Vortex bending and tightly packed vortex lattices in Bose-Einstein condensates

J. J. García-Ripoll* and V. M. Pérez-García†
Departamento de Matemáticas, ETSI Industriales, Universidad de Castilla-La Mancha, Avenida Camilo José Cela 3, 13071 Ciudad Real, Spain
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We study in detail the structure of the ground state of elongated rotating Bose-Einstein condensates. This ground state is composed of one or more vortex lines which bend even in completely symmetric setups. This symmetry breaking allows the condensate to smoothly adapt to rotation and to produce tightly packed arrays of vortex lines. The dependence of vortex bending on the relevant parameters is studied.

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I. INTRODUCTION

A stirred coffee climbing up the cup walls, a whirl of water spiraling down a drain, and a tornado moving over the earth are natural phenomena involving the rotation of a fluid, let it be coffee, water, or dusty air. In all of them the fluid is subject to a balance between an external force and the centrifugal force, and this balance is responsible for a depression of the density along the axis of rotation. These structures have no trivial symmetries as is evident in the case of tornados, which have continuously changing bent shapes. None of these structures have long lives because viscosity, imperfections, and other dissipative mechanisms play such an important role that rotation cannot be self-sustained.

This behavior contrasts with that of the so called superfluids, which represent a state of matter of negligible viscosity or *dry fluid* [1]. The lack of viscosity allows a superfluid to host rotation for long times with little or no external intervention. In addition, when a superfluid rotates it follows a special type of flow that is irrotational, $\boldsymbol{\nabla}\times\mathbf{v}=\mathbf{0}$, except for a finite number of extended singularities, which are called *vortex lines*. These vortex lines, first predicted for ^{4}He condensates, are the superfluid equivalent of robust whirls and tornados. However, due to centrifugal forces, the density of the superfluid [2] becomes zero along the vortex line, something that is difficult to achieve in ordinary liquids.

Until recently only two examples of true superfluids were known: the superfluid phases of ^{4}He and ^{3}He. In both cases, the strength of the many-body interactions obscures the signatures of superfluidity and makes their study difficult. The search for superfluid-type weakly interacting systems has led to the experimental realizations of Bose-Einstein condensates (BECs) using dilute gases of alkali-metal atoms [3]. These condensates are ruled at low temperature by a mesoscopic quantum wave function $\psi(\mathbf{r})=\sqrt{\rho(\mathbf{r})}e^{i\theta(\mathbf{r})}$, whose phase determines a velocity field $\mathbf{v}=\hbar\boldsymbol{\nabla}\theta(\mathbf{r})/m$ that may host vortices along the lines where θ is not well defined. The existence of vortices in BECs has been directly confirmed in two sets of experiments. The first one is based on the manipulation of the internal degrees of freedom of an almost spherical condensate [4]. In the second one, vortices are produced by "mechanical means" in rotating elongated traps with transverse asymmetries [5,6], a geometry that leads to intriguing effects on the vortex nucleation process.

In Ref. [7] we find several commonly accepted predictions about vortices in rotating traps, namely: (i) A rotating trap leads to the production of one or more vortices of unit topological charge m, where $m=(2\pi)^{-1}\oint\boldsymbol{\nabla}\theta$. (ii) Vortices appear in finite numbers, with straight shapes, forming lattices with p-fold symmetries. (iii) Certain critical rotation speeds $\Omega_p<\Omega_{p+1}$ should be surpassed before each new vortex is nucleated. (iv) Asymmetric states without rotational symmetry of any kind are found to be energetically unstable, which means that there are perturbations with no energetic cost that can destroy such states. However, Ref. [7] is based on a variational ansatz for the weak coupling limit, which lacks longitudinal degrees of freedom and implicitly induces the p-fold symmetry of the vortex lattice. Recent works abandoned these constraints but either focused on infinitely long condensates or studied almost spherically symmetric traps [8].

In this paper we study an elongated three-dimensional Bose-Einstein condensate subject to rotation. Our main result is that stationary vortex lines are bent even in completely symmetric setups and that the bending depends on the type of trap and on the number of atoms of the experiment. The fact that vortex lines may bend was mentioned in Ref. [9], where longitudinal excitation modes of a vortex line induce vortex bending. However, the physics in that paper differs drastically from our work, in which we consider the *ground state* and the generation of *stable and stationary* structures. Second, in a complementary paper [10] we have presented some evidence in favor of that bending and how it could be related to the experimental results of Madison *et al.* [5]. Here we extend that work to the analysis of arrays of bent vortices and their properties as well as the dependence of the bending of the vortices on the geometry of the trap.

Our work consists of two parts. In Sec. II we provide the mathematical foundations for the search for the most favorable configurations of the rotating condensate. To do so we first introduce the mean-field model used to describe the condensate together with some basic definitions and an energy functional whose minima are the possible ground states of the condensate. In this section we also develop an efficient minimization method for this functional.

All these tools are then applied in Sec. III to different

*Electronic address: jjgarcia@ind-cr.uclm.es
†Electronic address: vperez@ind-cr.uclm.es

configurations. We start with setups from current experiments [5] and show that in elongated traps with many bosons the vortex lines are bent and form Abrikosov lattices which are regular only in the core of the gas cloud and deform close to the boundaries. Next we study the dependence of the vortex bending on the free parameters in our theory. Finally in Sec. IV we offer our conclusions together with some open questions.

II. MATHEMATICAL TOOLS

A. The mean-field model

For current experiments it is an accurate approximation to use the zero temperature mean-field theory of the condensate, in which the atomic cloud is described by a single wave function $\psi(\mathbf{r},t)$ ruled by a Gross-Pitaevskii equation. In the case of rotating systems it is useful to consider the problem in the mobile reference frame that moves with the trap [11], in which the equation reads

$$i\frac{\partial \psi}{\partial t}=\left[-\frac{1}{2}\triangle+V_0(\mathbf{r})+g|\psi|^2-\Omega L_z\right]\psi. \quad (1)$$

Here $L_z=i(x\partial_y-y\partial_x)$ is the Hermitian operator that represents the angular momentum along the z axis, and the effective trapping potential is given by

$$V_0(\mathbf{r})=\frac{1}{2}\omega_\perp^2[(1-\varepsilon)x^2+(1+\varepsilon)y^2]+\frac{1}{2}\omega_z^2z^2]. \quad (2)$$

In Eq. (1) we have applied a convenient adimensionalization which uses the harmonic oscillator length $a_\perp=\sqrt{\hbar/m_{Rb}\omega_\perp}$ and period $\tau=\omega_\perp^{-1}$. With these units the nonlinear parameter becomes $g=4\pi a_S/a_\perp$.

B. Variational formulation

There are several conserved quantities associated with Eq. (1). The first one is the norm of the wave function, $N[\psi]=\int|\psi|^2d\mathbf{r}$, which is related to the number of bosons in the condensate. The second conserved quantity is the energy of the gaseous condensate

$$\begin{aligned}E[\psi]&=\int\bar{\psi}\left[-\frac{1}{2}\triangle+V_0(\mathbf{r})+\frac{g}{2}|\psi|^2-\Omega L_z\right]\psi d\mathbf{r}\\&=E_0(\psi)-\Omega L_z(\psi).\end{aligned} \quad (3)$$

In current experiments with stirred condensates [5,6,12], the gaseous cloud reaches certain long lived configurations with one or more vortices. In this work we are not interested in the precise dynamics of the nucleation process, but look for the final stationary configurations. From thermodynamical considerations we expect the condensate to achieve the least energy for given experimental parameters $\{N,g,\Omega\}$. Such a configuration is called a ground state.

Within the framework of our mean-field model, a stationary state is represented by a wave function with the form

$$\psi_\mu(\mathbf{r},t)=e^{-i\mu t}\phi(\mathbf{r}). \quad (4)$$

The first way to find such solutions is to introduce Eq. (4) into our mean-field model (1) and directly solve the resulting partial differential equation. This method has several disadvantages: (i) it is difficult due to the many degrees of freedom that it involves; (ii) the resulting equation is satisfied not only by the ground state but also by excited states, and (iii) there is no mathematical guarantee yet for the existence of solutions of this problem.

The second way to characterize a stationary solution is by studying the energy functional and using the fact that each stationary configuration is a critical point of the energy (3) for given $\{N,g,\Omega\}$:

$$\left.\frac{\partial E}{\partial \psi}\right|_N[\psi_\mu]=0. \quad (5)$$

Since we are indeed looking for the ground state, one should use a minimization procedure able to find the minima of the energy functional for given parameters.

C. Reshaping the energy functional

As we stated above, our objective is to find the solutions ψ_μ that are the minima of the energy subject to the restriction $\int|\psi|^2\equiv N$, for a fixed angular speed of the trap and for a given interaction. The existence of at least one minimum or ground state for this variational problem has been proved elsewhere [13]; the solutions are guaranteed to be stable and represent the energetically most favorable configurations for given N and Ω.

Such a problem is called a constrained optimization and we will refer to the solution ψ_μ as a constrained minimum. The existence of constraints in an optimization problem poses serious difficulties for traditional descent methods, since it is difficult to design an efficient minimization algorithm that takes care of the constraints at each step [14]. To take into account this restriction one should use Lagrange's multipliers, i.e., just add a fraction of the constraint $\omega(\mu,N)$ (also called a "penalizer") to the original functional:

$$F[\psi]=E[\psi]+\omega(\mu,N). \quad (6)$$

It is not difficult to show that $F[\psi]$ and $E[\psi]$ have the same stationary states, and that any absolute or relative minimum of $F[\psi]$ is also a constrained minimum of $E[\psi]$.

One expects that a wise choice of the penalizer will establish a one-to-one correspondence between the values of the chemical potential μ, the absolute minima of $F[\psi]$, and the ground states of our condensate. The advantage of $F[\psi]$ over $E[\psi]$ is that in $F[\psi]$ there is no need to be concerned about constraints: the value of N for the ground state is determined by the chemical potential μ, which is fixed throughout the minimization process.

The traditional choice for a Lagrange multiplier leads to the definition of the free energy

$$\mathcal{F}[\psi]=E[\psi]-\mu N[\psi]. \quad (7)$$

However, this functional is not bounded below and thus ψ_μ is at most a local minimum of $\mathcal{F}$. Consequently, this functional cannot be used to characterize the state of the condensate. This is an interesting result, since one could be tempted to think that the traditional definitions of the thermodynamical potentials are suitable to characterize all physical systems. We must remember, however, that our condensate is being described by a mean-field model, a simplification of a more complex model, and as such what works for the full problem need not work for the simpler one.

In our search for suitable functionals we have found a simple one which we call the *nonlinear free energy*:

$$F[\psi]=E[\psi]+\frac{1}{2}(N[\psi]-\lambda)^2. \tag{8}$$

First and most important, it can be proved that $F[\psi]$ has at least one finite norm absolute minimum for each value of λ [13], and that each of those minima corresponds to a constrained minimum of the energy $E[\psi]$. Second, there exists a simple and invertible relation between the usual thermodynamic variables, the chemical potential and the number of particles (μ,N), and our variables, Lagrange's constant and the number of particles (λ,N),

$$\mu=N[\psi_\mu]-\lambda. \tag{9}$$

An important feature of our functional (8) is that it is highly nonlinear with respect to ψ. This poses no additional difficulty, since our original equations (1) were already nonlinear. Indeed Eq. (8) has proved to be the most natural choice for many other problems, such as the propagation of incoherently coupled laser beams through saturable media [15].

D. Optimal methods for minimization

To minimize the nonlinear free energy (8) we follow Ref. [16]. First we choose the right function space, which in our case is the Sobolev space $H^1(R^3)\equiv\{\psi/\psi,\nabla\psi\in L^2\}$ of functions that admit at least one spatial derivative. This space is equipped with a scalar product

$$\langle\psi,\phi\rangle\equiv\int[\overline{\psi}(\mathbf{r})\phi(\mathbf{r})+\nabla\overline{\psi}(\mathbf{r})\cdot\nabla\phi(\mathbf{r})]d^n r \tag{10}$$

and a norm $\|\psi\|=\langle\psi,\psi\rangle$. To obtain an explicit expression for the gradient of the functional in H^1 we perform a first order expansion of $F[\psi]$ around a trial state ψ:

$$F[\psi+\epsilon\delta]=F[\psi]+2\epsilon\,\mathrm{Re}\int\left(\delta^*\frac{\partial E}{\partial\overline{\psi}}+\nabla\delta^*\frac{\partial E}{\partial\nabla\psi}\right)+O(\epsilon^2). \tag{11}$$

We have to turn this expression into something that looks like Frechet's definition of a derivative. This means that we have to find some ϕ such that

$$\mathrm{Re}\int\left[\overline{\delta}\frac{\partial E}{\partial\overline{\psi}}+\nabla\overline{\delta}\frac{\partial E}{\partial(\nabla\overline{\psi})}\right]=\mathrm{Re}\int[\overline{\delta}\phi+\nabla\overline{\delta}\nabla\phi]. \tag{12}$$

If we integrate by parts and impose the condition that this equality be satisfied for all perturbations δ, the problem has a formal solution which is given by a Lagrange equation,

$$(1-\triangle)\phi=\frac{\partial E}{\partial\overline{\psi}}-\nabla\frac{\partial E}{\partial(\nabla\overline{\psi})}. \tag{13}$$

In consequence, our formal expression for the Sobolev gradient of $F[\psi]$ finally reads $\nabla_S F\equiv(1-\triangle)^{-1}\nabla F$, where $\nabla_S F$ stands for the Sobolev gradient, ∇F is the ordinary one, and $(1-\triangle)^{-1}$ represents the inverse of a linear and strictly positive definite operator.

We may now use the Sobolev gradient of our functional as the direction of descent for a minimization procedure:

$$\frac{\partial\nu}{\partial\tau}(\mathbf{r},\tau)=(1-\triangle)^{-1}\left[-\frac{1}{2}\triangle+V+g|\nu|^2-\Omega L_z\right]\nu. \tag{14}$$

The preceding equation converges to some stationary state $\phi(\mathbf{r})=\lim_{\tau\to\infty}\nu(\mathbf{r},\tau)$. To grant convergence to the true ground state one needs to improve the descent method, using, for instance, the nonlinear conjugate gradient method instead of Eq. (14), and introducing some type of relaxation that helps avoid saddle node points. In any case, the minimization process must be performed on a suitable space of functions that we built using a discrete Fourier basis with $64\times64\times128$ modes.

III. RESULTS

A. Bent vortex lattices in current experiments

We have applied the numerical methods outlined above to different setups. The family of numerical experiments in this subsection resembles the experiments of Madison *et al.* [5,6]. For these experiments with ^{87}Rb condensates, the bosonic interaction is ruled by the scattering length $a_S\simeq5.5$ nm. For several pictures in this paper, the number of atoms was chosen to match, $Ng=9000$, which corresponds to a few times 10^5 Rb bosons, but our results remain qualitatively valid for an ample range of gN values as will be shown below.

The geometry of the trap is a very elongated one, $\omega_\perp>18.7\omega_z$, and for the small transverse deformation of the trap we have tried $\varepsilon=0.0$ as well as $\varepsilon=0.03$, the latter being the closest one to the actual experiment [5]. This means that we have studied both axially symmetric setups and completely asymmetric setups, and in both cases our study has provided essentially the same results. This is due to the fact that an intense nonlinear interaction $(Ng\sim10^4)$ effectively prolongs the existence of all solutions from the symmetric setup to the asymmetric setup [17].

The first result is that an increase of the angular speed causes the minimum of the energy to move from a nodeless ground state into states with one, two, three, and more vortices. Consequently, as was already predicted [7], there exists a cascade of increasing angular frequencies $\{\Omega_1<\Omega_2<\cdots\}$ for the nucleation of one, two, and more vortices—the larger the rotation frequency, the more vortices, and for

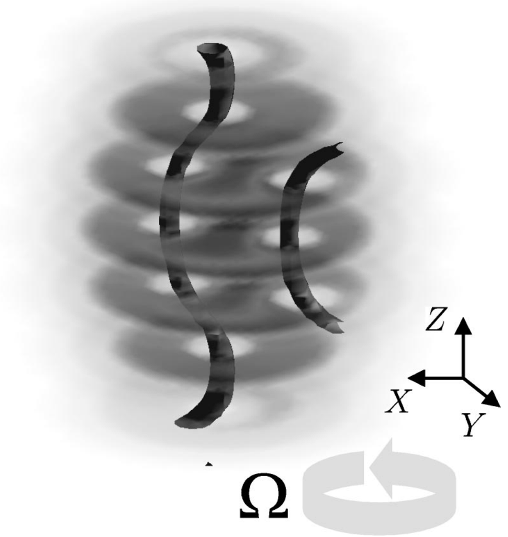

FIG. 1. Ground state of a condensate in an asymmetric trap ($\varepsilon=0.03$) such as the one from current experiments for an angular speed of $\Omega=0.55\omega_\perp$. Shown are the vortex cores and several transverse sections of the condensate. The gray scale indicates the modulus of the wave function. We also indicate the axis of rotation of the trap.

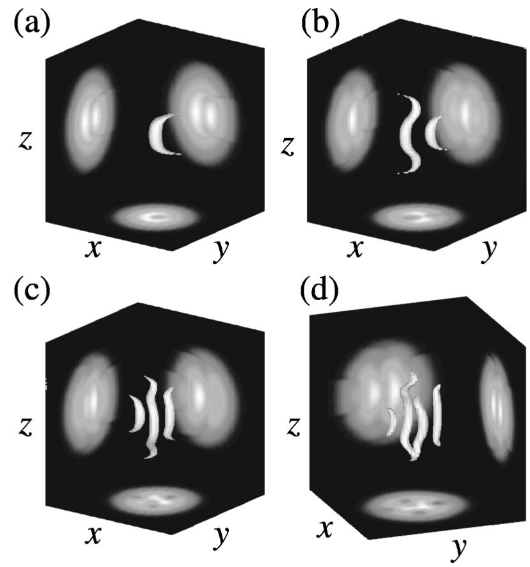

FIG. 2. Three-dimensional surface plots of one to four vortex lines for $\varepsilon=0.03$ and $\Omega/\omega_\perp=0.41,0.55,0.6,0.65$. The walls of the boxes reflect different integrated views of the gas cloud (i.e., as they would look if imaged with high resolution), and within each box we show the lines of lowest density within the condensate. Each plot represents a centered box $(x,y,z)\in[-6,6]\times[-6,6]\times[-60,60]$, with sizes in adimensional units. Parameters are close to experimental values: $Ng=9000$, $\varepsilon=0.03$, $\omega_\perp/\omega_z=219/11.7$.

our setup, $\Omega_1\simeq0.4\ (0)$, $\Omega_2\simeq0.5\ (3)$, $\Omega_3\simeq0.5\ (8)$, and $\Omega_4\simeq0.6\ (4)$. Figure 1 shows the structure of a ground state hosting two vortices at $\Omega=0.55$.

The second and most important result, which is already evident in Fig. 1, is that vortices are nucleated with a stable and stationary bent shape even in axially symmetric traps. In Fig. 2 we show three-dimensional pictures of a condensate with up to four bent vortex lines.

These contorted shapes are absolutely stable configurations that lack rotational symmetry of any kind, even discrete, something which was thought to be forbidden according to Ref. [7]. In that case asymmetric states were found only for discrete values of the angular speed, right on the transition between different numbers of vortices, and always exhibiting energetic instabilities. Furthermore, due to the bending, while the trunk of the condensate rotates, the caps remain almost still. This feature allows the gas to accommodate a fractional value of the angular momentum which is between 0 and 1, and which never reaches 1, as was already conjectured in [10]. As the speed of the trap increases, the angular momentum grows continuously by means of pulling the vortex line straighter and not only with discontinuous jumps.

B. Regularity of the vortex lattice

From old studies with liquid helium and more recent work with condensates of simpler geometry, it has been expected that vortices should form regular triangular lattices. On the other hand, our finding above seems to suggest that the vortex aggregate rather adopts an irregular shape. As we will show below, the condensate actually develops a medium range order, which is diluted in the vicinity of the cloud boundary.

In Fig. 3 we use the solutions of our previous section to simulate the pictures that should be seen in experiments. The first row of two-dimensional plots [Figs. 3(a–d)] are views of the condensate from above, and represent the column density of the bosonic cloud along the transverse directions. These pictures bear a close resemblance to the experimental photographs of Refs. [5,6], showing blurred clouds where the vortex cores seem partially filled.

In Figs. 3(i–l) we show the density of the condensate as seen at half the height of the condensate, i.e., $|\psi(x,y,z=0)|^2$. Very recently an experiment was made that observes this type of transverse cut of a stirred sodium condensate [12]. In that experiment a thin slice of bosons is promoted to a different internal state and then imaged separately from the rest of the cloud. The resulting pictures are like those in Figs. 3(i–l), where the regularity of the Abrikosov vortex lattice is made evident. The blurring has disappeared and the holes are arranged along the expected triangular lattice. In the above mentioned paper the bending of the vortex lines is also reported.

As further confirmation we suggest that using the setup of Ref. [5] and watching the condensate not from above, but from one side would allow a direct observation of bent vortex lines, leading to pictures such as those in Figs. 3(e–h).

C. What is the ultimate cause of bending?

The fact that the bending of the vortex line takes place not only in the asymmetric trap of current experiments but also in a radially symmetric trap $\varepsilon=0$ represents a surprising type of symmetry breaking in which the superfluid not only chooses the sense of rotation, but also a plane for its bending.

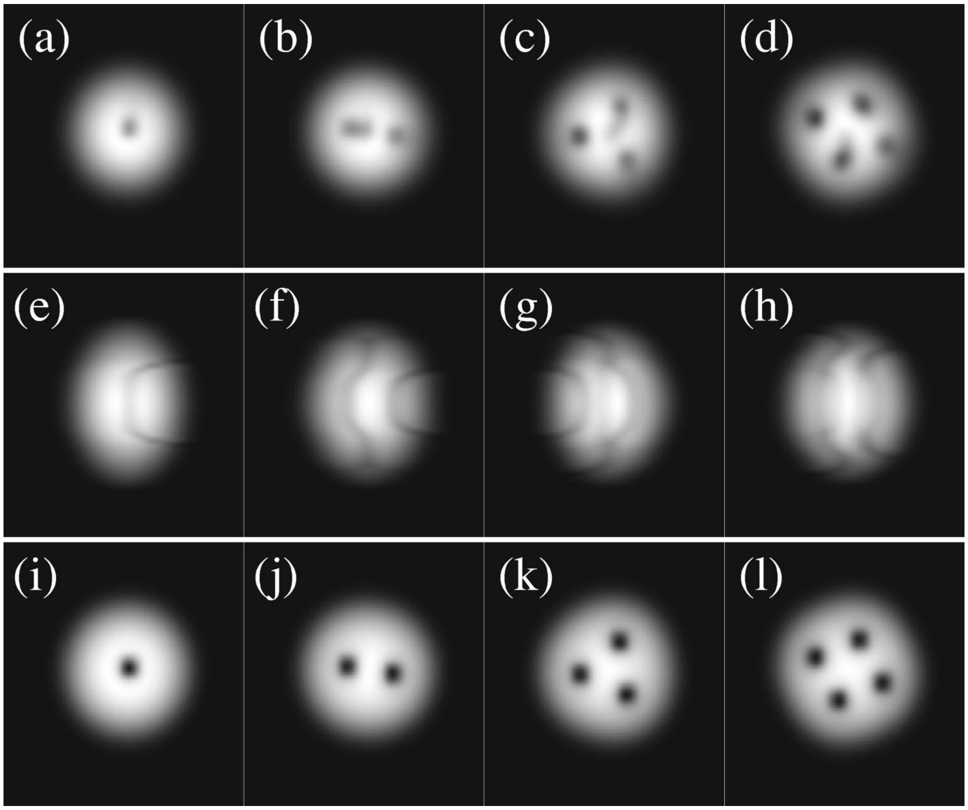

FIG. 3. Condensate shapes for the same states as in Fig. 2: $\Omega/\omega_\perp = 0.41$ (a,e,i), $\Omega/\omega_\perp = 0.55$ (b,f,j), $\Omega/\omega_\perp = 0.6$ (c,g,k), and $\Omega/\omega_\perp = 0.65$ (d,h,l). Shown are top (a–d) and side views (e–h) and (i–l) two-dimensional cuts of the cloud at half its height.

It is important to emphasize the counterintuitive nature of the vortex bending in the case of an isolated vortex line. First, in contrast to the case of ^{4}He, the BEC is not constrained by any recipient and there are no asymmetric boundary conditions which could easily explain the deformation of the fundamental solutions. Second, although qualitative reasons for such bending may be found *a posteriori*, they have never been reported before. And even though physical arguments may justify the bending of the vortex line, they can hardly support the fact that the bent vortex line is a stable stationary configuration. Both the stability and the stationarity can be demonstrated only with a direct study of the energy functional.

Nevertheless, it is legitimate to ask for the ultimate reasons for this rupture of symmetry. Since our search for states was based on energetic considerations, we have studied the dependence of the bending on each of the free parameters in our problem: N, Ω, g, ε, and the elongation $\gamma = \omega_\perp/\omega_z$.

First, the influence of ε is discarded. As we mentioned above, bending exists in either symmetric or asymmetric traps. Rather, the asymmetry seems to take part only in the dynamics of the nucleation process, by changing the values of the critical frequencies and inducing a type of hysteresis [10].

A different role is played by the elongation of the trap, γ, and the effective interaction $U = Ng$. Starting with a configuration and lowering either the elongation or the number of atoms, we see that the bending becomes smaller and eventually disappears [Figs. 4(a–c)].

We have tried to measure the bending so as to determine the minimal elongation and interaction that are required to induce this phenomenon. The problem is that, due to the small changes of L_z, the full minimization procedure does not allow us to bound this minimal elongation accurately. An alternative procedure is to study the relation between the angular speed that is required to stabilize a straight vortex,

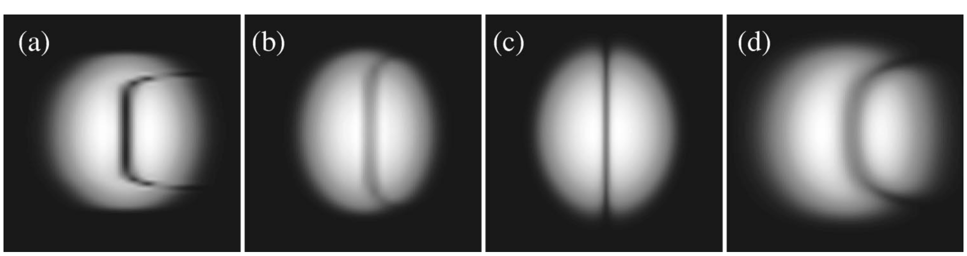

FIG. 4. Side views of the condensate for $N = 10^5$ atoms of ^{87}Rb, and decreasing elongation. The trap parameters and plot dimensions $(\Omega, \omega_\perp/2\pi, \omega_z/2\pi, r_{max}, z_{max})$ are (a) ($0.5\omega_\perp$, 219 Hz, 11.7 Hz, 2.63 nm, 49.09 nm), (b) ($0.4\omega_\perp$, 96 Hz, 25 Hz, 5.00 nm, 19.23 nm), and (c) ($0.3\omega_\perp$, 50 Hz, 50 Hz, 8.51 nm, 8.51 nm). (d) uses the same trap as in (a), but with $\{N = 10^4, \Omega = 0.53, r_{max} = 1.65, z_{max} = 30.98\}$.

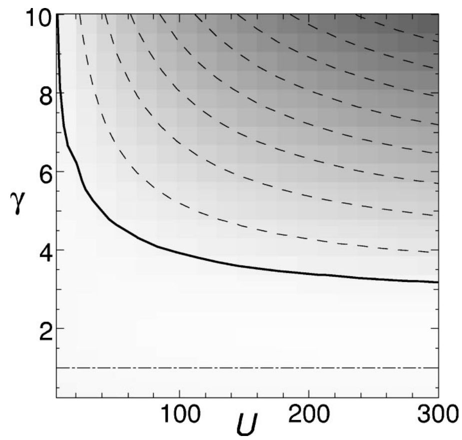

FIG. 5. Gray scale plot and contour lines for the difference $\overline{\Omega}_1-\Omega_1$ between the speed required to nucleate a vortex, Ω_1, and the speed required to stabilize a straight one, $\overline{\Omega}_1$, as a function of the elongation of the trap, $\gamma=\omega_\perp/\omega_z$, and the effective interaction $U=Ng$. A horizontal dash-dotted line marks the spherically symmetric trap, $\gamma=1$. For $\overline{\Omega}_1>\Omega_1$, i.e., above the thick solid line, the bending of vortices becomes energetically favorable. All figures are adimensional.

$\overline{\Omega}_1$, and the angular speed at which the ground state acquires some angular momentum, Ω_1. The first value arises from the study of normal modes around a straight vortex, while the second value is the energy difference between a straight vortex and a vortexless state [10]. When $\overline{\Omega}_1 \geqslant \Omega_1$ there are values of the angular speed where bending can be favorable. This phenomenon has been referred to in the literature as ‘‘anomalous modes’’ [18], and it is a signature of bending.

In Fig. 5 we plot the difference $\overline{\Omega}_1-\Omega_1$ as a function of the elongation γ and of the effective interaction $U=Ng$ for an axially symmetric trap. According to this study, a large elongation of the trap is required to make a bent vortex energetically favorable over a straight one.

IV. CONCLUSIONS

In this work we have studied vortex lines in a very elongated Bose-Einstein condensate in a rotating trap. By developing both a customized energy functional (Sec. II C) and an optimized descent method (Sec. II D) we have been able to find the ground state of a condensate in a family of different experimental configurations.

The first part of our study (Sec. III A) focused on realistic values of the experimental parameters, taken from the work by Madison *et al.* [5,6,19]. The main conclusion of this work is that by increasing the rotation speed the condensate achieves states with one, two, three, and more bent vortices. Such vortices form a regular Abrikosov lattice at half the height of the condensate and deform not very far from the condensate core.

Appart from the observation reported in [12] and while current experiments by Madison *et al.* [5,6,19] are carried out in a regime in which vortex lines should bend, this has not yet been observed. The reason is that the imaging of a condensate which is a few micrometers in size large requires a previous expansion in which the cloud becomes disk shaped and bending cannot be appreciated. Furthermore, the study of the elongated cloud without vortices [10,20] shows that the nucleation of vortices is subject to hysteresis and one must actually exceed some rotation frequency Ω_m, which is rather large and makes it difficult to selectively produce one, two, or more vortices.

On the other hand the bending of vortex lines has signatures which are also observed in current experiments. First, the bending by itself can explain the apparent filling of vortex lines when seen from above [5] and the regular pictures that appear in most recent observations with sliced sodium condensates [12]. Second, the bending also accounts for the continuous growth of the angular momentum with respect to the rotation speed, and the fractional values of the angular momentum $0<l<1$ that arise in the indirect measurements of [6] and [19]. Finally, the nucleation of many bent vortices and their subsequent interaction during the expansion phase may lead to turbulent structures that should explain the lack of regular pictures above a certain angular speed [5,6].

The second part of this paper studied the dependence of the bending on the parameters of the mean-field model (1). Here we conclude that it is both the elongation and interactions that induce the bending of the vortex line, while the transverse asymmetry plays no important role.

Our results imply that past studies devoted to the quasi-linear limit ($U\ll 1$) would become of no applicability for elongated traps due to the lack of bending in the simplified models. Second, along this line it would be nice to find *analytically* the geometry of the vortex line as a function of the interaction U and the elongation γ.

Our study also reveals that current experiments are being developed in a regime which is qualitatively different from the setups that have been studied up to date [7,8]. Values of the relevant parameters (γ,U) in current experiments are so far from most studies that new methods must be developed to accurately describe these amazing systems.

As an example, let us pose one of the technically difficult questions that arise in this work. From our numerical work it seems that the extremes of some of the vortex lines actually reconnect not too far away from the core of the cloud, forming what is called a vortex ring. Although it is not possible with current numerical methods to fully support this conjecture, vortex rings have already been observed in BECs [21]. The limits of zero and infinite radius of a vortex ring lead to a dimensionless zero and an isolated vortex line, respectively. Therefore vortex rings would be a nice tool for explaining the nucleation of vortices and they would allow interpretation of the states in this work as the result of an incomplete nucleation. The proof or refutation of this conjecture remains a mathematically challenging problem.

ACKNOWLEDGMENTS

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[1] Around 1900, John Von Neumann noticed that most mathematical models of the date did not take viscosity into account and thus could not explain the features of real fluids. He coined the term "dry water" to refer disrespectfully to those idealized models that did not care take account of dissipation. R. P. Feynmann, R. B. Leighton, and M. Sands, *Física. Vol. II: Electromagnetismo y Materia* (Addison-Wesley Iberoamericana, Wilmington, 1987), Chap. 40. Bose-Einstein condensates represent an experimental realization of such a "dry fluid" or superfluid.

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