Entanglement in a two-identical-particle system

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The definition of entanglement in identical-particle system is introduced. The separability criterion in two-identical-particle system is given. The physical meaning of the definition is analyzed. Applications to two-boson and two-fermion systems are made. It is found that some different entanglement and correlation phenomena in identical-boson systems exist, and they may have applications in the field of quantum information.

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There is no doubt that the phenomenon of quantum entanglement lies at the heart of the foundation of quantum mechanics. The original investigation on entanglement began from the famous Einstein-Podolsky-Rosen-Gedanken experiment. Entanglement has been widely applied in many aspects of quantum information such as quantum teleportation, quantum cryptography, and quantum computation. However, although it is well studied in distinguishable-particle systems, entanglement in identical-particle system has hardly been investigated, and even a proper definition is not given yet. It is noted that entanglement in certain systems such as quantum dots [1], Bose-Einstein condensates [2], and parametric down-conversion [3] must be dealt with in an identical-particle manner. Recently, this problem was noted by Schliemann *et al.* [1,4] and they discussed the entanglement in two-fermion system. They have found that entanglement in two-fermion system is analogous to that in a two-distinguishable-particle system, and the results obtained for two-distinguishable-particle system can be translated into the two-fermion system. However, due to the fundamental difference between bosons and fermions, the concept in two-boson systems is quite different. In this paper, we explore the definition of entanglement in indistinguishable-particle systems. We first show that factorization alone is not able to define entanglement in a two-boson system. Then we gave a general definition for entanglement in identical-particle systems. We show that this definition works well for two-boson systems as well as for two-fermion systems. Furthermore, the definition can be generalized into systems with more than two identical particles. We also address the concept of relative correlation.

Entanglement in distinguishable-particle systems has been well studied. For a system of two distinguishable particles possessing single-particle Hilbert space labeled by $\mathcal{H}_1$ and $\mathcal{H}_2$, the states can be described as vectors in the direct product space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$$|\mathbf{\Psi}\rangle_{12} = \sum_{i,j} c_{ij} |\phi_i\rangle_1 \otimes |\varphi_j\rangle_2, \tag{1}$$

where $\{|\phi_i\rangle\}$ is basis for $\mathcal{H}_1$ and $\{|\varphi_j\rangle\}$ is basis for $\mathcal{H}_2$, respectively. The state $|\mathbf{\Psi}\rangle$ is called separable if and only if it can be written as $|\mathbf{\Psi}\rangle_{12} = |\psi\rangle_1 \otimes |\psi'\rangle_2$, where $|\psi\rangle_1 \in \mathcal{H}_1$ and $|\psi'\rangle_2 \in \mathcal{H}_2$, otherwise it is entangled. We can define local operations as those operations acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A separable state cannot be transformed into an entangled state by any local operation and classical communication.

What will happen in indistinguishable-particle systems? Can the same definition be used? First, let us see the difference between distinguishable-particle systems and identical-particle systems. Suppose we have a two-photon Bell state,

$$|\leftrightarrow\rangle_1 |\updownarrow\rangle_2 + |\updownarrow\rangle_1 |\leftrightarrow\rangle_2, \tag{2}$$

where $|\leftrightarrow\rangle$ and $|\updownarrow\rangle$ stand for states with horizontal and vertical polarization, respectively. If the two photons are separable, say the two photons have different momentum though their frequencies are the same, then we can write state (2) in second-quantization formalism as $(a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger)|0\rangle$, where $a_1^\dagger|0\rangle$, $a_2^\dagger|0\rangle$, $a_3^\dagger|0\rangle$, and $a_4^\dagger|0\rangle$ stand for single-photon states $|\leftrightarrow\rangle_1$, $|\updownarrow\rangle_1$, $|\leftrightarrow\rangle_2$, and $|\updownarrow\rangle_2$, respectively. $|0\rangle$ is the vacuum state. Each photon in the system can be in one of the four modes $(a_i^\dagger|0\rangle, i=1,2,3,4)$, which span a four-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. $(a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger)|0\rangle$ is not separable and thus it is entangled. However, if the two photons are indistinguishable, then state (2) will be represented by $a_\leftrightarrow^\dagger a_\updownarrow^\dagger|0\rangle$, which is separable, and hence not entangled.

In identical-particle systems, it is impossible to distinguish the two particles. A direct-sum resolution of the single-particle state into two-constituent particle state is not possible. We can only say that there is one particle in a given state, but we cannot tell which of the two particles is in that state. Because of this, a separable state in identical particle system may be defined, analogous to the case of distinguishable particles.

In an identical two-particle system whose single-particle Hilbert space $\mathcal{H}$ is spanned by $\alpha_i^\dagger|0\rangle, i=1,2,\ldots,N$, a state is separable if it can be written as $c^\dagger d^\dagger|0\rangle$, where $c^\dagger|0\rangle, d^\dagger|0\rangle \in \mathcal{H}$. Otherwise it is entangled.

It will be shown next that this definition does not cover all the entangled states in two-boson systems. The state $|\mathbf{\Psi}\rangle$ of two identical bosons with a single-particle Hilbert space $\mathcal{H} = C^N$ can be described as follows:

$$|\mathbf{\Psi}\rangle = \sum_{i,j=1}^{N} \omega_{ij} a_i^\dagger a_j^\dagger |0\rangle, \tag{3}$$

where $\omega_{ij}=\omega_{ji}$ is an $N\times N$ complex symmetric matrix Ω and it can be decomposed to a diagonal matrix by the lemma below.

Lemma. For any symmetric $N\times N$ matrix S there exists a unitary transformation U, such that $S=UD_MU^T$, where the matrix D_M is diagonal,

$$D_M=\text{diag}[d_1,d_2,\ldots,d_M,Z], \tag{4}$$

and Z is a $(N-M)\times(N-M)$ null matrix. This lemma is analogous to a lemma given in Ref. [4] for an antisymmetric matrix in the identical-fermion case. The proof of this lemma will be given in the Appendix.

Now, we can diagonalize the state (3) by a unitary transformation U,

$$|\Psi\rangle=\sum_{j=1}^{M}\lambda_j c_j^\dagger c_j^\dagger|0\rangle,\quad c_i^\dagger=\sum_{j=1}^{N}u_{ji}a_j^\dagger, \tag{5}$$

where U is a representation transformation and we arrange the eigenvalues in absolute value descending order $|\lambda_1|\geqslant|\lambda_2|\geqslant\cdots\geqslant|\lambda_M|$. The above diagonal form can be regarded as a standard form because it is unique except for global phases in the definite two-boson basis states. Since the rank of the matrix Ω, M, does not change under unitary transformation, and it can be used as the criterion of entanglement for identical two-boson systems.

If $N=2$, the standard form of Eq. (5) can be written as $(r_1e^{i\varphi}c_1^\dagger c_1^\dagger+r_2e^{-i\varphi}c_2^\dagger c_2^\dagger)|0\rangle$ after neglecting an overall phase factor where r_1 and r_2 are non-negative. The state can be written as $[(r_1-r_2)f_1^\dagger f_1^\dagger+2\sqrt{r_1r_2}f_1^\dagger f_2^\dagger]|0\rangle$ by a representation transformation,

$$(c_1^\dagger\ \ c_2^\dagger)=(f_1^\dagger\ \ f_2^\dagger)\times\begin{pmatrix} e^{i\varphi/2}\sqrt{\dfrac{r_1}{r_1+r_2}} & -ie^{i\varphi/2}\sqrt{\dfrac{r_2}{r_1+r_2}} \\ -ie^{-i\varphi/2}\sqrt{\dfrac{r_2}{r_1+r_2}} & e^{-i\varphi/2}\sqrt{\dfrac{r_1}{r_1+r_2}} \end{pmatrix}. \tag{6}$$

If $r_1=r_2$, the state will be $f_1^\dagger f_2^\dagger|0\rangle$ whose rank of coefficient matrix is 2 [5]. It would be a separable state if the lemma is used.

It is easy to check that if Eq. (3) has the standard form as follows:

$$|\Psi\rangle=\sum_{i=1}^{L}z_i(e^{i\varphi_i}f_{1i}^\dagger f_{1i}^\dagger+e^{-i\varphi_i}f_{2i}^\dagger f_{2i}^\dagger)|0\rangle. \tag{7}$$

It can be transformed into $\Sigma_{i=1}^{L}2z_ic_{1i}^\dagger c_{2i}^\dagger|0\rangle$, which can be discussed as a system with distinguishable particles. If it has at least two nonzero z_i, it can be defined as a distinguishable entangled state because it is identical to entangled states in distinguishable-particle systems. In general, a separable state according to the lemma can be written as $c^\dagger(\alpha c^\dagger+\beta b^\dagger)|0\rangle$ with the rank being either 1 or 2, where $c^\dagger$ and $b^\dagger$ are orthogonal. States such as $c^\dagger(c^\dagger+b^\dagger)|0\rangle$ need special attention. It has no invariant particle number in the modes $c^\dagger$, $b^\dagger$, or $c^\dagger+b^\dagger$ and seems to have some persistent correlation. The density matrix of this state is not separable and it is an inseparable state. So the definition of entanglement has to be changed to let $c^\dagger(c^\dagger+b^\dagger)|0\rangle$ be entangled and hold for states such as $c^\dagger d^\dagger|0\rangle$ and $c^\dagger c^\dagger|0\rangle$ as separable. It is worth pointing out here that $c^\dagger c^\dagger|0\rangle$ can be regarded as if it were a single particle, and we treat them as separable. Of course, the definition should be generalizable to identical multiparticle systems.

Now we can give the following definition of separability and entanglement in identical-particle systems.

Definition 1. A state with identical k particles is separable if it can be written as $c_1^\dagger c_2^\dagger\cdots c_k^\dagger|0\rangle$, where $c_i^\dagger$ and $c_j^\dagger$ are either equal or orthogonal. Otherwise it is entangled.

This definition works for both identical-boson and identical-fermion systems, because in the fermion system, the Pauli principle prohibits two particles from occupying the same state, states with $c^\dagger(c^\dagger+b^\dagger)|0\rangle=c^\dagger b^\dagger|0\rangle$ that becomes the product of two orthogonal states automatically. It is interesting to point out that an equivalent definition can be formulated in the following: a state with identical k particle is separable if it is an eigenvector of a complete set of one-body Hermitian operators. Otherwise it is entangled. Complete operator sets can be generated by the operators $\{(a_i^\dagger a_j+a_j^\dagger a_i)/2,\ i,j=1,2,\ldots,n\}$. This alternative definition is consistent with the statement in [6,7], which says that any basis that is eigenvectors of complete set consists of one-body mechanical quantities must be separable in distinguishable-particle case.

For systems with two bosons, we have the standard form (5) to tell whether a state is entangled or not. A state must be entangled if the rank of its coefficient matrix Ω is greater than 2. If rank (Ω) is 2, it is also easy to judge whether it is entangled or not from its standard form according to definition 1. From a normalized standard form such as Eq. (5), we can define the entanglement measure as

$$-4\sum_{i=1}^{M/2}(|\lambda_{2i-1}\lambda_{2i}|)\ln(|4\lambda_{2i-1}\lambda_{2i}|) \tag{8}$$

for a two-boson system. It is similar to that in distinguishable-particle systems and if the standard form can be written as Eq. (7), which has a counterpart in distinguishable systems, the entanglement measure thus defined will be just the partial entropy in distinguishable-particle systems [11]. But for systems with $k(k>2)$ bosons, it is more difficult to give the standard forms to tell whether a state is separable. It is also noted that the quantitative description of entanglement in multiparticle systems is a very hard mathematical problem, and it is still not solved in distinguishable-particle systems. The problem to quantify entanglement in multi-identical-boson systems is even more difficult and remains an open challenge.

A different kind of entanglement has been found in identical-boson systems. They may have important applications different from those in distinguishable-particle systems. For example $(a^\dagger a^\dagger+b^\dagger c^\dagger)|0\rangle$ is an entangled state in an identical-boson system with $N=3$ single-particle space. It can denote a superposed state of two-photon states, one with two photons sent to Alice, and one with one photon sent to Bob and the other sent to Clare. Alice, Bob, and Clare measure the number of photons. If Alice gets two photons, no photon will reach Bob and Clare, and if no photon reaches Alice, either Bob or Clare will have one photon. If Bob gets one photon, the other photon will reach Clare. The situation of Clare is similar to that of Bob. So if one person of the three gets a result, the results of the other two are decided, which means that some kind of communication can be built with only two photons.

The above definition of entanglement also applies to many-identical fermion system. Schliemann *et al.* have discussed entanglement in two-fermion systems [4]. For a two-fermion system having the single-particle Hilbert space $\mathcal{C}^{2K}$, an arbitrary state has the form $|\Psi\rangle=\Sigma_{i,j=1}^{2K}\omega_{ij}a_i^\dagger a_j^\dagger|0\rangle$, where $\Omega=(\omega_{ij})$ are antisymmetric. It can be decomposed into a standard form $\Sigma_i^K z_i f_{1i}^\dagger f_{2i}^\dagger|0\rangle$ by a representation transformation [4], which implies that entanglement and separability in two-fermion systems are equivalent with those in distinguishable-particle systems. However, when generalizing this into many-fermion system, this elegant property in two-fermion system disappear in many-fermion system. This can be easily understood that the single-particle Hilbert space $\mathcal{H}$ for a state in system with k greater than two-identical fermions cannot be decomposed to a direct-sum resolution of k subspaces. For example, composing a three-fermion state with single-particle Hilbert space $\mathcal{C}^{3N}$ to three-orthogonal N-dimension subspaces requires $7N^3-9N^2+2N$ real equations satisfied while the group SU($3N$) can only provide $9N^2-1$ real parameters. The equations have the required number of parameters only when $N=1$ or $N=2$. It is easy to check that lemma and definition 1 are equivalent for identical-fermion systems because arbitrary two fermions cannot be in one state. Hence for identical-two-fermion systems, the rank of Ω is the criterion to judge entanglement or separability: rank $(\Omega)=2$ for separability and rank $(\Omega)>2$ for entanglement. It must be noted that rank $(\Omega)\neq 1$ for two-fermion systems.

Another important concept is quantum correlation. It is quite often used in literature of physics. A recent development has made this concept an important one. It is well known that quantum teleportation can be implemented with Bell states that are distinguishably entangled. Lee and Kim gave an experimental scheme, in which a state superposed by one photon and vacuum can be teleported with a single-photon state $(a^\dagger+b^\dagger)|0\rangle$ [8], where $a^\dagger$ and $b^\dagger$ are particle creation operators in path A and B. It is meaningless to discuss entanglement for a single particle, therefore, there is no entanglement in this teleportation scheme. They suggest that entanglement may not be necessary for quantum teleportation, because it can even be implemented with separable states. To study phenomenon like this, it is useful to define relative correlation.

Definition 2. A state is said to have correlation relative to a quantum-mechanical quantity F, if and only if the state is not an eigenvector of F.

According to the definition above, the quantity F is important for certain correlation. A correlation must be related to certain measurement corresponding to the mechanical quantity F. In fact, the so-called correlation is the correlation between eigenvectors with different eigenvalues of F. Operations in eigenvectors with the same eigenvalue may be called local operations. Operations in eigenvectors with different eigenvalues are nonlocal. It is obvious that local operations do not change eigenvectors' eigenvalues. There is correlation relative to particle number $a^\dagger a$ or $b^\dagger b$ in $(a^\dagger+b^\dagger)|0\rangle$ and no correlation relative to the two-particle number operators in $a^\dagger b^\dagger|0\rangle$, where $a^\dagger$ and $b^\dagger$ are orthogonal. For the state $a^\dagger b^\dagger|0\rangle$, local operations relative to $a^\dagger a$ do not affect the particle at mode $b^\dagger$, and the similar occurs to $b^\dagger b$. Hence $a^\dagger b^\dagger|0\rangle$ is called a separated state in distinguishable-particle systems. Entanglement must have some correlation, but correlation can happen in both separable and entangled states.

To summarize, entanglement in identical-particle systems can be well distinguished by definition 1. This definition reduces to that in distinguishable-particle system if the particles are distinguishable. It is noted that the entanglement definition of both distinguishable-particle and identical-particle systems can be dealt with using the definition given in this paper. Using identical-particle formalism to treat identical-particle systems is important, examples of such treatment can be found in Refs. [9,10]. There are also different phenomena in identical-particle systems, which may have future applications. Moreover, relative correlation is defined, and it may be physically more important than entanglement.

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APPENDIX: PROOF OF LEMMA

Similar to the proof in Ref. [4] for identical fermions, we prove the lemma here. Let S be a $N\times N$, complex, symmetric matrix, $S^T=S$. Hence $SS^*=SS^\dagger$ is Hermitian that can be diagonalized by a unitary transformation U': $SS^*=U'DU'^\dagger$, D diagonal. Let us define $C=U'^\dagger SU'^*$. It is easy to check that C is symmetric and normal $CC^\dagger=C^\dagger C$. Then we will decompose C into its real and imaginary parts $C=F+iG$. Since C is normal, F and G commute. Thus F and G are real, symmetric, and commuting matrices. Hence they can be simultaneously diagonalized by a real orthogonal transformation O, $F=OD^1O^T$, and $G=OD^2O^T$. Thus $C=OD_MO^T(D_M=D^1+D^2)$ and finally

$$S=U'OD_MO^TU'^T=UD_MU^T, \tag{A1}$$

where $U=U'O$.

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