Nonperturbative and perturbative treatments of parametric heating in atom traps

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We study the quantum description of parametric heating in harmonic potentials both nonperturbatively and perturbatively, having in mind applications to atom traps. The first approach establishes an explicit connection between classical and quantum descriptions; it also gives analytic expressions for properties such as the width of fractional frequency parametric resonances. The second approach gives an alternative insight into the problem and can be directly extended to take into account nonlinear effects. This is especially important for shallow traps.

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I. INTRODUCTION

Cooling techniques have allowed the trapping and manipulation of atoms by optical means. Such systems are used to perform experimental tests of fundamental principles and have important applications, such as very precise frequency standards or studies of cold atomic collisions or collective effects. Fluctuations of the electromagnetic fields used to trap or to modify the quantum state lead to decoherence in ion traps [1] and limit the trap stability in, e.g., far-off-resonance optical traps (FORTs) [2]. As a consequence, there has been in recent years an increasing interest in understanding the dynamics of noise-induced heating in atom traps.

In the harmonic model of the potential, fluctuations manifest themselves as either variations of the spring constant or on the equilibrium position. First-order perturbative studies [2,3] of harmonic parametric heating yield similar results to those classically expected: position noise is resonant at the vibrational frequency ω_0 leading to a constant heating rate, while intensity fluctuations are resonant at twice the vibrational frequency $2\omega_0$ leading to an exponential growth of the energy. Far-off-resonance optical traps are so sensitive to these fluctuations that parametric excitation has been used to accurately measure the trap parameters [4–6].

There are other interesting phenomena predicted by the classical theory of harmonic parametric excitation such as resonance effects at fractional frequencies [7]. They arise from intensity modulations of frequency $2\omega_0/n$ with n being any integer number. The $n=2$ resonance has actually been observed in FORTs [4–6] and resonances with $n\leqslant 10$ have been classically studied in ion traps [8].

The purpose of this paper is to analyze the quantum description of parametric heating in harmonic potentials both nonperturbatively and perturbatively. The first treatment shows an explicit connection with the classical problem that is valid only for harmonic potentials. It is based on well known algebraic techniques [9]. It explains observed features of parametric heating predicted both classically and quantum mechanically. The second approach has the advantage that it can be directly extended to anharmonic potentials. This is especially important for shallow traps.

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II. NONPERTURBATIVE TREATMENT OF HARMONIC PARAMETRIC HEATING

A. Time-dependent equilibrium position

The effective Hamiltonian that describes a harmonic oscillator with fluctuations in the trap equilibrium position is

$$H=\frac{p^2}{2m}+\frac{1}{2}M\omega_0^2[q+\epsilon_q(t)]^2, \tag{1}$$

where $\epsilon_q(t)$ measures such a fluctuation. When the standard creation $a^\dagger$ and annihilation a operators are introduced, so that

$$q=\sqrt{\frac{\hbar}{2m\omega_0}}(a+a^\dagger),$$

$$p=-i\sqrt{\frac{\hbar m\omega_0}{2}}(a-a^\dagger), \tag{2}$$

this Hamiltonian can be written in the form

$$H=\hbar\omega_0(a^\dagger a+1/2)+\hbar f_q(t)(a+a^\dagger)+\hbar g_q(t) \tag{3}$$

with

$$f_q(t)=\sqrt{\frac{m\omega_0^3}{2\hbar}}\epsilon_q(t), \quad g_q(t)=\frac{m\omega_0^2}{2\hbar}\epsilon_q^2(t). \tag{4}$$

The evolution operator $U(t)$ satisfies the equation

$$i\hbar\frac{\partial U}{\partial t}=HU \tag{5}$$

with the initial condition

$$U(0)=1. \tag{6}$$

Due to the fact that $\{a^\dagger a, a^\dagger, a, 1\}$ form a closed algebra, it is reasonable to write [9]

$$U=e^{-i\lambda_1 a^\dagger}e^{-i\lambda_2 a}e^{i\lambda_3 a^\dagger a}e^{-i\lambda_4}. \tag{7}$$

Using the well known relation

$$e^{\lambda A}Be^{-\lambda A}=B+\frac{\lambda}{1!}[A,B]+\frac{\lambda^2}{2!}[A,[A,B]]+\frac{\lambda^3}{3!}[A,[A,[A,B]]]+\cdots \tag{8}$$

it can be directly shown that the λ_i's satisfy the equations

$$\begin{aligned}\dot{\lambda}_1&=-i\dot{\lambda}_3\lambda_1+f_q(t),\\ \dot{\lambda}_2&=i\dot{\lambda}_3\lambda_2+f_q(t),\\ \dot{\lambda}_3&=\omega_0,\\ \dot{\lambda}_4&=\dot{\lambda}_3\lambda_2\lambda_1-i\dot{\lambda}_2\lambda_1+g_q(t),\end{aligned} \tag{9}$$

with the solutions

$$\begin{aligned}\lambda_1(t)&=e^{-i\omega_0 t}\int_0^t f_q(t')e^{i\omega_0 t'}dt',\\ \lambda_2(t)&=e^{i\omega_0 t}\int_0^t f_q(t')e^{-i\omega_0 t'}dt'=\lambda_1^*(t),\\ \lambda_3(t)&=\omega_0 t,\\ \lambda_4(t)&=\int_0^t[\omega_0\lambda_2\lambda_1-i\dot{\lambda}_2(t')\lambda_1(t')+g_q(t')]dt',\end{aligned} \tag{10}$$

which guarantee the initial condition (6). We are able, therefore, to evaluate the time evolution of any physical quantity. For instance, it turns out that the energy operator evolves as

$$\begin{aligned}\hat{\mathcal{E}}(t)&=\hbar\omega_0 U^{-1}(t)(a^\dagger a+1/2)U(t)\\ &=\hat{\mathcal{E}}(0)+\hbar\omega_0[i\lambda_2 a-i\lambda_1 a^\dagger+\lambda_2\lambda_1].\end{aligned} \tag{11}$$

If $\epsilon_q(t)$ is a fluctuating field, we are interested in averages over a time T short compared with the time scale over which the measurable physical quantities change, but large compared with the correlation time of the fluctuations. Let us consider fluctuations with zero mean value

$$\langle\epsilon_q(t)\rangle=\frac{1}{T}\int_0^T dt'\,\epsilon_q(t')=0, \tag{12}$$

and with a stationary correlation function

$$\langle\epsilon_q(t)\epsilon(t+\sigma)\rangle=\frac{1}{T}\int_0^T dt'\,\epsilon_q(t')\epsilon_q(t'+\sigma)=\eta_q(|\sigma|). \tag{13}$$

Then

$$\langle\hat{\mathcal{E}}(t)\rangle=\hat{\mathcal{E}}(0)+\frac{m\omega_0^4}{2}\int_0^t dt'\int_0^t dt''\langle\epsilon_q(t')\epsilon_q(t'')\rangle e^{i\omega_0(t'-t'')}. \tag{14}$$

Introducing the one-sided spectrum of the position fluctuations in the trap equilibrium position,

$$S_q(\omega)=\frac{2}{\pi}\int_0^\infty d\sigma\cos(\omega\sigma)\,\eta_q(|\sigma|), \tag{15}$$

and taking $t\gg T$, it follows:

$$\langle\mathcal{E}(t)\rangle=\mathcal{E}(0)+\frac{\pi}{2}m\omega_0^4 S_q(\omega_0), \tag{16}$$

recovering the asymptotic expression found by Savard *et al.* [2]. However, we have obtained it as valid in the nonperturbative regime with only the assumption that the fluctuating fields satisfy the equations (12) and (13). Notice that Eq. (14) is also valid whenever the state of the system is a Fock state. This possibility has been studied in Ref. [10] for the vacuum state where a detailed analysis of the short time behavior for different expressions of the correlation function $\lambda_1\lambda_2$ is performed.

On the other hand, if one considers driven fluctuations with a well specified $\epsilon_q(t)$ instead of noise fields, the exact equation for the energy evolution (11) can be used. For instance, if

$$\epsilon_q(t)=\epsilon_0\cos\omega_q t, \tag{17}$$

λ_1 and λ_2 are trivially calculated and the energy of, say, a coherent state $|\alpha\rangle$ evolves as

$$\begin{aligned}\langle\alpha|\hat{\mathcal{E}}(t)|\alpha\rangle&=\hbar\omega_0\left(|\alpha|^2+\frac{1}{2}\right)+\frac{t}{2}\hbar\omega_0\epsilon_0\{i\alpha[\zeta((\omega_q-\omega_0)t)\\ &\quad+\zeta^*((\omega_q+\omega_0)t)]-i\alpha^*[\zeta^*((\omega_q-\omega_0)t)\\ &\quad+\zeta((\omega_q+\omega_0)t)]\}+\frac{t^2}{4}\hbar\omega_0\epsilon_0^2\\ &\quad\times\left[|\zeta((\omega_q+\omega_0)t)|^2+|\zeta((\omega_q-\omega_0)t)|^2\right.\\ &\quad\left.+2\cos(\omega_0 t)\frac{\cos(\omega_0 t)-\cos(\omega_q t)}{\omega_0^2-\omega_q^2}\right],\end{aligned} \tag{18}$$

where we have defined

$$\zeta(y)=e^{iy/2}\frac{\sin(y/2)}{y/2}. \tag{19}$$

Equation (18) shows the expected resonances at $\omega=\omega_q$ and emphasizes the relevance of the parameter $\epsilon_0\alpha$ for characterizing the expected heating rates.

B. Time-dependent frequency

In this case the effective Hamiltonian is

$$H=\frac{p^2}{2m}+\frac{1}{2}M\omega_0^2[1-\epsilon(t)]q^2, \tag{20}$$

which takes the form

$$H=\hbar\omega_o(a^\dagger a+1/2)-\frac{\hbar\omega_0\epsilon(t)}{4}(a+a^\dagger)^2. \tag{21}$$

Analogously to the former case, the closed algebra nature of $\{a^\dagger a+1/2,a^{\dagger 2},a^2\}$ guarantees that the evolution operator can be written in the form

$$U(t)=e^{c_0(a^\dagger a+1/2)/2}e^{c_-a^{\dagger 2}/2}e^{c_+a^2/2}, \tag{22}$$

whenever the differential equations

$$i\dot{c}_0-2c_-\dot{c}_+=2\omega_0[1-1/2\epsilon(t)],$$

$$i(\dot{c}_--2c_-^2\dot{c}_+)e^{c_0}=\frac{1}{2}\omega_0\epsilon(t),$$

$$i\dot{c}_+e^{-c_0}=\frac{1}{2}\omega_0\epsilon(t) \tag{23}$$

are satisfied together with the initial conditions

$$c_0(0)=c_-(0)=c_+(0)=0. \tag{24}$$

A connection between the functions $\{c_0,c_+,c_-\}$ and a pair of solutions $h_1(t)$ and $h_2(t)$ of the classical equations of motion

$$\ddot{h}_k(t)+\omega_0^2(1-\epsilon(t))h_k(t)=0 \tag{25}$$

with the initial conditions

$$h_1(0)=1,\quad \dot{h}_1(0)=0,$$

$$h_2(0)=0,\quad \dot{h}_2(0)=-\omega_0 \tag{26}$$

is explicitly given by [11]

$$c_0=-2\ln(M_1),$$

$$c_+=M_2^*/M_1,$$

$$c_-=-M_2M_1, \tag{27}$$

where

$$M_1=\frac{1}{2}\left[(h_1-ih_2)-\frac{1}{\omega_0}(\dot{h}_2+i\dot{h}_1)\right],$$

$$M_2=\frac{1}{2}\left[-(h_1-ih_2)-\frac{1}{\omega_0}(\dot{h}_2+i\dot{h}_1)\right]. \tag{28}$$

The M_1 and M_2 functions satisfy the condition $|M_1|^2-|M_2|^2=1$. Equations. (22), and (25)–(28) show that the theory of classical response of harmonic oscillators to noise can be useful in the study of their quantum dynamics.

Using Eq. (8), an expression for the time evolution of the energy is found

$$\begin{aligned}\hat{\mathcal{E}}(t)&=\hbar\omega_0U^\dagger(t)(a^\dagger a+1/2)U(t)\\&=\hbar\omega_0[(1+2|M_2|^2)(a^\dagger a+1/2)-M_1M_2a^{\dagger 2}\\&\quad-M_1^*M_2^*a^2],\end{aligned} \tag{29}$$

as well as a nonperturbative expression for nonzero transition probabilities between Fock states $|k\rangle$ and $|s\rangle$:

$$\begin{aligned}|\langle k|U|s\rangle|^2=&\frac{|M_2|^{k-s}}{2^{k-s}|M_1|^{k+s+1}}\\&\times\left[\sum_{m=0}^{[s/2]}\frac{(-1)^m\sqrt{s!k!}}{m!(s-2m)![m+(k-s)/2]!}\right.\\&\left.\times\left(\frac{|M_2|}{2}\right)^{2m}\right]^2,\end{aligned} \tag{30}$$

where s and k have the same parity and for definiteness we have taken $k\geqslant s$.

Let us focus on the particular case

$$\epsilon(t)=\epsilon_0\cos(\omega t), \tag{31}$$

corresponding to controlled parametric excitation. Then, the classical equations of motion can be written in the Mathieu canonical form

$$\frac{d^2h_k}{d^2z}+[\alpha-2q\cos(2z)]h_k=0, \tag{32}$$

with $z=\omega t/2$, $\alpha=(2\omega_0/\omega)^2$, and $q=\epsilon_0\alpha/2$. It is well known [12,13] that depending on the values of α and q, the solutions h_k are stable or unstable. In the context of parametric heating the transition curves separating regions of stability and instability define the width of the corresponding resonance. For $q\ll\alpha$ the resonances are located at $\alpha\sim n^2$, i.e., $\omega=2\omega_0/n$. Their width can be found as a power series in ϵ_0 using, e.g., Eqs. 20.2.25 of Ref. [12]. Thus, the resonance at $\omega=2\omega_0$ has a width $\Delta_{2\omega_0}\sim\epsilon_0\omega_0/2$ while the resonance $\omega=\omega_0$ has a width $\Delta_{\omega_0}\sim\epsilon_0^2\omega_0/6$.

Using the normal form of the unstable solutions of the classical equations of motion, it is found that

$$M_i=\mu_i^{(+)}e^{\gamma\omega t/2}\phi_0(t)+\mu_i^{(-)}e^{-\gamma\omega t/2}\phi_0(-t), \tag{33}$$

with ϕ_0 a periodic function $\phi_0(t)=\phi_0(t+2\pi/\omega)$, and $\gamma=\gamma_r+i\gamma_i$ a complex number known as the characteristic exponent. In general, γ can be obtained using numerical methods or, for small values of q, by means of approximate expressions such as 20.3.15 of Ref. [12]. The complex numbers $\mu_i^{\pm}$ are determined by the initial conditions (26) and the general form of M_i given by Eq. (28). Thus, for $|\gamma_r\omega|t\gg1$, the energy (29) behaves as

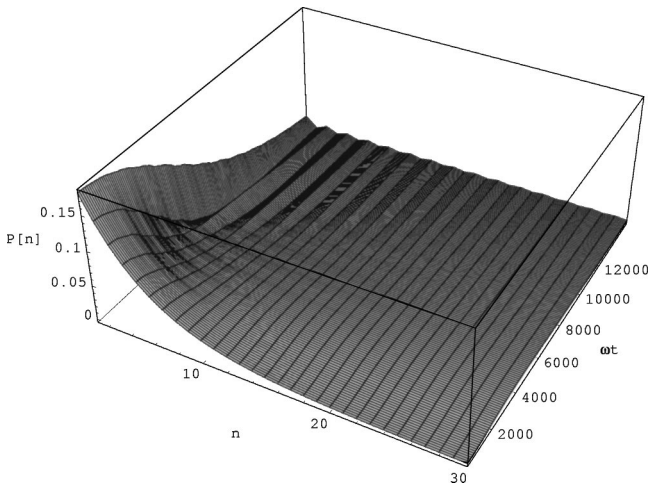

FIG. 1. Nonperturbative time evolution of the probability $P(n)$ of finding an atom in the n Fock state due to parametric heating with a time dependence $\epsilon(t)=\epsilon_0\cos(\omega t)$, when $\epsilon_0=0.05$ and the first fractional resonance condition $\omega=\omega_0$ is satisfied. The initial state of the system is a thermal one, Eq. (35), with a temperature $k_BT=5\hbar\omega_0$.

$$\hat{\mathcal{E}}(t)\rightarrow\hbar\omega_0[(1+2|\mu_2^{(+)}\phi_0(t)|^2e^{|\gamma_r|\omega t})(a^\dagger a+1/2) - \mu_2^{(+)}\mu_1^{(+)}\phi_0^2(t)e^{|\gamma_r|\omega t}a^{\dagger 2} - \mu_2^{(+)*}\mu_1^{(+)*}\phi_0^{*2}(t)e^{|\gamma_r|\omega t}a^2] \tag{34}$$

in the resonance region.

If the system is initially in thermal equilibrium with its environment

$$\hat{\rho}_T(t=0)=A\sum_{n=0}^{\infty}e^{(\hbar\omega_0/k_BT)(n+1/2)}|n\rangle\langle n|,$$

$$A^{-1}=\frac{2k_BT}{\hbar\omega_0}\mathrm{sech}\left(\frac{\hbar\omega_0}{2k_BT}\right) \tag{35}$$

a direct calculation shows that the mean value of the energy evolves as

$${}_T\langle\hat{\mathcal{E}}(t)\rangle_T=(1+2|M_2|^2)_T\langle\hat{\mathcal{E}}(0)\rangle_T. \tag{36}$$

If resonance conditions are achieved, the energy will exhibit an exponential growth with a rate determined both by the characteristic exponent and the resonance frequency:

$${}_T\langle\hat{\mathcal{E}}(t)\rangle_T\rightarrow 2e^{|\gamma_r|\omega t}|\mu_2^{(+)}\phi_0(t)|_T^2\langle\hat{\mathcal{E}}(0)\rangle_T. \tag{37}$$

In order to illustrate these ideas, Fig. 1 shows the time evolution of the probability $P(n)$ of finding an atom in the nth Fock state due to parametric heating with an intensity $\epsilon(t)=\epsilon_0\cos\omega t$, when $\epsilon_0=0.05$ and the first fractional resonance condition $\omega=\omega_0$ is satisfied. The initial state of the system is a thermal one, Eq. (35), with a temperature $k_BT=5\hbar\omega_0$. In Fig. 2 the corresponding evolution for the mean energy $\langle\mathcal{E}(t)\rangle$ is shown; the same values for the parameters k_BT, ω, and ϵ_0 are used. Essentially an exponential growth superposed to the oscillatory behavior of function $\phi_0(t)$, in Eq. (33), is observed. These graphs have been obtained using a numerical solution of Mathieu equation.

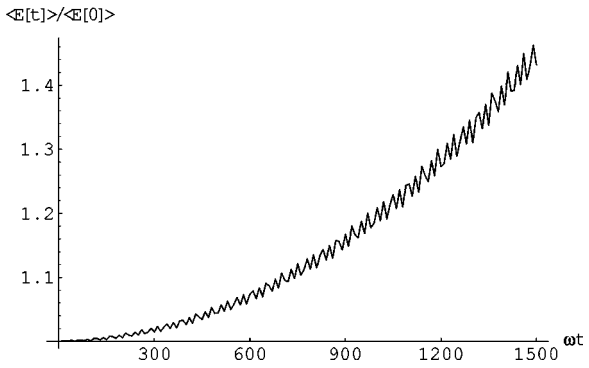

FIG. 2. Nonperturbative evolution of the mean energy $\langle\mathcal{E}(t)\rangle$ of a time-dependent harmonic oscillator of natural frequency ω_0 in units of $\langle\mathcal{E}(0)\rangle$. The system is initially in a thermal state with a temperature $k_BT=5\hbar\omega_0$ and the intensity time dependence is given by $\epsilon(t)=\epsilon_0\cos(\omega t)$, $\epsilon_0=0.05$, $\omega=\omega_0$, so that the first fractional resonance condition is satisfied. Superposed to the exponential growth, the oscillatory behavior of the function $\phi(t)$ that appears in the Floquet solution to Mathieu equation is clearly manifested.

III. PERTURBATIVE APPROACH TO PARAMETRIC HEATING

The nonperturbative approach to parametric heating made in the preceding section was useful to understand the connection between classical and quantum descriptions of heating, due to either controlled or stochastic variations of the parameters defining a harmonic oscillator. Unfortunately, this approach is valid only for quadratic potentials while anharmonic effects may be crucial in experiments with shallow confining potentials [8,14]. The purpose of this section is to study some high order perturbative effects due to variations on the strength of a confining potential. The resulting equations will be applied to a harmonic oscillator but with a straightforward extension to anharmonic potentials. Besides, this will allow us to understand fractional frequency resonances from an alternative point of view.

The system is described by a Hamiltonian

$$H=\frac{p^2}{2m}+V(x)(1+\epsilon_v(t)),\quad \epsilon_v(t)\ll 1. \tag{38}$$

Following standard time-dependent perturbation theory we define the unperturbed Hamiltonian

$$H_0=\frac{p^2}{2m}+V(x) \tag{39}$$

and work in the interaction picture in which the equation of motion of the state is

$$i\hbar\frac{d|\tilde{\Psi}(t)\rangle}{dt}=\epsilon_v(t)\tilde{V}(x,t)|\tilde{\Psi}(t)\rangle. \tag{40}$$

The transformed state $|\tilde{\Psi}(t)\rangle$ in the interaction picture is obtained from the Schrödinger picture state vector by a time-dependent unitary operator

$$|\tilde{\Psi}(t)\rangle = e^{iH_0 t/\hbar}|\Psi(t)\rangle, \tag{41}$$

while the interaction operator $\tilde{V}(x)$ is given by

$$\tilde{V}(x,t) = e^{iH_0 t/\hbar} V(x) e^{-iH_0 t/\hbar}. \tag{42}$$

In this picture the evolution operator satisfies the integral equation

$$\tilde{U}(t) = 1 - \frac{i}{\hbar}\int_0^t \epsilon_v(t)\tilde{V}(x,t)\tilde{U}(t')dt'. \tag{43}$$

Let us consider the transition probability amplitude $\langle k|U(t)|s\rangle$ between given eigenstates of the unperturbed Hamiltonian

$$H_0|n\rangle = E_n|n\rangle. \tag{44}$$

An iterative treatment of Eq. (43) gives

$$\begin{aligned}\langle k|\tilde{U}(t)|s\rangle = \delta_{ks} &- \frac{i}{\hbar}V_{ks}\int_0^t dt'\,\epsilon_v(t')e^{i\omega_{ks}t'} \\ &- \frac{1}{\hbar^2}\sum_n V_{kn}V_{ns}\int_0^t dt'\,\epsilon_v(t')e^{i\omega_{kn}t'} \\ &\times \int_0^{t'} dt''\,\epsilon_v(t'')e^{i\omega_{ns}t''} + \cdots \end{aligned} \tag{45}$$

with $\omega_{kn} = (E_k - E_n)/\hbar$ and

$$V_{kn} =: \langle k|V|n\rangle = E_k\delta_{kn} - \langle k|\frac{p^2}{2m}|n\rangle. \tag{46}$$

Notice that an n-order perturbation treatment of heating induced by stochastic noise $\epsilon_v^s(t)$ requires the knowledge of the spectrum of the $(1,\ldots,2n)$-point correlation functions, $\langle \epsilon_v^s(t)\rangle, \ldots, \langle \epsilon_v^s(t)\epsilon_v^s(t+\tau_1)\ldots\epsilon(t+\tau_{2n-1})\rangle$ defined in complete analogy with Eqs. (12) and (13). This results from the direct evaluation of the average rate to make a transition from a state $|k\rangle$ to state $|s\rangle$ in a time T,

$$R_{s\leftarrow k} = \frac{1}{T}|\langle k|\tilde{U}^{(n)}(T)|s\rangle|^2. \tag{47}$$

If heating is induced by a controlled modulation of the confining potential

$$\epsilon_v(t) = \epsilon_0\cos(\omega t), \tag{48}$$

then up to second order in ϵ_0

$$\begin{aligned}\langle k|\tilde{U}^{(2)}(t)|s\rangle = \delta_{ks} &+ \frac{i}{2\hbar}\epsilon_0 t V_{ks}[\zeta((\omega_{ks}+\omega)t) \\ &+ \zeta((\omega_{ks}-\omega)t)] - \left(\frac{\epsilon_0}{2\hbar}\right)^2 t \\ &\times \sum_n V_{kn}V_{ns}\Bigg[\frac{1}{i(\omega_{ns}+\omega)}[\zeta((\omega_{ks}+2\omega)t) \\ &+ \zeta(\omega_{ks}t) - \zeta((\omega_{kn}+\omega)t) - \zeta((\omega_{kn}-\omega)t)] \\ &+ \frac{1}{i(\omega_{ns}-\omega)} \\ &\times [\zeta((\omega_{ks}-2\omega)t) + \zeta(\omega_{ks}t) - \zeta((\omega_{kn}+\omega)t) \\ &- \zeta((\omega_{kn}-\omega)t)]\Bigg]. \end{aligned} \tag{49}$$

These expressions have physical meaning only if the changes in the wave function induced by $\tilde{U}(t)$ are small in the interval $(0,t)$.

For a harmonic oscillator with frequency ω_0, one finds that

$$\begin{aligned}V_{kn} = \frac{\hbar\omega_0}{4}[&(2k+1)\delta_{kn} + \sqrt{k(k-1)}\,\delta_{k,n+2} \\ &+ \sqrt{(k+1)(k+2)}\,\delta_{k,n-2}]. \end{aligned} \tag{50}$$

As a consequence, the following products $V_{kn}V_{ns}$ may be different from zero.

(i) $V_{k,k\pm2}V_{k\pm2,k\pm4}$. In Eq. (49), the resonant terms appear in the combination $\zeta((4\omega_0 \pm 2\omega)t) - \zeta((2\omega_0 \pm \omega)t)$ so that this transition is highly suppressed.

(ii) $V_{kn}V_{nk}$. Resonances are located at $\omega = 0, 2\omega_0$. For $\omega \sim 2\omega_0$ the contribution of the transition amplitude is of the form

$$\begin{aligned}\frac{\epsilon^2 t}{2\hbar\omega_0}\Bigg[&|V_{k,k+2}|^2\left(\frac{\zeta(0) - \zeta((\omega - 2\omega_0)t)}{i(\omega-\omega_0)}\right) \\ &+ |V_{k,k-2}|^2\left(\frac{\zeta(0) - \zeta((2\omega_0 - \omega)t)}{i(\omega-\omega_0)}\right)\Bigg]. \end{aligned} \tag{51}$$

(iii) $V_{k,k\pm2}V_{k\pm2,k\pm2}$ and $V_{k,k}V_{k,k\pm2}$. These are transitions that may be viewed as a combination of two virtual transitions $k \to s \to s$ and $k \to k \to s$. The corresponding resonance frequency according to Eq. (49) is the fractional frequency $\omega = |\omega_{ks}|/2 = \omega_0$. In fact the transition probability for a modulating frequency $\omega \sim \omega_0$ is

$$\begin{aligned}|\langle k|U^{(2)}|s\rangle|^2 &\sim \left(\frac{\epsilon}{2\hbar}\right)^4 \frac{t^2}{\omega_0^2}|V_{ks}|^2(V_{kk} - V_{ss})^2\frac{\sin^2(\omega-\omega_0)t/2}{(\omega-\omega_0)^2t^2/4} \\ &\sim \frac{\epsilon^4\omega_0^2 t^2}{1024}(k-s)^2[k(k-1)\delta_{k,s+2} \\ &+ (k+1)(k+2)\delta_{k,s-2}]\frac{\sin^2(\omega-\omega_0)t/2}{(\omega-\omega_0)^2t^2/4}. \end{aligned} \tag{52}$$

In all cases, the nonresonant terms $\zeta(\omega' t)$, with $\omega' \neq 0$, give rise to an oscillatory behavior of the transition probability which is consistent with that found using the exact evolution of the transition probability $|\langle 0|U|2\rangle|^2$. If one considers sufficiently long times, $\omega' t \gg 1$, having in mind the δ function representation

$$\delta(\omega) = \frac{2}{\pi} \lim_{t\to\infty} \frac{\sin^2(\omega t/2)}{t\omega^2}, \tag{53}$$

it is clear that just the resonant terms have a significant contribution. In such a limit the transition probability rates $R_{s\leftarrow k}$ are constant.

Now, some of the general behavior of higher order corrections can be inferred. Thus, the dominant transition probability of a fractional frequency resonance $\omega = 2\omega_0/n$ arises at n-order perturbation theory. It can be interpreted as an n-step procedure consisting of n-virtual transitions , where $n-1$ of them do not change the state and one changes it. Thus, we expect the expression (52) to describe approximately the transition probabilities $k \to k \pm 2$ when the source has a frequency $\omega = \omega_0$. This can be verified by comparing with the exact results of Sec. II.

IV. PERTURBATIVE TREATMENT OF ANHARMONIC EFFECTS: A USEFUL EXAMPLE

In the case of a one-dimensional (1D) optical lattice along the z axis, the Stark-shift potential is given by

$$V(r_\perp, z) = U_0 e^{-2r_\perp^2/w(z)^2} \cos^2(kz) \tag{54}$$

with $w(z)$ the beam radius and k its wave number. Atoms are trapped at the antinodes of the standing wave where the potential is usually approximated by a harmonic oscillator potential $V(r_\perp, z) = m\omega_r^2 r_\perp^2/2 + m\omega_z z^2/2$. This kind of approximation fails for describing some features of the atoms dynamics for shallow traps. Nevertheless, experimental conditions usually reveal an approximate separability of the transverse and longitudinal motion. In those circumstances the potentials

$$V_c(z) = A\cos^2(kz),$$

$$V_g(r_\perp) = -V_0 e^{-2r_\perp^2/\overline{w^2}}, \quad V_0 > 0 \tag{55}$$

result quite useful for modeling the confining interaction.

The one-dimensional stationary Schrödinger equation corresponding to the sinusoidal potential can be solved exactly in terms of Mathieu functions. A perturbative approach to parametric heating must take into account the band structure of the energy spectra for describing the excitation of atoms occupying the highest bound levels. A detailed treatment of this problem can be found in Ref. [14].

The stationary Schrödinger equation for the Gaussian transverse potential $V_g(r_\perp)$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} r_\perp \frac{\partial}{\partial r_\perp} + \frac{\partial^2}{\partial \phi^2} \right) + V_g(r_\perp) \right] \Psi_g(r_\perp, \phi) = E\Psi_g(r_\perp, \phi) \tag{56}$$

cannot be solved, to our knowledge, in an analytical form. However, a variational approach in terms of a properly chosen bidimensional harmonic oscillator basis set $\{\psi_h^{(N,m_\phi)}(r_\perp, \phi)\}$ can be easily implemented. That is, we can write

$$\Psi_g^{(E_k, m_\phi)}(r_\perp, \phi) \sim \sum_{N=0}^{N_{max}} c_{kN}^{(m_\phi)} \psi_h^{(N,m_\phi)}(r_\perp, \phi) \tag{57}$$

with the coefficients $c_{kN}^{(m_\phi)}$ determined by the Ritz variational method. The frequency of the harmonic basis is taken as

$$\omega_r = 2\sqrt{\frac{V_0}{m\overline{w^2}}} \tag{58}$$

in order to fit the Gaussian potential at its bottom.

Heating induced by a modulation of the confining potential with a time dependence given by Eq. (48) can be studied using the equations obtained in Sec. III. The relevant matrix elements are

$$\begin{aligned} V_{km_\phi; nm'_\phi} &=: \langle km_\phi | V_g | nm'_\phi \rangle \\ &= -\delta_{m_\phi m'_\phi} \frac{V_0}{2} \sum_{L=0}^{L_{max}} \sum_{J=0}^{J_{max}} c_{kL} c_{nJ} \\ &\times \frac{\Gamma((J+L)/2+1)}{((J-|m_\phi|)/2)!\,((L-|m_\phi|)/2)!} \\ &\times \frac{\beta^{[(J+L)/2]-|m_\phi|}}{(1+\beta)^{[(J+L)/2]+1}} F\left(-\frac{J-|m_\phi|}{2}, -\frac{L-|m_\phi|}{2}, \right. \\ &\left. -\frac{J+L}{2}, \frac{\beta^2-1}{\beta^2} \right) \end{aligned} \tag{59}$$

with J and L of the same parity, $\beta = (2\hbar)/(m\omega_r w^2)$ and $F(\alpha, \beta, \gamma, x)$ the confluent hypergeometric function [15]. The long time, $\omega_r t \gg 1$, evolution of the probability of finding a single atom in the state $|n, m_\phi\rangle$, $P(n, m_\phi)$ is determined by the transition rates $R_{km_\phi \leftarrow n, m_\phi}$, Eq. (47), through the expression

$$\dot{P}(n, m_\phi) = \sum_k R_{km_\phi \leftarrow nm_\phi} [P(k, m_\phi) - P(n, m_\phi)], \tag{60}$$

subject to the condition

$$\sum_{n, m_\phi} P(n, m_\phi) = 1. \tag{61}$$

In Fig. 3 we illustrate the kind of results that can be obtained from this analysis. There we show the fraction F of

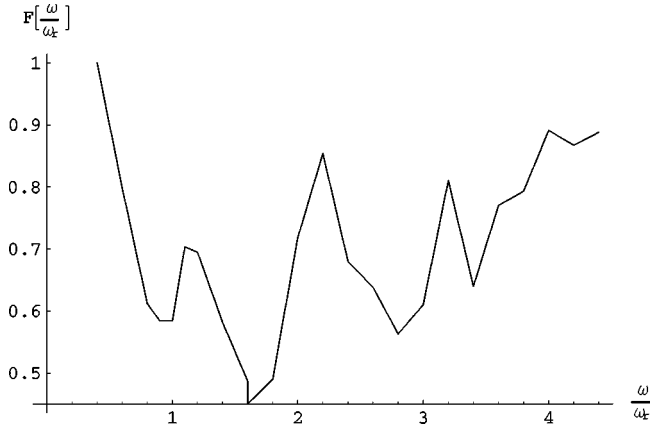

FIG. 3. Fraction F of trapped atoms as a function of the frequency ω used to induce parametric heating in a shallow Gaussian trap. The trap parameters $V_0=15\hbar\omega_r$ and $\overline{w^2}=\hbar/15m\omega_r$ were chosen to yield a low number of bound states. A broadering of the spectral lines of $0.1\hbar\omega_r$ has been assumed. The modulation amplitude is $\epsilon_0=0.01V_0$, and the time of application of this perturbation is $\tau=10^5/\omega_r$. The initial state of the system is a thermal one with $k_BT=10\hbar\omega_r$.

trapped atoms as a function of the frequency ω used to induce parametric heating in a shallow Gaussian trap. The trap parameters $V_0=15\hbar\omega_r$ and $\overline{w^2}=\hbar/15m\omega_r$ were chosen to yield a low number of bound states. As expected, the discrete part of the spectra is not equidistant: the energy difference between the lowest m_ϕ levels ranges from $\sim\hbar\omega_r$ to $\sim\hbar\omega_r/2$. We have assumed a broadening of the spectral lines of $0.1\hbar\omega_r$. In actual experiments line broadening could arise from the coupling of axial and radial modes, beam instabilities, as well as collisions with residual gas or the atom sample. We have taken a modulation amplitude $\epsilon_0=0.01V_0$, and the time of application of this perturbation is $\tau=10^5/\omega_r$. The initial state of the system is a thermal one with $k_BT=10\hbar\omega_r$. We notice that the response of the system to parametric heating as a function of frequency exhibits a rich structure with the deepest resonance at $\omega\sim1.8\omega_r$ as well as subharmonic and superharmonic resonances. Experimentally, the subharmonic resonance $\omega\sim\omega_r$ has been observed [4] while reports on the observation of superharmonic radial resonances typical of anharmonic potentials could not be found.

V. DISCUSSION AND OUTLOOK

In this work we have performed a perturbative and a nonperturbative analysis of quantum parametric oscillators. The first approach is based on standard algebraic techniques and gives a direct connection between classical and quantum results. In the case of controlled driving terms of the form $\epsilon(t)=\epsilon_0\cos\omega t$, the analytic solutions were used to evaluate time-dependent observables such as the energy growth due to parametric heating. This is especially important for far-off-resonance traps (FORTs) when the harmonic oscillator approximation is valid. In that case, parametric heating is used as a technique to measure the characteristic frequency of the trap. In such experiments, the fractional frequency resonances are usually observed and it is clear, within our formalism, how to perform their quantum description. In particular, using well known results of the theory of Mathieu functions, it was shown how to evaluate the fractional resonances width and how to obtain explicit expressions for the exponential growth of the energy.

The possibility of using the nonperturbative analysis to describe noise heating effects is a subject that deserves more analysis. This idea has recently been exploited [10] in the case of fluctuations in the trap center. In the case of intensity fluctuations, the problem is more complicated since a rigorous description would require us to study the correlation functions of the classical solutions $\{h_1,h_2\}$ and their derivatives $\{\dot{h}_1,\dot{h}_2\}$ as they appear in the expressions for the time evolution of a given physical quantity. For instance, to study the time evolution of the energy, it would be necessary to evaluate the classical correlation functions $|M_2|^2$ and M_1M_2. These correlation functions would depend on the noise correlations.

An alternative would consist of using solutions $\{h_1,h_2\}$ of the classical equations in order to describe in an effective way the coupling between the harmonic oscillator and the fluctuating fields. The usefulness of such an approach would be limited by its ability to reproduce experimental effects. For instance, classical harmonic oscillators with an intensity variation $\epsilon(t)=\epsilon_0\cos\omega' t$ and subject to a damping force $-\gamma\dot{x}$ lead to fractional frequency resonances only if ϵ_0 is greater than a certain threshold that depends on γ and the order of the resonance. Such thresholds have been observed in the classical collective motion of ions in Paul traps [8]. It can be expected that a similar phenomena occurs in the quantum regime of motion. The experimental study of these thresholds could be used to evaluate effective damping effects in atom traps.

In this paper the perturbative analysis of heating induced by variations of the intensity of a potential was studied in detail for a harmonic oscillator. This analysis gave a different insight into the problem. Fractional frequencies of order n appear in nth order perturbation theory. From the quantum theory point of view, they are a direct consequence of (i) the fact that the harmonic potential has diagonal matrix elements $\langle k|V|k\rangle$ different from zero; (ii) the equidistant spectrum of the harmonic oscillator.

The extension of the perturbative analysis to other confining potentials is straightforward. In fact, this approach has already been implemented for FORTs with shallow sinusoidal potentials [14]. The example of Gaussian potentials has been explicitly studied. A complete treatment of the Stark-shift interaction, Eq. (54), implies understanding the coupling between radial and longitudinal modes. This can be done using the results mentioned above but requires a numerical effort beyond the scope of this paper.

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