Damping of trapped Bose-Einstein condensate oscillations at zero temperature

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We provide evidence for the zero-temperatuture damping of condensate radial oscillations in the elongated cylindrical trap. The origin of this damping is a parametric resonance leading to an energy transfer from the coherent oscillations of condensate to longitudinal sound waves in certain frequency interval.

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The damping of the oscillations of the Bose-condensed gas in a trap isolated from environment is one of the most-interesting problems of the physics of the Bose-Einstein condensation. So far, the experimental [1–4] and the theoretical [5–7] investigations have been reduced to the study of the damping due to the interaction of oscillations and thermal excitations. An ensemble of such excitations plays a role of a heat bath. In all cases the relatively high temperatures $T \gg \hbar\omega_0$ have been considered (ω_0 is the frequency of a parabolic trap). However, the principal question of the origin of the irreversible damping in the oscillating-trapped condensate at $T=0$ requires a special study.

In the present work it will be shown that such damping, in any case under definite conditions, does exist. We consider the radial oscillations of the condensate caused by varying frequency $\omega(t)$ of the transverse isotropic parabolic potential in an elongated trap of the cylindrical symmetry at zero temperature. The study of the problem under such conditions has a series of advantages. As is found in [8], there exists an exact scaling solution of the Gross-Pitaevskii equation for an isotropic two-dimensional parabolic potential with an arbitrary dependence $\omega(t)$. This solution describes a space-time evolution of the condensate, based only on the solution in the initial static potential at $\omega=\omega_0$. It is essential that this solution holds for the quasi-two-dimensional case corresponding to the cylindrical symmetry of a trap. In particular, a ratio between interparticle interaction and kinetic energy remains constant. As a result, if for the initial static trap the Thomas-Fermi approximation is applicable, it continues to be valid at all stages of the gas evolution.

If the change of radial potential is related with the transition from the frequency ω_0 to ω_1, then the condensate oscillations with the frequency $2\omega_1$ arises [8]. The variation of condensate density is accompanied by the oscillations of the sound velocity c. As will be shown below, the phenomenon of parametric resonance (see e.g., [9]), appears under these conditions. The essence of the phenomenon is that the amplitude of the waves propagating along the axial z axis with frequency close to ω_1 is undergone exponential enhancement. At zero temperature the initial amplitudes, in fact, are due to zero-point oscillations. As a result, the dynamical energy of the coherently oscillating condensate decreases, transforming into the energy of the longitudinal oscillations.

Considering cylindric configuration of the external field with the longitudinal size $L \gg R$ and neglecting the edge effects, we can represent the general equation for the Heisenberg field operator $\Psi(\vec{r},z,t)$ as

$$i\hbar\frac{\partial\hat{\Psi}}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla_r^2-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+\frac{1}{2}m\omega^2(t)r^2\right]\hat{\Psi}+U_0\hat{\Psi}^+\hat{\Psi}\hat{\Psi}, \quad (1)$$

here $U_0=4\pi a\hbar^2/m$ where a is the scattering length. The only simplification in this equation is an assumption of the local character of the interparticle interaction.

Let us introduce the spatial scaling parameter $b(t)$ and define new variable $\vec{\rho}=\vec{r}/b$. In addition, we introduce the time variable $\tau(t)$. The field operator can be represented as

$$\hat{\Psi}(\vec{r},t)=\frac{1}{b}\hat{\chi}(\vec{\rho},\tau,z)\exp[i\Phi]. \quad (2)$$

Inserting Eq. (2) into Eq. (1) and using the results obtained in [8], we find for the phase Φ of the wave function

$$\Phi(r,t)=\frac{mr^2}{2\hbar b}\frac{db}{dt} \quad (3)$$

and the equation for the operator $\hat{\chi}$

$$i\hbar\frac{\partial\hat{\chi}}{\partial\tau}=\left[-\frac{\hbar^2}{2m}\nabla_\rho^2+\frac{1}{2}m\omega_0^2\rho^2\right]\hat{\chi}+U\hat{\chi}^+\hat{\chi}\hat{\chi}-\frac{\hbar^2}{2m}b^2\frac{\partial^2\hat{\chi}}{\partial z^2}. \quad (4)$$

This equation acquires such form provided that the $b(t)$ and $\tau(t)$ satisfy equations

$$\frac{d^2b}{dt^2}+\omega^2(t)b=\omega_0^2b^{-3}, \qquad \tau(t)=\int_0^t\frac{dt'}{b^2}, \quad (5)$$

where $\omega_0=\omega(-\infty)$ is the initial frequency and $b(-\infty)=1$.

For $T=0$, employing the symmetry of the problem, we can treat the condensate wave function χ_0 as independent of z. Making a standard substitution of operator $\hat{\chi}$ for the macroscopic wave function χ_0 in Eq. (4), we have

$$i\hbar\frac{\partial\chi_0}{\partial\tau}=\left[-\frac{\hbar^2}{2m}\nabla_\rho^2+\frac{1}{2}m\omega_0^2\rho^2\right]\chi_0+U\chi_0^*\chi_0^2=0. \quad (6)$$

Equation (6) is formulated in terms of variables ρ and τ for the condensate wave function in a static two-dimensional parabolic potential with frequency ω_0. In this case χ_0 has the known-time behavior

$$\chi_0(\vec{\rho},\tau)=\tilde{\chi}_0(\vec{\rho})e^{-i\mu\tau/\hbar}, \tag{7}$$

where μ is the initial chemical potential and a real function $\tilde{\chi}_0(\vec{\rho})$ is a solution of the equation

$$-\frac{\hbar^2}{2m}\Delta_\rho\tilde{\chi}_0+\left(\frac{1}{2}m\omega_0^2\rho^2+U_0\tilde{\chi}_0^2-\mu\right)\tilde{\chi}_0=0. \tag{8}$$

We restrict ourselves by considering the case when the inequality $\mu\gg\hbar\omega_0$ is valid and, therefore, the Thomas-Fermi approximation holds for. In this case Eq. (8) has the familiar solution

$$\tilde{\chi}_0=\left(\frac{\mu}{U_0}\right)^{1/2}\left(1-\frac{\rho^2}{R^2}\right)^{1/2}, \tag{9}$$

where $R=\sqrt{2\mu/m\omega_0^2}$. Thus, to describe the space-time evolution of the condensate wave function (2) it is sufficient to find a solution of Eqs. (5) and use derived $b(t)$ and $\tau(t)$ for the definition of $\Phi(r,t)$ [Eq. (3)] and χ_0 (7), [Eq. (9)].

Let us assume a fast transition from the frequency ω_0 to ω_1. Then the solution of Eq. (5) yields

$$b^2(t)=\frac{1}{2}(\beta^2+1)-\frac{1}{2}(\beta^2-1)\cos 2\omega_1 t, \tag{10}$$

where $\beta=\omega_0/\omega_1>1$.

Consider the excited states of the system at the background of the coherently oscillating condensate. These states can be found as oscillations of the "classical" field of the condensate (see, e.g., [10]). For this purpose we replace operator $\hat{\chi}$ with the function $\chi=(\tilde{\chi}_0+\tilde{\chi}')\exp(-\imath\mu\tau/\hbar)$ in Eq. (4) and carry out a linearization in $\tilde{\chi}'$

$$i\hbar\frac{\partial\tilde{\chi}'}{\partial\tau}=\left(-\frac{\hbar^2}{2m}\nabla_\rho^2+\frac{1}{2}m\omega_0^2\rho^2+2G-\mu\right)\tilde{\chi}'+G\tilde{\chi}'^{+}-\frac{\hbar^2}{2m}b^2(\tau)\frac{\partial^2\tilde{\chi}'}{\partial z^2}, \tag{11}$$

where

$$G=U_0\tilde{\chi}_0^2. \tag{12}$$

We are interested in the longitudinal excitations having a quasicontinuous spectrum at $L\gg R$. The solution of Eq. (11), corresponding to these excitations, is sought as (see, e.g., [10])

$$\tilde{\chi}'(\rho,\tau)=u(\rho,\tau)e^{ikz}-v^*(\rho,\tau)e^{-ikz}. \tag{13}$$

The substitution of this expression into Eq. (11) yields

$$i\hbar\frac{\partial}{\partial\tau}u=\left[-\frac{\hbar^2}{2m}\nabla_\rho^2+\frac{1}{2}m\omega_0^2\rho^2+2G-\mu+\frac{b^2\hbar^2k^2}{2m}\right]u-Gv, \tag{14}$$

$$-i\hbar\frac{\partial}{\partial\tau}v=-Gu+\left[-\frac{\hbar^2}{2m}\nabla_\rho^2+\frac{1}{2}m\omega_0^2\rho^2+2G-\mu+\frac{b^2\hbar^2k^2}{2m}\right]v. \tag{15}$$

At $k\to 0$, as is clear from the physical reasons, the lowest branch of the longitudinal excitations is related to a uniform shift along the z axis with the radial distribution of the density determined by amplitude $\tilde{\chi}_0$. Hence this branch is gapless. For small but finite value of k the transverse distribution of the density in such wave changes weakly. This is clear, e.g., from results obtained for a static elongated trap in [11] and [12]. Keeping it in mind, we put

$$u=U(\rho,\tau)\tilde{\chi}_0(\rho),\quad v=V(\rho,\tau)\tilde{\chi}_0(\rho), \tag{16}$$

and assume that for longitudinal sound excitations $U(\rho,\tau)$ and $V(\rho,\tau)$ are weakly dependent on ρ. Neglecting derivatives in ρ of $U(\rho,\tau)$ and $V(\rho,\tau)$ in Eqs. (14) and (15), we find for functions $f=U+V$ and $F=U-V$

$$i\hbar\frac{\partial F}{\partial\tau}=\frac{b^2\hbar^2k^2}{2m}f,$$

$$i\hbar\frac{\partial f}{\partial\tau}=\left(2G+\frac{b^2\hbar^2k^2}{2m}\right)F. \tag{17}$$

Let us average both equations over ρ, denoting the corresponding average by a dash over the functions. The weak dependance of F on ρ allows to find approximately

$$\langle GF\rangle\simeq\overline{GF}.$$

In the Thomas-Fermi approximation, we have involved Eqs. (9) and (12), $\overline{G}=\mu/2$.

Next, we go over to variable t in a set of Eqs. (17), taking into account Eq. (5) after direct transformation we find

$$\frac{\partial^2\overline{F}}{\partial t^2}+\Omega_k^2(t)\overline{F}=0, \tag{18}$$

$$\Omega_k^2(t)=\frac{k^2}{2m}\left(\frac{2\overline{G}}{b^2(t)}+\frac{\hbar^2k^2}{2m}\right). \tag{19}$$

In the case of statical potential $(b=1)$ the quantity Ω_k is directly the frequency of the Bogoliubov excitation spectrum determined for the averaged value of the condensate density. In the problem under consideration b and therefore Ω_k depend on time.

In what follows, longitudinal phonons of $\Omega_k\approx\omega_1\ll\mu$ prove to be involved into the general dynamics of the system. In this case $\Omega_k(t)$ lies within the sound range

$$\Omega_k(t)=\frac{\overline{c}k}{b(t)},\quad \overline{c}=\left(\frac{\mu}{2m}\right)^{1/2}. \tag{20}$$

Note that the value (20) for the velocity of the longitudinal sound in an elongated trap was found for the first time in [11] and [12].

We restrict ourselves by considering a relatively weak variation of the frequency of a trap, i.e., $g=\beta-1\ll 1$. From Eq. (10) one has in this case

$$b^{-2}\simeq 1-g+g\cos(2\omega_1 t). \tag{21}$$

Finally, Eq. (18) acquires a form of the Mathieu equation

$$\frac{\partial^2\overline{F}}{\partial t^2}+\omega_k^2[1+g\cos(2\omega_1 t)]\overline{F}=0, \tag{22}$$

where

$$\omega_k=(1-g/2)\overline{c}k. \tag{23}$$

This equation determines actually the parametric resonance connecting the coherent transverse oscillations of the condensate with longitudinal phonons of frequency ω_k close to ω_1. From the physical point of view the parametric resonance originates due to a periodic variation of the condensate density and, therefore, variation of the sound velocity.

For $g\ll 1$, one can employ the standard algorithm of solving the Mathieu equation (see, e.g., [9]). We seek for the solution of the equation as

$$\overline{F}=a(t)\cos(\omega_1 t)+b(t)\sin(\omega_1 t). \tag{24}$$

Of course in an exact solution there are terms with the multiple harmonics but they correspond to the amplitudes of higher powers in g. The coefficients $a(t)$, $b(t)$ are slowly varying functions of t. So, on the substitution of Eq. (24) into Eq. (22) we retain only the terms of zero and first orders in g and neglect second derivatives for $a(t)$ and $b(t)$, that are $\sim g^2$. As a result, we arrive at a set of two-linear differential equations with constant coefficients. This system has exponentially increasing solution $[a(t),b(t)\sim\exp(\gamma_k t)]$ with the increment

$$\gamma_k=\frac{1}{2}\left[\left(\frac{1}{2}g\omega_1\right)^2-\varepsilon_k^2\right]^{1/2},\quad \varepsilon_k=2(\omega_k-\omega_1). \tag{25}$$

Thus, the parametric resonance takes place within the narrow range near ω_1 with the width

$$\delta\omega=\frac{1}{2}g\omega_1. \tag{26}$$

Within this range there occurs a growth of the sound-wave amplitudes, leading to a reduction of the energy of coherent oscillations of the condensate.

A square of the modulus of the wave function (24) increases as $\exp(2\gamma_k t)$.

Obviously, the energy grows similarly. At $T=0$, the initial energy of the longitudinal mode is zero-point energy. Then the total-energy transferred into the longitudinal modes as a result of the parametric resonance can be estimated as

$$E(t)\simeq\frac{L\hbar}{4\pi}\int dk\,\omega_k(e^{2\gamma_k t}-1)\,\theta\left(\frac{1}{2}g\omega_1-\varepsilon_k\right). \tag{27}$$

Let us rewrite this expression introducing instead of k a new variable $x=2\varepsilon_k/g\omega_1$. Then, with regard to Eq. (23) and (25) we have

$$E(t)\simeq B\mu J(t),\quad J(t)=\int_{-1}^{1}dx\left\{\exp\left[\frac{t}{t_1}(1-x^2)^{1/2}\right]-1\right\}, \tag{28}$$

where

$$B=\frac{Lg\omega_1}{4\pi\overline{c}}\frac{\hbar\omega_1}{\mu},\quad t_1=2/g\omega_1. \tag{29}$$

Let us compare $E(t)$ [Eq. (28)] with the energy of the coherent condensate oscillations E_c. Within the Thomas-Fermi approximation the energy of the condensate in a trap of frequency ω reads $E(\omega)=\frac{2}{3}\mu(\omega)N_0$. The chemical potential $\mu(\omega)$ is connected with the number of particles N_0 in the cylindrical trap by the relation

$$N_0=\frac{L}{a}\frac{\mu^2(\omega)}{4(\hbar\omega)^2}. \tag{30}$$

For a fixed number of particles $\mu(\omega)\sim\omega$.

At fast transition of the trap frequency from ω_0 to ω_1 the condensate keeps its space configuration at the first moment, being already in the new external field. Consequently, the condensate energy changes from $E_0=E(\omega_0)$ to some value E_0'.

The oscillation energy E_c is equal to the difference between E_0' and the static condensate energy corresponding to the frequency ω_1. Direct calculations give

$$E_c=E_0'-E(\omega_1)=\frac{1}{3}g^2\mu N_0 \tag{31}$$

[we have conserved a notation $\mu(\omega_0)=\mu$].

The damping of the condensate oscillations is characterized by a ratio

$$\frac{E(t)}{E_c}\simeq\frac{a}{R}\left(\frac{\hbar\omega_0}{\mu}\right)^3\frac{1}{g}J(t). \tag{32}$$

The factor in front of integral $J(t)$ for actual values of g is much smaller than unity. The damping becomes noticeable at times $t\gg t_1$. Calculating integral $J(t)$ in this limit, we find

$$\frac{E(t)}{E_c}\simeq\frac{a}{R}\left(\frac{\hbar\omega_0}{\mu}\right)^3\frac{1}{g}\left(\frac{2\pi t_1}{t}\right)^{1/2}\exp(t/t_1). \tag{33}$$

Hence, within a logarithmic accuracy the characteristic damping time equals

$$t_*=t_1\ln\left\{\frac{R}{a}\left(\frac{\mu}{\hbar\omega_0}\right)^3 g\left(\frac{t_*}{2\pi t_1}\right)^{1/2}\right\}. \tag{34}$$

This time proves to be large compared with the period $2\pi/\omega_1$ of the condensate oscillations not only due to smallness of g but also due to very-large factor in the argument of logarithm. Thus, the damping of the condensate oscillations may be slow, though inevitably emerging at $T=0$ for the conditions concerned.

A continuous character of the consideration assumes implicitly that at least several modes of the longitudinal sound excitations fall within the energy bandwidth (26) of the parametric resonance. The spacing between the neighbor modes equals $\Delta\omega = 2\pi\overline{c}/L$. Since $R\approx 2\overline{c}/\omega_1$, this condition holds when an inequality

$$L > 2\pi R/g \tag{35}$$

takes place.

Thus, obtained results demonstrate the existence of the damping at $T=0$ for the radial condensate oscillation in cylindrical trap. This damping is a result of the parametric resonance that couples the coherent condensate oscillations with the longitudinal sound waves.

At finite temperatures $\hbar\omega_1 < T \ll \mu$ the same resonance amplification occurs, but now the initial number of phonons of energy $\hbar\omega_k$ equals $T/\hbar\omega_k$ and the initial energy of resonance modes equals $\sim\hbar\omega_k(T/\hbar\omega_k)$. In this case in Eq. (27) the quantity $\hbar\omega_k$ should be replaced with T. Such replacement increases insignificantly the argument of logarithm in Eq. (34).

The parametric resonance is accompanied by a production of the large number of sound excitations $\sim E(t)/\hbar\omega_1$. This leads to reducing the number of particles in the condensate. In principle, for the times compared with t_* [Eq. (34)], this could require a solution of the self-consistent problem. From the general considerations it is clear that the escape of particles from the condensate can be neglected if the vibrational energy per particle $g\mu$ is small compared with the temperature of the Bose condensation T_c.

Let us estimate the characteristic damping time t_* for quantum gases Rb and Na. We consider an elongated cylindrical trap with $\nu_0 = 400$ Hz (Rb) or $\nu_0 = 10^3$ Hz (Na), $L = 2\times10^{-2}$ cm, $N_0 = 2\times10^5$ and assume that the dimensionless parameter $g\approx 0.15$. Then, from Eq. (30) we find Rb gas $\mu\approx 90$ nK. Accordingly, the Thomas-Fermi radius equals $R\approx 1.7\times10^{-4}$ cm. For the parameter t_1 in Eq. (29) and taking into account the relation $\omega_1 = (1-g)\omega_0$, we have $t_1\approx 6.2\times10^{-3}$ s. The direct calculation estimates the logarithmic factor in Eq. (34) as 10. Thus, evaluating the damping time we find $t_*\leqslant 0.1$ s. Being less than the ordinary lifetime of the systems, this time is realistic for revealing the damping.

Similar results can be obtained for sodium. In this case the estimations give: $\mu\approx 140$ nK, $R\approx 1.6\times10^{-4}$ cm, $t_1 \approx 2.5\times10^{-3}$ s. As a result, $t_*\approx 0.03$ s. It is important that in both cases the condition (35) proves to be fulfilled. At the same time, $g\mu \ll T_c$, and it means that after complete damping the depletion of condensate is quite small.

The results obtained demonstrate the feasibility of observing the damping at $T\ll\mu$ when the interaction with the thermal excitations can be ignored.

In conclusion, it is worth while to make a remark concerning a relation between the damping under consideration and Beliaev mechanism of phonon decay in a uniform weakly interacting Bose gas (see, e.g., [13]) Beliaev had studied the decay of a real phonon with the specific character of dispersion law. In the present paper, the coherent evolution of the condensate under conditions of complete absence of real phonons is considered. The damping in our case is caused by the parametric resonance as a result of the sound-velocity oscillation. It is instructive that the parametric resonance leads to the creation of the longitudinal sound waves filling the *finite*-energy interval [see Eqs. (26) and (27)], evidencing the nonlinear origin of the phenomenon.

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