cos^2\theta}, & \cos\theta\in\left[0,\dfrac{1}{2}\right)\\[2ex] (\cos^2\theta+2\cos\theta+3)/6, & \cos\theta\in\left[\dfrac{1}{2},1\right].\end{cases} \tag{15}$$

In contrast to this, the fidelity of the optimum unitary operation is given by the bottom expression for all angles θ and thus coincides with the optimum value if $\cos\theta\geqslant 1/2$. We note that the optimum unitary operation is the identity map $u=v=0$ (Bob leaves particle 3 alone) for all θ. The analytical solution confirms the results obtained numerically with the help of our iterative algorithm shown in Fig. 1. The most interesting Alice's measurement (11) is that for which optimum unitary operation just gives the classical limit $\langle F\rangle=2/3\approx 0.667$, but enables nonclassical teleportation using a nonunitary CP map. This happens for $\cos\theta=\sqrt{2}-1$, optimum fidelity being $\langle F_{\mathrm{opt}}\rangle=(3+8\sqrt{2})/21\approx 0.6816$.

The enhancement of quantum teleportation by nonunitary process has recently been discussed by Badziąg *et al.* [11] in a slightly different context. They considered teleportation protocols with a perfect Bell analyzer but with imperfect preparation of the shared pair of particles, and found out that teleportation protocols could sometimes be enhanced by the interaction of the teleportation device with environment (damping). The most pronounced example yielded improvement corresponding to our $\cos\theta=\sqrt{2}-1$ case. In fact, the choice of our POVMs (11) that are Bell states being subject to a kind of decohering process has been inspired by their result. Now we can use the language of extremal CP maps to explain why the result of Badziąg *et al.* is so exceptional. A general teleportation protocol is, in fact, a CP map composed of two different maps: The first one Ψ being the Alice's transformation of the input state to the output state sent to Bob; the second being the operation Φ on the output state applied by Bob. The teleportation protocol Ω becomes perfect if the two maps make up the identity map:

$$\Omega(\overline{\rho_1})\equiv\sum_a p_a\Phi^a[\Psi^a(\overline{\rho_1})]=\overline{\rho_3},\quad \forall\ \overline{\rho_1}. \tag{16}$$

Obviously, the most interesting protocols are protocols where both parts Ψ and Φ are *extremal* CP maps, because they contain optimal maps, see discussion after Eq. (10), and because all remaining protocols are just convex combinations of such "extreme" protocols. Among the protocols consisting of two extremal maps, one can find the standard teleportation, both maps here being unitary operations, but this is also the case of our example (11). This is immediately seen using another equivalent representation of our Ψ [13]: $\Psi(\overline{\rho_1})=\mathrm{Tr}_1\{\overline{\rho_1}^T\chi_{13}\}$, where $\chi_{13}=O_{13}^{\tilde{T}}$, and $\tilde{T}$ is partial transposition with respect to system 1. For POVMs from our example (11) and shared singlets, we have that the operator χ is at most rank 2 operator and hence the map Ψ it generates is extremal.

The trigonometric parametrization of optimum CP maps hints on a possible generalization of the situation discussed in [11]. One could think of a protocol where the first CP map Ψ would not be degenerated with respect to angles u and v [unlike in (13)]. Such nondegenerated Alice's POVMs would not, however, lead to substantially new physics, since also in this case the optimum Bob's operation would be a kind of a damping channel. [12].

In conclusion, we have derived an iterative algorithm for finding optimum CP maps for quantum teleportation and have identified situations where a unitary transformation on the third particle is not optimal and should be replaced by a more general completely positive map.

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