# Fidelity trade-off for finite ensembles of identically prepared qubits

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We calculate the trade-off between the quality of estimating the quantum state of an ensemble of identically prepared qubits and the minimum level of disturbance that has to be introduced by this procedure in quantum mechanics. The trade-off is quantified using two mean fidelities: the operation fidelity, which characterizes the average resemblance of the final qubit state to the initial one, and the estimation fidelity, describing the quality of the obtained estimate. We analyze properties of quantum operations saturating the achievability bound for the operation fidelity versus the estimation fidelity, which allows us to reduce substantially the complexity of the problem of finding the trade-off curve. The reduced optimization problem has the form of an eigenvalue problem for a set of tridiagonal matrices, and it may be easily solved using standard numerical tools.

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## I. INTRODUCTION

It is a well-known fact that given a single copy of a quantum system, it is, in general, impossible to determine its quantum state exactly [1]. This principle is closely related to the no-cloning theorem [2], which prevents one from producing multiple faithful copies from a single unknown state. Of course, the situation becomes different when one is given an ensemble of identically prepared quantum systems. With the increasing size of such an ensemble, one can extract more and more precise information on its preparation [3].

An important effect that usually accompanies an attempt to determine the unknown preparation of a quantum system is the disturbance of its original state. The state disturbance is generally a penalty for gaining classical information from a quantum system. This fundamental feature of quantum mechanics has been discussed from many different points of view, depending on the particular physical scenario considered [4,5]. A recent paper [6] presented the description of the trade-off between the information gain and the quantum state disturbance motivated by the problem of quantum state estimation [3,7–10]. In this approach, the classical outcome gained from an operation on the quantum system is converted into a guess what the original state was. According to the general quantum-mechanical rule linking the information gain with the state disturbance, the better guess can be made on average, the less the final state of the system should resemble the initial one. Natural parameters for quantifying the trade-off in this context are mean fidelities, defined using the scalar product between the relevant state vectors, and averaged over many realizations of the scenario. The first quantity of interest is the operation fidelity $F$, which parametrizes the average resemblance of the state of the system after the operation to the original one. The second quantity, known as the estimation fidelity $G$, tells us how good an estimate one can provide on the basis of the classical outcome of a given operation. In the plane of the fidelities $F$ and $G$, quantum mechanics imposes a bound in the form of a trade-off curve limiting the maximum values of $F$ and $G$ that can be achieved simultaneously for any quantum operation. For a single copy of a $d$-level system, the trade-off curve between $F$ and $G$ may be described in simple analytical terms [6].

In this paper, we discuss a more general case of the fidelity trade-off, when one is given a finite ensemble of identically prepared systems. We shall assume that these systems are qubits prepared in an arbitrary, randomly selected, pure state. Of course, the penalty—i.e., the state disturbance—for gaining information is expected to be smaller for an ensemble compared to the single system case. In the limit of an infinite number of copies, we may determine the quantum state exactly and reset the state of all the qubits according to this precisely known information. Then, in principle, no disturbance needs to occur. Our interest here will be focused on the intermediate case between the single-copy operations and nearly perfect estimation of large ensembles. We shall discuss the trade-off curve between the operation fidelity and the estimation fidelity in the most general case when the ensemble consists of a finite number $N$ of qubits.

The two fidelities used in our paper have fundamentally different practical meanings. Informally speaking, the operation fidelity $F$ deals with the intrinsic quantum information remaining in the state of a quantum system. In contrast, the estimation fidelity $G$ describes the classical information gained from the measurement. This classical information allows us, for example, to generate arbitrarily many copies of the initial state with the same fidelity $G$. Consequently, for finite ensembles, the fidelities $F$ and $G$ take values from different ranges. For example, $F$ may easily be equal to one for any size of the ensemble (which simply means that nothing is done to the qubits) but, in contrast, $G=1$ means that we are able to generate arbitrarily many perfect copies using the

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result of the estimation. Previous work on quantum state estimation [3] demonstrated that the maximum attainable estimation fidelity for an ensemble consisting of $N$ qubits is $G=(N+1)/(N+2)$. As we justify later, this is also the value for the operation fidelity in the limit of optimal quantum state estimation. Of interest here is what happens below this value for $G$ and above this value for $F$, as this describes the region where we try to convert part of the quantum information contained in the initial state into a classical guess.

Finding the trade-off curve for an ensemble of $N$ qubits presents, in principle, a rather complicated problem. A general quantum operation can map the initial state of the qubits onto the full Hilbert space with the dimensionality $2^N$, which grows exponentially with the size of the ensemble. Furthermore, the classical operation outcomes used for the estimation can *a priori* assume values from an arbitrarily large set. In this paper, we demonstrate in several steps that it is possible to reduce the general problem of finding the trade-off curve to $\mathcal{O}(\sqrt{N})$ independent constrained optimization problems, each involving $\mathcal{O}(N)$ real variables. The reduced optimization problems have the quadratic form, and they may be solved numerically by finding the eigenvectors of certain tridiagonal matrices. This is a substantial reduction of the complexity of the problem compared to its original formulation, which allows us to deal numerically with much larger ensembles. Furthermore, numerical evidence strongly suggests that, in general, just one from $\mathcal{O}(\sqrt{N})$ optimization problems gives the full trade-off curve, but a strict mathematical proof of this conjecture is lacking. Most importantly, we achieve the reduction of the complexity without imposing any restrictions on the generality of quantum operations considered, and the calculated trade-off curves are both tight and universal.

The results of our paper are summarized in Fig. 1, where we depict the trade-off curves calculated using our approach for several exemplary values of $N$. All the curves have a common extreme point attained for $F=1$ and $G=1/2$. This point is reached when the ensemble is simply left intact, and the corresponding value for the estimation fidelity $G=1/2$ describes making a completely random guess about the state of the qubits. The other extreme point for each trade-off curve, corresponding to optimal quantum state estimation, is given by $F=G=(N+1)/(N+2)$. In this limit, the fidelity of the qubits remaining after the operation is exactly the same as the fidelity of our guess. One can give a simple intuitive argument that this should be the case: $F$ is always an upper bound on $G$, since one can always set the state of the qubits equal to the guess state. It is plausible that this bound may be achieved in the limit where we only care about maximizing $G$. Between the two extreme points, the depicted trade-off curves illustrate how with increasing $N$, the extraction of classical information has less of an effect on the state of the qubits after the operation.

The problem considered in this paper may be viewed as a special case of quantum cloning, i.e., generating a larger number of imperfect copies from a given ensemble [11]. As we noted, given a classical estimate of the quantum state, we may use it to generate an infinite number of qubits with the same fidelity as the estimate [12]. Thus, the trade-off curves

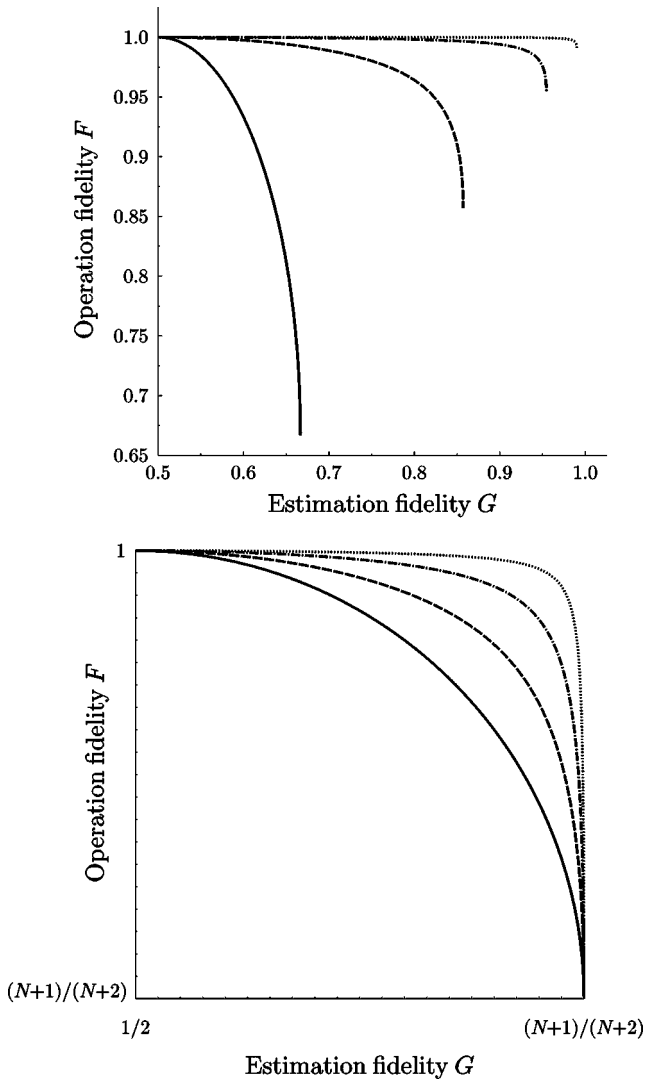


FIG. 1. The trade-off curves between the operation fidelity $F$ and the operation fidelity $G$, depicted using absolute (top) and relative (bottom) scaling of the axes, for several values of the ensemble size: $N=1$ (solid), $N=5$ (dashed), $N=20$ (dotted dashed), and $N=100$ (dotted).

presented in this paper describe the optimal performance of an asymmetric quantum copying machine that given $N$ identical pure qubits produces $N$ clones with the fidelity $F$, and in addition, arbitrarily many clones with the fidelity $G$.

The paper is organized as follows. First, in Sec. II, we formulate the problem of the fidelity trade-off in quantitative terms. In Sec. III, we simplify the formulas for the fidelities using the angular momentum representation of the rotation group. This provides explicit expressions for the fidelities that are suitable for further calculations. In Sec. IV, we argue that in order to find the trade-off curve it suffices to consider a single-quantum operation element, thus substantially reducing the complexity of the problem. We further demonstrate in Sec. V, that it is sufficient to consider operations that map the initial state of the qubits only onto the fully symmetric subspace. With these results in hand, we define in Sec. VI the reduced optimization problem that yields the actual trade-off curve. The numerical solution to this problem is

discussed in Sec. VII. Next, we show in Sec. VIII that the calculated trade-off curves are achievable, by constructing explicit quantum operations attaining the derived bound. Section IX concludes the paper.

## II. FORMULATION OF THE PROBLEM

We begin with an ensemble of $N$ qubits all prepared in the same pure state $|\Omega\rangle$. We shall use the following notation:

$$|\Omega\rangle = \hat{U}(\Omega)|\uparrow\rangle, \tag{1}$$

i.e., the state $|\Omega\rangle$ is represented as a result of a unitary operation $\hat{U}(\Omega)$ on a reference state $|\uparrow\rangle$, which we will take for concreteness to be the spin-up state along the $z$ axis, $\hat{\sigma}^z|\uparrow\rangle = |\uparrow\rangle$. The group of the unitary transformations $\hat{U}(\Omega)$ may be conveniently parametrized using the two-dimensional irreducible representation of the rotation group. Thus, $\Omega$ may be considered as an abbreviation for the triplet of the Euler angles $(\phi,\theta,\zeta)$, and

$$\hat{U}(\Omega) = \exp(-i\phi\hat{\sigma}^z/2)\exp(-i\theta\hat{\sigma}^y/2)\exp(-i\zeta\hat{\sigma}^z/2). \tag{2}$$

The third Euler angle $\zeta$ introduces a trivial overall phase factor in the definition of the states $|\Omega\rangle$ and, in principle, it could be set to zero. However, we will keep it as an independent variable in order to adhere strictly to standard angular momentum algebra notation that we will use later. The canonical volume element in the group of unitary transformations $\hat{U}(\Omega)$ is given by

$$d\Omega = \frac{1}{8\pi^2}\sin\theta\, d\theta\, d\phi\, d\zeta. \tag{3}$$

We assume that the initial state $|\Omega\rangle$ of the ensemble of $N$ qubits is randomly selected according to the probability distribution given by this measure.

Initially, the composite state of the ensemble of the qubits is described by a tensor product $|\Omega\rangle\langle\Omega|^{\otimes N}$. We assume that the qubits are submitted to an action of a certain quantum operation, which may, in general, act collectively on the whole ensemble. Such an operation is described by a set of operators $\{\hat{A}_{rs}\}$ acting in the $2^N$-dimensional tensor product Hilbert space of all the qubits [13]. The classical outcome of the operation is given by the index $r$, and it is correlated with the final quantum state of the qubits. The probability $p_r(\Omega)$ of obtaining the result $r$ is given by

$$p_r(\Omega) = \sum_s \mathrm{Tr}(\hat{A}_{rs}^\dagger \hat{A}_{rs}|\Omega\rangle\langle\Omega|^{\otimes N}). \tag{4}$$

The conditional transformation of the ensemble corresponding to the outcome $r$ is described by the formula

$$\hat{\varrho}_r^{\mathrm{out}}(\Omega) = \frac{1}{p_r(\Omega)}\sum_s \hat{A}_{rs}|\Omega\rangle\langle\Omega|^{\otimes N}\hat{A}_{rs}^\dagger. \tag{5}$$

The summation over the index $s$ maps in general pure states onto mixed ones, and it may be viewed as responsible for introducing excess stochastic fluctuations [14]. In general, the index $s$ may assume values from a different set for each $r$. For the operation to be trace preserving, the set of the operators $\{\hat{A}_{rs}\}$ must satisfy the completeness relation of the form

$$\sum_{rs} \hat{A}_{rs}^\dagger \hat{A}_{rs} = \hat{\mathbb{1}}. \tag{6}$$

With the above notation for quantum operations, we can now define explicitly the two quantities central to this paper: the operation fidelity $F$ and the estimation fidelity $G$. The operation fidelity $F$ quantifies the average resemblance of the state after the operation to the original one. Let us consider the reduced single-qubit density matrix after the operation, averaged over all $N$ qubits

$$\hat{\varrho}_r^{\mathrm{red}}(\Omega) = \frac{1}{N}[\mathrm{Tr}_{2\text{-}N}\hat{\varrho}_r^{\mathrm{out}}(\Omega) + \mathrm{Tr}_{1,3\text{-}N}\hat{\varrho}_r^{\mathrm{out}}(\Omega) + \cdots + \mathrm{Tr}_{1\text{-}(N-1)}\hat{\varrho}_r^{\mathrm{out}}(\Omega)], \tag{7}$$

where the subscript of the trace symbol Tr labels the range of qubits over which the trace operation is performed. The expectation value of the above expression on the initial state $\langle\Omega|\hat{\varrho}_r^{\mathrm{red}}(\Omega)|\Omega\rangle$ tells us how much on average the state of a single qubit after the operation resembles its initial value. This quantity, summed over the possible outcomes $r$ of the operation with the corresponding weights $p_r$ and averaged over the initial state of the qubits, yields the mean operation fidelity for an ensemble of identically prepared qubits

$$F = \int d\Omega \sum_r p_r(\Omega)\langle\Omega|\hat{\varrho}_r^{\mathrm{red}}(\Omega)|\Omega\rangle. \tag{8}$$

The second quantity of interest is the estimation fidelity $G$. Given the classical outcome $r$ of the operation, we can make a guess $|\Omega_r\rangle$ what the original state of the qubits was. A natural way to quantify the quality of the guess is to take the squared absolute value of the scalar product between the guess and the original state, equal to $|\langle\Omega_r|\Omega\rangle|^2$. The estimation fidelity is obtained by averaging this expression over the sets of possible operation outcomes $r$ and the input states $|\Omega\rangle$:

$$G = \int d\Omega \sum_r p_r(\Omega)|\langle\Omega_r|\Omega\rangle|^2. \tag{9}$$

The estimation fidelity depends not only on the quantum operation $\{\hat{A}_{rs}\}$ itself, but also on the estimation rule used to make the guess, described by the mapping $r \mapsto |\Omega_r\rangle$. We will demonstrate in the following how to define this mapping in a way that optimizes the estimation fidelity for a given arbitrary quantum operation.

Our goal is now to find the inequality that bounds the fidelities $F$ and $G$, assuming the most general form of the

operation $\{\hat{A}_{rs}\}$. The only requirement that has to be satisfied by the set of the operators $\hat{A}_{rs}$ is the completeness condition described by Eq. (6).

## III. FIDELITIES

In this section, we will simplify the expression for the fidelities $F$ and $G$ to the form that makes them more manageable in the optimization procedure. Our first step will be an explicit calculation of the integrals over the space of pure states $|\Omega\rangle$, which can be done with the help of tools developed in the theory of representations of the rotation group. Throughout this section, we shall follow strictly the notation of Ref. [15] for the angular momentum algebra and the elements of rotation matrices.

To simplify subsequent expressions, we begin with a general observation that the index $s$ appearing in Eqs. (4) and (5) could, in principle, also be known classically after the operation. Summation over the index $s$ in Eqs. (4) and (5) means that the operation is imperfect, and it averages statistically different output states. This results in the loss of a fraction of the information extracted from the initial quantum state. As we are interested in the optimal operations saturating the quantum-mechanical bound on the fidelities, we can assume with no loss of generality that both $r$ and $s$ are known. In such a case, we may use a single index to label both $r$ and $s$. For this reason, we will assume in the following that the index $s$ is trivial, i.e., it assumes only a single value for each $r$, and that consequently, it may be dropped from further notation. Thus, we restrict our attention to quantum operations that are known in the literature as ideal [16] or efficient [5].

In the following calculations, it will be convenient to use the decomposition of the complete Hilbert space of $N$ qubits into subspaces with the fixed value of the total angular momentum operator. This decomposition has the form [17]

$$\bigotimes_{i=1}^{N}\mathcal{H}=\bigoplus_{\alpha=1}^{\binom{2j}{\lfloor j\rfloor}}\mathcal{H}_{j_\alpha}, \tag{10}$$

where $j=N/2$ is the largest value of the angular momentum appearing in the decomposition, the $j_\alpha$'s assume values in the set $\{j,j-1,\ldots,j-\lfloor j\rfloor\}$ and are arranged in a nonascending order. Here, $\mathcal{H}_{j'}$ denotes the subspace corresponding to the total angular momentum value $j'$. We note that $j_1=j$ and $j_\alpha<j$ for $\alpha>1$, as there is only one representation with $j=N/2$ corresponding to the fully symmetric subspace. For completeness, in Appendix A, we present a simple derivation of the multiplicities of the angular momentum representations appearing in Eq. (10). The action of a tensor product of the unitary operators $\hat{U}^{\otimes N}(\Omega)$ in each of the subspaces $\mathcal{H}_{j'}$ is given by the corresponding representation of the rotation group. In order to simplify the notation, we will henceforth use the same symbol $\hat{U}(\Omega)$ to denote the action of $\hat{U}^{\otimes N}(\Omega)$ on the whole ensembles of qubits, as for a single qubit.

### A. Operation fidelity

Let us start with the operation fidelity $F$. The matrix element $\langle\Omega|\hat{\varrho}_r^{\mathrm{red}}(\Omega)|\Omega\rangle$ appearing in the definition of $F$ in Eq. (8) is equivalently given by the expectation value over the density matrix $\hat{\varrho}_r^{\mathrm{out}}(\Omega)$ defined in Eq. (5) of the following operator:

$$\hat{P}(\Omega)=\frac{1}{N}(|\Omega\rangle\langle\Omega|\otimes\hat{1}\otimes\cdots\otimes\hat{1}+\hat{1}\otimes|\Omega\rangle\langle\Omega|\otimes\cdots\otimes\hat{1}+\hat{1}\otimes\hat{1}\otimes\cdots\otimes|\Omega\rangle\langle\Omega|). \tag{11}$$

Using the above definition, the mean estimation fidelity may be written compactly as

$$F=\int d\Omega\sum_r \mathrm{Tr}[\hat{P}(\Omega)\hat{A}_r|\Omega\rangle\langle\Omega|^{\otimes N}\hat{A}_r^\dagger]. \tag{12}$$

We will now express $\hat{P}(\Omega)$ in terms of the angular momentum operators. Expressing the projection $|\Omega\rangle\langle\Omega|$ as a combination of the Pauli matrices

$$|\Omega\rangle\langle\Omega|=\frac{1}{2}[\hat{1}+\sin(\theta)\cos(\phi)\hat{\sigma}^x+\sin(\theta)\sin(\phi)\hat{\sigma}^y+\cos(\theta)\hat{\sigma}^z] \tag{13}$$

we can write the operator $\hat{P}(\Omega)$ in terms of the total angular momentum operators for the composite system of $N$ qubits

$$\begin{aligned}\hat{P}(\Omega)&=\frac{1}{2N}\sum_{i=1}^{N}[\hat{1}+\sin(\theta)\cos(\phi)\hat{\sigma}_i^x+\sin(\theta)\sin(\phi)\hat{\sigma}_i^y+\cos(\theta)\hat{\sigma}_i^z]\\&=\frac{1}{2}\hat{1}+\frac{1}{N}[\sin(\theta)\cos(\phi)\hat{J}^x+\sin(\theta)\sin(\phi)\hat{J}^y+\cos(\theta)\hat{J}^z],\end{aligned} \tag{14}$$

where the index $i$ enumerates the qubits. In the following, it will be convenient to switch to the pair of the angular momentum raising and lowering operators $\hat{J}^\pm=\hat{J}^x\pm i\hat{J}^y$, using what we have

$$\hat{P}(\Omega)=\frac{1}{2}\hat{1}+\frac{1}{N}\left(\frac{1}{2}e^{i\phi}\sin(\theta)\hat{J}^-+\frac{1}{2}e^{-i\phi}\sin(\theta)\hat{J}^++\cos(\theta)\hat{J}^z\right). \tag{15}$$

With this expression for the operator $\hat{P}(\Omega)$, we may write the mean fidelity $F$ as

$$F=\frac{1}{2}+\frac{1}{N}\sum_r \mathrm{Tr}\left(\frac{1}{2}\hat{J}^-\hat{A}_r\hat{K}_{-1}\hat{A}_r^\dagger+\frac{1}{2}\hat{J}^+\hat{A}_r\hat{K}_1\hat{A}_r^\dagger+\hat{J}^z\hat{A}_r\hat{K}_0\hat{A}_r^\dagger\right), \tag{16}$$

where all the terms involving the Euler angles $\Omega$ have been collected to three integrals $\hat{K}_\tau$, $\tau=-1,0,1$, defined as

$$\hat{K}_\tau=\int d\Omega\, k_\tau(\Omega)|\Omega\rangle\langle\Omega|^{\otimes N}, \tag{17}$$

with the functions $k_\tau(\Omega)$ given by

$$\begin{aligned} k_{-1}(\Omega)&=e^{i\phi}\sin\theta,\\ k_0(\Omega)&=\cos\theta,\\ k_1(\Omega)&=e^{-i\phi}\sin\theta. \end{aligned} \tag{18}$$

Explicit calculation of the integrals $\hat{K}_\tau$, performed in Appendix B, yields the following expressions:

$$\begin{aligned} \hat{K}_{\pm1}&=\frac{1}{(j+1)(2j+1)}\hat{J}_j^{\mp},\\ \hat{K}_0&=\frac{1}{(j+1)(2j+1)}\hat{J}_j^z, \end{aligned} \tag{19}$$

where $j=N/2$ and $\hat{J}_j^{\mp}$, $\hat{J}_j^z$ denote the angular momentum operators restricted to the completely symmetric subspace of the $N$ qubits defined by the angular momentum $j=N/2$. The operators $\hat{K}_\tau$ vanish outside this space, as they are defined as integrals of totally symmetric projection operators $|\Omega\rangle\langle\Omega|^{\otimes N}$.

Inserting the explicit form of the operators $\hat{K}_\tau$ into Eq. (16) yields

$$F=\frac{1}{2}+\frac{1}{2j(j+1)(2j+1)}\times\sum_r \mathrm{Tr}\left(\frac{1}{2}\hat{J}^-\hat{A}_r\hat{J}_j^+\hat{A}_r^\dagger+\frac{1}{2}\hat{J}^+\hat{A}_r\hat{J}_j^-\hat{A}_r^\dagger+\hat{J}^z\hat{A}_r\hat{J}_j^z\hat{A}_r^\dagger\right). \tag{20}$$

The operation fidelity may be equivalently expressed in terms of the Hermitian operators $\hat{J}^x$, $\hat{J}^y$, and $\hat{J}^z$ as

$$F=\frac{1}{2}+\frac{1}{2j(j+1)(2j+1)}\times\sum_r \mathrm{Tr}(\hat{J}^x\hat{A}_r\hat{J}_j^x\hat{A}_r^\dagger+\hat{J}^y\hat{A}_r\hat{J}_j^y\hat{A}_r^\dagger+\hat{J}^z\hat{A}_r\hat{J}_j^z\hat{A}_r^\dagger). \tag{21}$$

We note that this expression is completely symmetric with respect to the three Cartesian components of the angular momentum. Furthermore, it may be easily checked that each of the traces appearing in the sum over $r$ in the above formula is invariant with respect to the transformation of the operator $\hat{A}_r$ according to

$$\hat{A}_r\to\hat{U}(\Omega)\hat{A}_r\hat{U}^\dagger(\Omega), \tag{22}$$

where $\hat{U}(\Omega)$ is an arbitrary rotation matrix.

### B. Estimation fidelity

We will now evaluate the integral over the space of pure states in the expression for the estimation fidelity $G$, given by

$$G=\int d\Omega\sum_r \mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r|\Omega\rangle\langle\Omega|^{\otimes N})|\langle\Omega_r|\Omega\rangle|^2. \tag{23}$$

As we discuss in Appendix B, the projection on the product state $|\Omega\rangle\langle\Omega|^{\otimes N}$ may be represented as

$$|\Omega\rangle\langle\Omega|^{\otimes N}=\hat{U}(\Omega)|j;j\rangle\langle j;j|\hat{U}^\dagger(\Omega), \tag{24}$$

where $j=N/2$ and $|j;j\rangle$ belongs to the subspace with the total angular momentum $j$ and is the eigenvector of $\hat{J}^z$ corresponding to the eigenvalue $j$. Thus, we have

$$G=\int d\Omega\sum_r |\langle\uparrow|\hat{U}^\dagger(\Omega_r)\hat{U}(\Omega)|\uparrow\rangle|^2\times\mathrm{Tr}[\hat{A}_r^\dagger\hat{A}_r\hat{U}(\Omega)|j;j\rangle\langle j;j|\hat{U}^\dagger(\Omega)]. \tag{25}$$

We may now swap the order of the integration over $\Omega$ and the summation over $r$, and change for each $r$ the integration variables from $\Omega$ to $\Omega'$, such that

$$\hat{U}(\Omega')=\hat{U}^\dagger(\Omega_r)\hat{U}(\Omega). \tag{26}$$

Using this parametrization, we have $\langle\uparrow|\hat{U}^\dagger(\Omega_r)\hat{U}(\Omega)|\uparrow\rangle|^2=(1+\cos\theta')/2$ and

$$G=\frac{1}{2}\sum_r\int d\Omega'(1+\cos\theta')\mathrm{Tr}[\hat{A}_r^\dagger\hat{A}_r\hat{U}(\Omega_r)\hat{U}(\Omega')|j;j\rangle\times\langle j;j|\hat{U}^\dagger(\Omega')\hat{U}^\dagger(\Omega_r)]. \tag{27}$$

This expression may be decomposed into two parts according to the two terms in the factor $1+\cos\theta'$. The first part involves the integral

$$\int d\Omega'\hat{U}(\Omega')|j;j\rangle\langle j;j|\hat{U}^\dagger(\Omega')=\frac{1}{2j+1}\hat{\mathbb{1}}_j, \tag{28}$$

which is proportional to the identity operator $\hat{\mathbb{1}}_j$, truncated to the completely symmetric subspace. Consequently, the summation over $r$ for this term may be easily performed that yields the constant value 1/2. The second part may be expressed with the help of the operator $\hat{K}_0$ defined in Eq. (17), which gives

$$G=\frac{1}{2}\sum_r\frac{\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{\mathbb{1}}_j)}{2j+1}+\frac{1}{2}\sum_r\mathrm{Tr}[\hat{A}_r^\dagger\hat{A}_r\hat{U}(\Omega_r)\hat{K}_0\hat{U}^\dagger(\Omega_r)]. \tag{29}$$

Inserting the explicit form of $\hat{K}_0$ yields

$$G=\frac{1}{2}+\frac{1}{2(j+1)(2j+1)}\sum_r\mathrm{Tr}[\hat{A}_r^\dagger\hat{A}_r\hat{U}(\Omega_r)\hat{J}_j^z\hat{U}^\dagger(\Omega_r)]. \tag{30}$$

This expression for the estimation fidelity allows one to derive easily the optimal estimation strategy, i.e., the mapping from the set of outcomes $r$ to guesses $\Omega_r$ for a given quantum operation $\{\hat{A}_r\}$. In order to derive this strategy, we first note that

$$\hat{U}(\Omega_r)\hat{J}_j^z\hat{U}^\dagger(\Omega_r)=\sin(\theta_r)\cos(\phi_r)\hat{J}_j^x+\sin(\theta_r)\sin(\phi_r)\hat{J}_j^y+\cos(\theta_r)\hat{J}_j^z. \tag{31}$$

This allows us to write each of the traces in the sum over $r$ in the form of a scalar product $\mathbf{A}_r^T\mathbf{\Omega}_r$ between two three-dimensional real vectors $\mathbf{A}_r$ and $\mathbf{\Omega}_r$ defined as

$$\mathbf{A}_r=\begin{pmatrix}\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^x)\\ \mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^y)\\ \mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^z)\end{pmatrix} \tag{32}$$

and

$$\mathbf{\Omega}_r=\begin{pmatrix}\sin(\theta_r)\cos(\phi_r)\\ \sin(\theta_r)\sin(\phi_r)\\ \cos\theta_r\end{pmatrix}. \tag{33}$$

It is seen that $\mathbf{\Omega}_r$ is a unit vector pointing in the direction defined by the Euler angles $\Omega_r$. Obviously, the scalar product $\mathbf{A}_r^T\mathbf{\Omega}_r$ is maximized when the two vectors are parallel. Thus, for a given $r$, the corresponding Euler angles $\phi_r$ and $\theta_r$ should be chosen, such that

$$\mathbf{\Omega}_r=\frac{\mathbf{A}_r}{\sqrt{\mathbf{A}_r^T\mathbf{A}_r}}. \tag{34}$$

Of course, the value of the third Euler angle $\zeta_r$ may be arbitrary, as it introduces only an irrelevant overall phase factor. If the vector $\mathbf{A}_r$ is zero, then the guess has to be made completely randomly. Thus, the maximum value of the trace for a specific $r$ in Eq. (30) is given by $\sqrt{\mathbf{A}_r^T\mathbf{A}_r}$, and the maximum value of the estimation fidelity attainable for a given operation $\{\hat{A}_r\}$ is defined by

$$G=\frac{1}{2}+\frac{1}{2(j+1)(2j+1)}\sum_r\sqrt{[\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^x)]^2+[\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^y)]^2+[\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^z)]^2}. \tag{35}$$

As in the case of the operation fidelity, this expression is invariant with respect to an arbitrary transformation of the operators $\hat{A}_r$ according to the rotation group.

## IV. DECOMPOSITION

Let us now summarize the constrained optimization problem describing the trade-off between the operation fidelity $F$ and the estimation fidelity $G$. These two quantities may be written as

$$F=\frac{1}{2}+\frac{1}{2j(j+1)}f \tag{36}$$

and

$$G=\frac{1}{2}+\frac{1}{2(j+1)}g, \tag{37}$$

where the terms $f$ and $g$ depending explicitly on the operation $\{\hat{A}_r\}$ are given by

$$f=\frac{1}{2j+1}\sum_r\mathrm{Tr}(\hat{J}^x\hat{A}_r\hat{J}_j^x\hat{A}_r^\dagger+\hat{J}^y\hat{A}_r\hat{J}_j^y\hat{A}_r^\dagger+\hat{J}^z\hat{A}_r\hat{J}_j^z\hat{A}_r^\dagger) \tag{38}$$

and

$$g=\frac{1}{2j+1}\sum_r\sqrt{[\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^x)]^2+[\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^y)]^2+[\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^z)]^2}. \tag{39}$$

It is worthwhile to look first at the extreme cases. The maximum value of the operation fidelity itself is of course $F=1$. This corresponds to $f=j(j+1)$. This limit is achieved by the identity operation, for which it is easy to check that $g=0$. Hence, we may assume in the following that $g$ is a nonnegative quantity. The other extreme case is the optimization of the estimation fidelity alone, which has been studied previously [3]. According to these results, in the limit of optimal quantum estimation we obtain $g=j$, which sets the upper bound on the region of interest for $g$. We will demonstrate in Sec. VII that the maximum value of $f$ attainable in the case of optimal estimation is equal to $f=j^2$. Expressing this in terms of the fidelities, we get that $F=G$.

The operators $\hat{A}_r$ must satisfy, in general, the completeness condition described in Eq. (6). However, since the initial state of the $N$ qubits lies in the completely symmetric

subspace, we need to consider the action of these operators only on the fully symmetric subspace described by $\mathcal{H}_j$. Consequently, for the purpose of our discussion we shall assume that $\hat{A}_r:\mathcal{H}_j\rightarrow\mathcal{H}^{\otimes N}$, and that the trace-preserving condition takes the form

$$\sum_r \hat{A}_r^\dagger\hat{A}_r=\hat{\mathbb{1}}_j. \tag{40}$$

Henceforth, we replace the trace-preserving condition by the milder requirement, obtained by taking the trace of the above equation

$$\mathrm{Tr}\left(\sum_r \hat{A}_r^\dagger\hat{A}_r\right)=2j+1. \tag{41}$$

We will perform the optimization under this weakened constraint and then show that the optimal $(f,g)$ curve may be attained by a set of $\hat{A}_r$ that is actually trace preserving.

Weakening the completeness condition allows us to introduce an important simplification in further calculations. As we noted in Sec. III A, the expression for the operation fidelity is invariant with respect to rotations performed on the operators $\hat{A}_r$. This is also the case of the weaker trace-preserving condition described in Eq. (41). Consequently, we may always modify each of the operators $\hat{A}_r$ by an operation of the form $\hat{A}_r\rightarrow\hat{U}(\Omega)\hat{A}_r\hat{U}^\dagger(\Omega)$, such that the vector defined in Eq. (32) is aligned along the $z$ axis, and moreover, its $z$ component is nonnegative. In explicit terms, we assume that $\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}\hat{J}_j^x)=\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}\hat{J}_j^y)=0$, and $\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}\hat{J}_j^z)\geqslant 0$. This allows us to replace the square root appearing in Eq. (39) by a much simpler expression $\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r\hat{J}_j^z)$. An important advantage of this step is that the latter expression is quadratic in the matrix elements of the operators $\hat{A}_r$.

Our next step will be the representation of $f$ and $g$ as linear combinations

$$f=\sum_r \lambda_r f(\hat{A}_r),$$

$$g=\sum_r \lambda_r g(\hat{A}_r), \tag{42}$$

where

$$f(\hat{A})=\frac{1}{\mathrm{Tr}(\hat{A}^\dagger\hat{A})}\mathrm{Tr}(\hat{J}^x\hat{A}\hat{J}_j^x\hat{A}^\dagger+\hat{J}^y\hat{A}\hat{J}_j^y\hat{A}^\dagger+\hat{J}^z\hat{A}\hat{J}_j^z\hat{A}^\dagger), \tag{43a}$$

$$g(\hat{A})=\frac{1}{\mathrm{Tr}(\hat{A}^\dagger\hat{A})}\mathrm{Tr}(\hat{A}^\dagger\hat{A}\hat{J}_j^z), \tag{43b}$$

and the nonnegative coefficients $\lambda_r$ are given by

$$\lambda_r=\frac{\mathrm{Tr}(\hat{A}_r^\dagger\hat{A}_r)}{2j+1}. \tag{44}$$

The values of $(f(\hat{A}_r),g(\hat{A}_r))$ are insensitive to the rescaling of $\hat{A}_r$, so $\lambda_r$ are free variables up to the constraint

$$\sum_r \lambda_r=1, \tag{45}$$

resulting from the weaker trace preserving condition. Hence, each $(f,g)$ point is a *convex* combination of independent $(f(\hat{A}_r),g(\hat{A}_r))$. As we are interested in the boundary of allowed $(f,g)$ points, it suffices to examine the case when the $\lambda_r$ are all zero except for one value of $r$. This means that our problem is solved by using only one operation element, which we denote by $\hat{A}$, obeying the constraint $\mathrm{Tr}(\hat{A}^\dagger\hat{A})=j+1$. This observation has also been used in quantum rate-distortion theory [18]. After deriving the bound for $(f,g)$ based on a single element $\hat{A}$, we will demonstrate in Sec. VIII that the operator $\hat{A}$ may be used in a canonical way to construct a quantum operation satisfying the original full completeness condition.

## V. FULLY SYMMETRIC SUBSPACE

We will now show that the optimization problem may be simplified even further: namely, that it is sufficient to consider operators $\hat{A}$ that do not transfer the state of $N$ qubits beyond the fully symmetric subspace.

According to our discussion of the structure of the Hilbert space of the whole ensemble, the operator $\hat{A}:\mathcal{H}_j\rightarrow\mathcal{H}^{\otimes N}$ may be viewed as consisting of entries $\hat{A}_\alpha$, each entry acting from $\hat{A}_\alpha:\mathcal{H}_j\rightarrow\mathcal{H}_{j_\alpha}$. As the angular momentum operators are block diagonal and they do not mix subspaces with different $\alpha$'s, we may introduce the following decomposition of the quantities $f$ and $g$:

$$f=\sum_{\alpha=1}^{\binom{2j}{\lfloor j\rfloor}} \lambda_\alpha f_{j_\alpha}(\hat{A}_\alpha) \tag{46}$$

and

$$g=\sum_{\alpha=1}^{\binom{2j}{\lfloor j\rfloor}} \lambda_\alpha g_{j_\alpha}(\hat{A}_\alpha), \tag{47}$$

where the positive coefficients $\lambda_\alpha$ are given by

$$\lambda_\alpha=\mathrm{Tr}(\hat{A}_\alpha^\dagger\hat{A}_\alpha), \tag{48}$$

and the functions $f_{j'}(\hat{B})$ and $g_{j'}(\hat{B})$ are defined as

$$f_{j'}(\hat{B})=\frac{1}{\mathrm{Tr}(\hat{B}^\dagger\hat{B})}\mathrm{Tr}(\hat{J}_{j'}^x\hat{B}\hat{J}_j^x\hat{B}^\dagger+\hat{J}_{j'}^y\hat{B}\hat{J}_j^y\hat{B}^\dagger+\hat{J}_{j'}^z\hat{B}\hat{J}_j^z\hat{B}^\dagger),$$

$$g_{j'}(\hat{B})=\frac{1}{\mathrm{Tr}(\hat{B}^\dagger\hat{B})}\mathrm{Tr}(\hat{B}^\dagger\hat{B}\hat{J}_j^z). \tag{49}$$

Here, $\hat{J}^x_{j'}$, $\hat{J}^y_{j'}$, and $\hat{J}^z_{j'}$ denote the angular momentum operators truncated to the subspace $\mathcal{H}_{j'}$. As before, the values of $(f_{j_\alpha}(\hat{A}_\alpha), g_{j_\alpha}(\hat{A}_\alpha))$ are insensitive to the rescaling of $\hat{A}_\alpha$ by a multiplicative factor, so consequently $\lambda_\alpha$ are free variables up to the constraint

$$\sum_{\alpha=1}^{\binom{2j}{\lfloor j \rfloor}} \lambda_\alpha = 1. \tag{50}$$

Hence, again, each $(f,g)$ point is a *convex* combination of independent $(f_{j_\alpha}(\hat{A}_\alpha), g_{j_\alpha}(\hat{A}_\alpha))$. Consequently, it is sufficient to examine the much simpler case when all the $\lambda_\alpha$ are all zero except for one value of $\alpha$. This means that out problem reduces to finding the upper boundaries of individual $(f_{j'}, g_{j'})$ regions, where $j' \in j, j-1, \ldots, j-\lfloor j \rfloor$.

Henceforth, we drop the index $\alpha$ and consider a single operator $\hat{A}_{j'}$ mapping the fully symmetric subspace $\mathcal{H}_j$ onto a certain subspace $\mathcal{H}_{j'}$ with the total angular momentum equal to $j'$. We may also assume with no loss of generality, that the operator $\hat{A}_{j'}$ is normalized in such a way that

$$1 = \mathrm{Tr}(\hat{A}^\dagger_{j'} \hat{A}_{j'}) = \sum_{m'=-j'}^{j'} \sum_{m=-j}^{j} |\langle j';m'|\hat{A}_{j'}|j;m\rangle|^2. \tag{51}$$

The explicit expressions for the functions $f_{j'}(\hat{A}_{j'})$ and $g_{j'}(\hat{A}_{j'})$ then take the following form:

$$f_{j'}(\hat{A}_{j'}) = \sum_{m'=-j'}^{j'} \sum_{m=-j}^{j} m' m |\langle j';m'|\hat{A}_{j'}|j;m\rangle|^2 + \sum_{m'=-j'}^{j'-1} \sum_{m=-j}^{j-1} \sqrt{(j'+m'+1)(j'-m')(j+m+1)(j-m)} \times \mathrm{Re}[\langle j';m'|\hat{A}_{j'}|j;m\rangle (\langle j';m'+1|\hat{A}_{j'}|j;m+1\rangle)^*] \tag{52}$$

and

$$g_{j'}(\hat{A}_{j'}) = \sum_{m'=-j'}^{j'} \sum_{m=-j}^{j} m |\langle j';m'|\hat{A}_{j'}|j;m\rangle|^2. \tag{53}$$

As the next step to simplify the problem, we note that the phases of the matrix elements $\langle j';m'|\hat{A}_{j'}|j;m\rangle$ may be set to make all of them real and nonnegative. This maximizes the second term in Eq. (52) while leaving unchanged the expressions for $g_{j'}(\hat{A}_{j'})$ and $\mathrm{Tr}(\hat{A}^\dagger_{j'} \hat{A}_{j'})$.

We will now demonstrate that among all the curves bounding the allowed regions of $(f_{j'}, g_{j'})$, the curve for $j' = j$ encompasses the largest region in the $(f,g)$ plane, which includes all other bounds obtained for $j' < j$. For this purpose, we will show that given an arbitrary operator $\hat{A}_{j'}: \mathcal{H}_j \to \mathcal{H}_{j'}$ satisfying the condition $g_{j'}(\hat{A}_{j'}) \geqslant 0$, it is possible to construct an operator $\hat{A}': \mathcal{H}_j \to \mathcal{H}_j$ mapping the fully symmetric space, such that

$$f_j(\hat{A}') \geqslant f_{j'}(\hat{A}_{j'}), \tag{54a}$$

$$g_j(\hat{A}') = g_{j'}(\hat{A}_{j'}), \tag{54b}$$

$$\mathrm{Tr}(\hat{A}'^\dagger \hat{A}') = \mathrm{Tr}(\hat{A}^\dagger_{j'} \hat{A}_{j'}). \tag{54c}$$

Hence, the operator $\hat{A}'$ will be always more optimal that the original operator $\hat{A}_{j'}$.

The explicit construction of the operator $\hat{A}'$ is given by

$$\langle j;n|\hat{A}'|j;m\rangle = \begin{cases} \langle j';n-j+j'|\hat{A}_{j'}|j;m\rangle, & \text{if } n \geqslant j-2j' \\ 0, & \text{if } n < j-2j'. \end{cases} \tag{55}$$

It is straightforward to check that the operator $\hat{A}'$ defined above automatically satisfies conditions given by Eqs. (54b) and (54c). In order to prove that condition (54a) is also satisfied, let us express $f_j(\hat{A}')$ in terms of the matrix elements of the operator $\langle j';m'|\hat{A}_{j'}|j;m\rangle$

$$f_j(\hat{A}') = \sum_{m'=-j'}^{j'} \sum_{m=-j}^{j} (m'+j-j') m |\langle j';m'|\hat{A}_{j'}|j;m\rangle|^2 + \sum_{m'=-j'}^{j'-1} \sum_{m=-j}^{j-1} \sqrt{(2j-j'+m'+1)(j'-m')(j+m+1)(j-m)} \times \mathrm{Re}[\langle j';m'|\hat{A}_{j'}|j;m\rangle (\langle j';m'+1|\hat{A}_{j'}|j;m+1\rangle)^*]. \tag{56}$$

The second term of the above formula majorizes the second term of Eq. (52), since for $j>j'$ we have

$$\sqrt{2j-j'+m'+1}\geqslant\sqrt{j'+m'+1}, \tag{57}$$

and all the other factors have been assumed to be nonnegative. This observation may be combined with the decomposition of the first term in Eq. (56) into two parts proportional to $m'$ and $j-j'$, which yields

$$f_j(\hat{A}')\geqslant f_{j'}(\hat{A}_{j'})+(j-j')g(\hat{A}_{j'}). \tag{58}$$

Since $j>j'$ and we have assumed that $g(\hat{A}_{j'})\geqslant 0$, this proves that Eq. (54a) is indeed satisfied. Of course, the condition $g(\hat{A}_{j'})\geqslant 0$ is fulfilled automatically for all the operators relevant to the trade-off, as according to our discussion from Sec. IV, the whole region of interest for $g$ is $0\leqslant g\leqslant j$.

## VI. OPTIMIZATION

We will now show that the search for the trade-off curve may be decomposed into a set of even simpler independent, constrained optimization problems. To proceed further, it will be convenient to introduce vector notation. Let us define

$$l_k=-j+\max(0,k),$$

$$u_k=j+\min(k,0), \tag{59}$$

where the index $k$ is from the range $-2j\leqslant k\leqslant 2j$. For brevity, we also denote

$$a_m^k=\langle j;m-k|\hat{A}_j|j;m\rangle, \tag{60}$$

where $l_k\leqslant m\leqslant u_k$, and we assume that all the matrix elements are real and nonnegative. We can now introduce $4j+1$ real vectors

$$\mathbf{a}_k=(a_{l_k}^k,a_{l_k+1}^k,\ldots,a_{u_k}^k). \tag{61}$$

The length of the vector with the index $k$ is equal to $2j+1-|k|$. These vectors are diagonal stripes of the matrix $\langle j;m|\hat{A}_j|j;n\rangle$. Using the vector notation, we have

$$f=\sum_{k=-2j}^{2j} f^k(\mathbf{a}^k), \tag{62a}$$

$$g=\sum_{k=-2j}^{2j} g^k(\mathbf{a}^k), \tag{62b}$$

$$1=\sum_{k=-2j}^{2j} h^k(\mathbf{a}^k), \tag{62c}$$

with

$$f^k(\mathbf{a}^k)=\sum_{m=l_k}^{u_k} m(m-k)(a_m^k)^2+\sum_{m=l_k}^{u_k-1}\gamma_m^k a_m^k a_{m+1}^k, \tag{63a}$$

$$g^k(\mathbf{a}^k)=\sum_{m=l_k}^{u_k} m(a_m^k)^2, \tag{63b}$$

$$h^k(\mathbf{a}^k)=\sum_{m=l_k}^{u_k} (a_m^k)^2, \tag{63c}$$

and

$$\gamma_m^k=\sqrt{(j-m)(j+m+1)(j+k-m)(j-k+m+1)}. \tag{64}$$

We may now use the same reasoning as before to restrict our interest to a single vector $\mathbf{a}^k$ with a fixed value of $k$. This vector should be normalized to unity, $(\mathbf{a}^k)^T\mathbf{a}^k=1$. The allowed region for $f$ and $g$ is defined as a union of regions bounded by curves obtained for different $k$'s, with the index $k$ running from $-2j$ to $2j$. We will now discuss several properties of the curves depending on the sign of $k$, which will allow us to restrict our search for the optimality curve to a smaller set of $k$'s.

### A. Case $k<0$

For negative $k$, we note that given an arbitrary vector $\mathbf{a}^k$, one may use its elements in the same order to construct a vector $\mathbf{a}^{-k}$ simply by taking $\mathbf{a}^{-k}=\mathbf{a}^k$. A simple calculation shows that

$$f^{-k}(\mathbf{a}^{-k})=f^k(\mathbf{a}^k),$$

$$g^{-k}(\mathbf{a}^{-k})=g^k(\mathbf{a}^k)-k. \tag{65}$$

Thus, one may obtain from any trade-off curve for $k<0$ a certain trade-off curve for $-k>0$, which is shifted along the $g$ axis towards higher values. All the trade-off curves for $k<0$ are, hence, suboptimal, and we may further restrict our attention only to the case of $k\geqslant 0$.

### B. Case $k\geqslant 0$

We will now show that the trade-off curves obtained for $k$'s greater or equal to $\sqrt{2j}$ lie completely within the region bounded by the curve corresponding to $k=0$. This will allow us to exclude all $k\geqslant\sqrt{2j}$ from further analysis. In order to prove the above lemma, we will demonstrate that the maximum value of $f$ attained by the trade-off curves for $k\geqslant\sqrt{2j}$ lies below the minimum value of $f$ on the trade-off curve obtained for $k=0$.

We start from the observation that the complete trade-off curve for $k=0$ lies above the value $f=j^2$. Indeed, let us define the vector $\mathbf{a}^0=(\sin\chi,0,\ldots,0,\cos\chi)$ with $\chi$ from the range 0 to $\pi/4$. It is straightforward to check that we have $f^0(\mathbf{a}^0)\geqslant j^2$ over this range of $\chi$, whereas $g^0(\mathbf{a}^0)=j\cos 2\chi$, which can assume any value between 0 and $j$. Thus, for any $g$ from the range $0\leqslant g\leqslant j$ relevant to the trade-off curve we have a vector such that the corresponding value of $f$ is larger or equal to $j^2$. Consequently, the complete trade-off curve occupies the region of the $(f,g)$ plane defined by the condition $f\geqslant j^2$.

Next, we prove in Appendix C that for an arbitrary normalized vector $\mathbf{a}^k$, the function $f^k(\mathbf{a}^k)$ is bounded by

$$f^k(\mathbf{a}^k) \leqslant j(j+1) - \frac{k^2}{2}. \tag{66}$$

The right-hand side of the above bound may be compared with the minimum value of $f^0 = j^2$ on the trade-off curve for $k=0$. If for a given $k$ the general upper bound on $f^k$ given by the right-hand side of Eq. (66) is below the value $j^2$, then the trade-off curve obtained for this specific $k$ will definitely be majorized by the trade-off curve corresponding to $k=0$ over the whole region of interest. Hence, we may exclude all $k$'s satisfying $j(j+1) - k^2/2 \leqslant j^2$, which after simplification yields $k \geqslant \sqrt{2j}$. Consequently, we may restrict our attention to nonnegative $k$'s from the range

$$0 \leqslant k < \sqrt{2j}. \tag{67}$$

Recalling that $2j = N$, it is thus sufficient to consider $k$'s from a finite set of only $\lceil \sqrt{N} \rceil$ values. The number of independent real variables that have to be optimized for a given $k$ is equal to $N+1-k$.

## VII. NUMERICAL PROCEDURE

Our task is now reduced to finding the trade-off curves for a set of $k$'s defined in Eq. (67). To complete this task, we shall resort to numerical means. For a given $k$, define $(u_k - l_k + 1) \times (u_k - l_k + 1)$ real symmetric matrices $\mathcal{F}^k$ and $\mathcal{G}^k$: the matrix $\mathcal{F}^k$ has $l_k(l_k-k), (l_k+1)(l_k-k+1), \ldots, u_k(u_k-k)$ on the diagonal, and $\gamma^k_{l_k}/2, \gamma^k_{l_k+1}, \ldots, \gamma^k_{u_r-1}$ on either side of the diagonal. The matrix $\mathcal{G}^k$ has $l_k, l_k+1, \ldots, u_k$ on the diagonal. We may now use the method of Lagrange multipliers to find the maximum of $f$ having fixed the value of $g$. Specifically, we need to optimize the expression

$$f^k(\mathbf{a}^k) + \lambda g^k(\mathbf{a}^k) - \mu h^k(\mathbf{a}^k) = (\mathbf{a}^k)^T (\mathcal{F}^k + \lambda \mathcal{G}^k - \mu \mathbb{1}) \mathbf{a}^k \tag{68}$$

with the constraints

$$g^k(\mathbf{a}^k) = g,$$
$$h^k(\mathbf{a}^k) = 1, \tag{69}$$

and $\lambda, \mu$ being the Lagrange multipliers.

Differentiating the right hand side of Eq. (68) with respect to the elements of the vector $\mathbf{a}^k$ we obtain that the maximum occurs when

$$(\mathcal{F}^k + \lambda \mathcal{G}^k - \mu I) \mathbf{a}^k = \mathbf{0}, \tag{70}$$

that is, $\mathbf{a}^k$ is an eigenvector of the matrix $\mathcal{F}^k + \lambda \mathcal{G}^k$ corresponding to its maximum eigenvalue. Assuming that this eigenvector is normalized to one, the value of $g$ is given by the product $(\mathbf{a}^k)^T \mathcal{G}^k \mathbf{a}^k$, which implicitly depends on $\lambda$ through the vector $\mathbf{a}^k$. In order to plot the trade-off curve as a function $f^k(g^k)$ we would need to invert this relation. However, we may equivalently consider the trade-off curve as parametrized by the Lagrange multiplier $\lambda$ running from 0 to $\infty$.

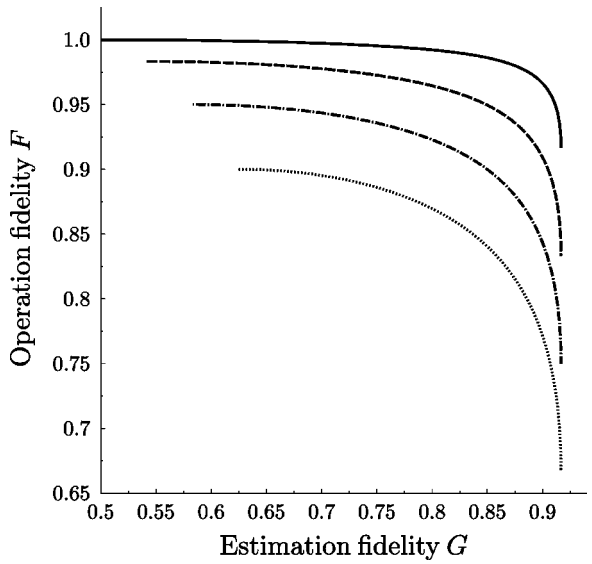


FIG. 2. The trade-off curves for the ensemble of $N=10$ qubits obtained by solving the optimization problem for $k=0$ (solid), $k=1$ (dashed), $k=2$ (dotted dashed), and $k=3$ (dotted). The allowed region is a union of the regions bounded by each of these curves.

The value $\lambda = 0$ corresponds to optimizing $f^k$ without any constraint on $g^k$, whereas in the limit $\lambda \to \infty$, the optimization gives the maximum attainable value of $g^k$ and the corresponding value of $f^k$. Thus, the procedure of calculating the trade-off curve for a given $k$ may be summarized as follows: for $\lambda$'s from the range $0 \leqslant \lambda < \infty$ find the normalized eigenvector of $\mathcal{F}^k + \lambda \mathcal{G}^k$ corresponding to the largest eigenvalue, and use it to calculate $(\mathbf{a}^k)^T \mathcal{F}^k \mathbf{a}^k$ and $(\mathbf{a}^k)^T \mathcal{G}^k \mathbf{a}^k$ which give the point $(f^k(\lambda), g^k(\lambda))$ on the trade-off curve.

We have performed numerically the task of finding the eigenvectors $\mathbf{a}^k$. In numerical calculations, it is convenient to change the parametrization of the trade-off curve according to $\lambda = x/(1-x)$ with $x$ running from 0 to 1, and to diagonalize the rescaled matrix $(1-x)\mathcal{F}^k + x\mathcal{G}^k$. We note that this matrix is of the tridiagonal form, for which there exist efficient numerical algorithms [19]. After finding the normalized eigenvectors, we can plot the parametrized trade-off curve $(f^k(x), g^k(x))$ for all relevant $k$'s from the range given by Eq. (67), and next find on this basis the region of fidelities allowed in quantum mechanics. To illustrate this procedure, we plot in Fig. 2 the trade-off curves for $N=10$ and the relevant range of $k$'s. It is seen that the quantum mechanically allowed region is bounded by a single trade-off curve obtained for $k=0$. This was the case also in all other cases we investigated numerically. The actual bounds on the operation fidelity $F$ versus the estimation fidelity $G$ for several values of $N$ are shown in Fig. 1.

On the basis of our numerical studies, we conjecture that the trade-off curve for $k=0$ is always optimal. This conjecture is supported by the fact that both the extreme points corresponding to $x=0$ and $x=1$ are attained only for $k=0$. Indeed, we have seen in Sec. VI B that for a given nonnegative $k$ the whole trade-off curve lies below $j(j+1) - k^2/2$. Consequently, the maximum value of $f = j(j+1)$ corresponding to unit operation fidelity is achieved only for $k$

$=0$. This is also the case of the other extreme point corresponding to the optimal estimation: it is straightforward to note that for any $k \geqslant 0$, the expression for $g^k$ has the same maximum value equal to $g=j$. This value is obtained for the unique vector of the form $\mathbf{a}^k=(0,0,\ldots,1)$. The corresponding value of $f^k$ is $f^k=j(j-k)$. Thus, for optimized $g$, the largest attainable value of $f$, equal to $j^2$, is obtained only for $k=0$.

## VIII. ATTAINABILITY OF THE BOUND

We will now show that the trade-off curves computed in the previous section are tight, i.e., that they may be attained by physically realizable operations. So far, we have considered only the trade-off curve generated by a single operator $\hat{A}:\mathcal{H}_j \rightarrow \mathcal{H}^{\otimes N}$ satisfying the condition $\mathrm{Tr}(\hat{A}^\dagger \hat{A})=1$. The critical step that allowed us to focus on a single operator was the replacement of the full trace-preserving condition in Eq. (40) by its trace. We will now present a method for constructing a complete quantum operation from a single operator $\hat{A}$, such that it generates the same point on the fidelity trade-off curve.

The classical outcome of the operation we construct has the continuous form of a triplet of Euler angles that we will denote by $\Xi$. The operation element corresponding to a specific outcome $\Xi$ is given by

$$\hat{A}_\Xi = \sqrt{2j+1}\,\hat{U}(\Xi)\hat{A}\hat{U}^\dagger(\Xi). \tag{71}$$

It is straightforward to verify that the operation fidelity $F$ and the estimation fidelity $G$ for this operation are given, respectively, by

$$F = \frac{1}{2} + \frac{1}{2j(j+1)} f(\hat{A}) \tag{72}$$

and

$$G = \frac{1}{2} + \frac{1}{2(j+1)} g(\hat{A}), \tag{73}$$

where $f(\hat{A})$ and $g(\hat{A})$ are defined in Eqs. (43a) and (43b). This confirms that we obtain the same point on the trade-off curve as for the operator $\hat{A}$ itself. The only condition we need to check is the completeness of the operation on the fully symmetric subspace $\mathcal{H}_j$

$$\int d\Xi\, \hat{A}_\Xi^\dagger \hat{A}_\Xi = \hat{\mathbb{1}}_j. \tag{74}$$

Of course, outside $\mathcal{H}_j$, the value of this integral vanishes, as the operator $\hat{A}$ is assumed to be zero there and the rotation matrices $\hat{U}(\Xi)$ do not mix subspaces with different values of the angular momentum.

In order to prove that the completeness condition given by Eq. (74) is indeed satisfied, let us consider the matrix element of the left-hand side of the above expression in the eigenbasis of the angular momentum operator $\hat{J}_j^z$

$$\begin{aligned}\int d\Xi \langle j;m|\hat{A}_\Xi^\dagger \hat{A}_\Xi|j;n\rangle &= (2j+1)\sum_{m',n'=-j}^{j} \int d\Xi \langle j;m|\hat{U}(\Xi)|j;m'\rangle \\ &\times \langle j;m'|\hat{A}^\dagger\hat{A}|j;n'\rangle\langle j;n'|\hat{U}^\dagger(\Xi)|j;n\rangle. \end{aligned}\tag{75}$$

The integral over the product of the elements of rotation matrices $\langle j;m|\hat{U}(\Xi)|j;m'\rangle = \mathcal{D}^j_{mm'}(\Xi)$ may be performed explicitly using the standard formula

$$\int d\Xi \mathcal{D}^j_{mm'}(\Xi)[\mathcal{D}^j_{nn'}(\Xi)]^* = \frac{\delta_{mn}\delta_{m'n'}}{2j+1}. \tag{76}$$

With the help of the above identity, we have

$$\begin{aligned}\int d\Xi \langle j;m|\hat{A}_\Xi^\dagger \hat{A}_\Xi|j;n\rangle &= (2j+1)\sum_{m',n'=-j}^{j} \int d\Xi\, \mathcal{D}^j_{mm'}(\Xi) \\ &\times [\mathcal{D}^j_{nn'}(\Xi)]^* \langle j;m'|\hat{A}^\dagger\hat{A}|j;n'\rangle \\ &= \sum_{m',n'=-j}^{j} \delta_{mn}\delta_{m'n'}\langle j;m'|\hat{A}^\dagger\hat{A}|j;n'\rangle \\ &= \delta_{mn}\,\mathrm{Tr}(\hat{A}^\dagger\hat{A}) = \delta_{mn}. \end{aligned}\tag{77}$$

This completes the proof that the operation $\{\hat{A}_\Xi\}$ satisfies the full trace-preserving condition in the symmetric subspace of the $N$-qubit Hilbert space. Consequently, the trade-off curve calculated for a single operator $\hat{A}$ is attained also by complete quantum operations.

## IX. DISCUSSION

In this paper, we calculated the fidelity trade-off for finite ensembles of identically prepared qubits. This trade-off relates the quality of estimating the quantum state of the qubits to the minimum disturbance of the original that has to be introduced in course of this procedure. The obtained trade-off curves may also be viewed as a characterization of a specific asymmetric quantum cloning procedure, which given $N$ qubits produces the same number of clones with a decreased fidelity $F$, and additionally an arbitrarily large number of clones with a lower fidelity $G$.

The calculation of the trade-off curve was based on a combination of analytical techniques and numerical calculations. The results obtained analytically allowed us to reduce significantly the complexity of the optimization problem. One should note that a single operator $\hat{A}:\mathcal{H}_j \rightarrow \mathcal{H}^{\otimes N}$ mapping the fully symmetric subspace onto the whole Hilbert space of $N$ qubits contains $2^{N+1}(N+1)$ independent real variables. If one wanted to find numerically the trade-off curve assuming such general form of the operator $\hat{A}$, the number of parameters in the optimization problem would

explode exponentially with the size of the ensemble. Fortunately, we were able to demonstrate that the problem of finding the trade-off curve can be reduced to $\lceil\sqrt{N}\rceil$ independent optimization problems, each involving only no more than $N+1$ real parameters. This is a substantial reduction of the problem, which enables one to handle numerically much larger ensembles of qubits.

There are several elements of our paper that could be investigated further. First, it would be interesting to prove our conjecture that the trade-off curve obtained for $k=0$ is always optimal. This would reduce further the complexity of the problem remaining to solve numerically. We have made several observations that might be helpful in this proof. First, numerical calculations suggest that the eigenvalues of the matrix $\mathcal{F}^k$ considered in Sec. VII belong to the analytically defined set $\{-\nu(\nu-1)/2+2j\nu-j^2 | \nu=0,1,\ldots,u_k-l_k\}$. The largest of these eigenvalues is $j(j+1)-k(k+1)/2$, which itself improves the upper bound given in Eq. (66). Inspection of numerical results suggests also that the value of $g^k(\mathbf{a}^k)$ corresponding to maximized $f^k(\mathbf{a}^k)$ is equal to $k/2$. Thus, both the extreme points of all trade-off curves for $k>0$ lie beneath the one obtained for $k=0$. This observation, combined with a demonstration that the curves have appropriate monotonicity and convexity properties, might prove the universally optimal character of the $k=0$ curve.

Another interesting direction is investigating in more detail quantum operations that saturate the trade-off inequality. We have shown that given a single operator that generates the values of $F$ and $G$ lying on the trade-off curve, one may construct a complete quantum operation that satisfies the full trace-preserving condition. The described operation had a continuous classical outcome in the form of a triplet of Euler angles. It would be interesting to investigate operations with a finite [7] (and possibly minimal [8]) number of outcomes that also saturate the quantum mechanical bound on the fidelities.

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## APPENDIX A: MULTIPLICITIES OF ANGULAR MOMENTUM REPRESENTATIONS

In this Appendix, we derive the multiplicities $\mu_{j'}$ of subspaces with the fixed value of the angular momentum $j'$ appearing in the decomposition of the Hilbert space of $N$ qubits. Let us consider the operator

$$\hat{Z}(\beta)=\bigotimes_{i=1}^{N}\exp(\beta\hat{\sigma}_i^z)=\exp(2\beta\hat{J}^z), \tag{A1}$$

where the subscript $i$ enumerates the qubits. From the tensor-product representation given on the left-hand side of the above formula we immediately obtain that

$$\mathrm{Tr}[\hat{Z}(\beta)]=(2\cosh\beta)^N. \tag{A2}$$

On the other hand, summation of the trace of the operator $\exp(2\beta\hat{J}^z)$ in all the subspaces yields

$$\begin{aligned}\mathrm{Tr}[\hat{Z}(\beta)]&=\sum_{j'=j-\lfloor j\rfloor}^{j}\mu_{j'}\sum_{m=-j'}^{j'}e^{2m\beta}\\&=\sum_{j'=j-\lfloor j\rfloor}^{j}\mu_{j'}\frac{\sinh\beta(2j'+1)}{\sinh\beta}.\end{aligned} \tag{A3}$$

Comparing equal powers of $e^{\beta}$ in this expression with the expansion of the left-hand side of Eq. (A2) given by $(2\cosh\beta)^N$ yields

$$\mu_{j'}=\binom{2j}{2+j'}-\binom{2j}{j+j'+1}=\frac{2j'+1}{2j+1}\binom{2j+1}{j-j'}. \tag{A4}$$

It is seen that the total number of subspaces in the decomposition (10) is $\Sigma_{j'=j-\lfloor j\rfloor}^{j}\mu_{j'}=\binom{2j}{\lfloor j\rfloor}$.

## APPENDIX B: EVALUATION OF INTEGRALS $\hat{K}_\tau$

In this appendix, we calculate explicitly the integrals $\hat{K}_\tau$ defined in Eq. (17). For this purpose, it is convenient to switch to the angular momentum representation resulting from the decomposition of the tensor product Hilbert space of the $N$ qubits into the direct sum of subspaces with the fixed value of the total angular momentum operator. In this representation, the state $|\Omega\rangle^{\otimes N}$ lies in the completely symmetric subspace characterized by the angular momentum $j=N/2$ and it is given by

$$|\Omega\rangle^{\otimes N}=\hat{U}(\Omega)|j;j\rangle, \tag{B1}$$

where $|j;j\rangle$ is the eigenvector of $\hat{J}^z$ corresponding to the eigenvalue $j$, and $\hat{U}(\Omega)$ is the rotation matrix for the angular momentum $j$. Of course, rotations cannot transfer the state $|j;j\rangle$ beyond the fully symmetric subspace. Consequently, the operators $\hat{K}_\tau$ are nonzero only in this subspace. In the basis of the eigenvectors of the operator $\hat{J}^z$, the matrix elements of these operators are given by

$$\begin{aligned}\langle j;m|\hat{K}_\tau|j;n\rangle=\int d\Omega k_\tau(\Omega)\langle j;m|\hat{U}(\Omega)|j;j\rangle\\\times\langle j;j|\hat{U}^\dagger(\Omega)|j;n\rangle.\end{aligned} \tag{B2}$$

We shall use the standard notation from Ref. [15] to denote the matrix elements of the unitary rotation operators appearing in the above formula

$$\begin{aligned}\langle j;m|\hat{U}(\Omega)|j;j\rangle&=\mathcal{D}^j_{mj}(\Omega),\\\langle j;j|\hat{U}^\dagger(\Omega)|j;n\rangle&=[\mathcal{D}^j_{nj}(\Omega)]^*=(-1)^{n-j}\mathcal{D}^j_{-n-j}(\Omega),\end{aligned} \tag{B3}$$

where in the second line, we have made use of the symmetry properties of the rotation matrix elements.

The functions $k_\tau(\Omega)$ may also be expressed as elements of rotation matrices for the value of the total angular momentum equal to one:

$$k_{-1}(\Omega)=\sqrt{2}\mathcal{D}^1_{-10}(\Omega),$$
$$k_0(\Omega)=\mathcal{D}^1_{00}(\Omega), \tag{B4}$$
$$k_1(\Omega)=-\sqrt{2}\mathcal{D}^1_{10}(\Omega).$$

With this notation, we may use the standard expression for the integrals of triple products of rotation matrix elements in terms of the Wigner 3-$j$ symbols

$$\langle j;m|\hat{K}_{\pm1}|j;n\rangle = \mp\sqrt{2}(-1)^{n-j}\int d\,\Omega\mathcal{D}^j_{mj}(\Omega)\mathcal{D}^j_{-n-j}(\Omega)\mathcal{D}^1_{\pm10}(\Omega)$$
$$=\sqrt{2}(-1)^{n-j}\begin{pmatrix} j & j & 1\\ m & -n & \pm1\end{pmatrix}\begin{pmatrix} j & j & 1\\ j & -j & 0\end{pmatrix} \tag{B5}$$

and

$$\langle j;m|\hat{K}_0|j;n\rangle=(-1)^{n-j}\int d\,\Omega\mathcal{D}^j_{mj}(\Omega)\mathcal{D}^j_{-n-j}(\Omega)\mathcal{D}^1_{00}(\Omega)$$
$$=(-1)^{n-j}\begin{pmatrix} j & j & 1\\ m & -n & 0\end{pmatrix}\begin{pmatrix} j & j & 1\\ j & -j & 0\end{pmatrix}. \tag{B6}$$

Inserting the explicit form of the 3-$j$ symbols yields

$$\langle j;m|\hat{K}_{\pm1}|j;n\rangle=\delta_{m\pm1,n}\frac{\sqrt{(j\pm n)(j\mp n+1)}}{(j+1)(2j+1)} \tag{B7}$$

and

$$\langle j;m|\hat{K}_0|j;n\rangle=\delta_{mn}\frac{m}{(j+1)(2j+1)}. \tag{B8}$$

The expressions appearing on the right-hand sides of the above formulas are proportional to the operators $\hat{J}^\mp$ and $\hat{J}^z$ restricted to the completely symmetric subspace with the total angular momentum $j$. Thus, the integrals $\hat{K}_\tau$ may be conveniently written as

$$\hat{K}_{\pm1}=\frac{1}{(j+1)(2j+1)}\hat{J}^\mp_j \tag{B9}$$

and

$$\hat{K}_0=\frac{1}{(j+1)(2j+1)}\hat{J}^z_j, \tag{B10}$$

where by the subscript $j$ we have denoted the angular momentum operators truncated to the subspace characterized by the angular momentum $j$.

## APPENDIX C: DERIVATION OF THE BOUND ON $F^k(\mathbf{a}^K)$

Here, we derive an upper bound on the functions $f^k(\mathbf{a}^k)$, defined in Eq. (63a), for $k\geqslant0$. For this purpose, let us rewrite $\gamma^k_m$ defined in Eq. (64) to the form

$$\gamma^k_m=\sqrt{\left[\left(j-m+\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)^2\right]\left[\left(j+m+1-\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)^2\right]}. \tag{C1}$$

Using the inequality

$$\sqrt{(x_1^2-x_2^2)(x_3^2-x_4^2)}\leqslant x_1x_3-x_2x_4, \tag{C2}$$

valid for any real $x_1\geqslant x_2\geqslant0$ and $x_3\geqslant x_4\geqslant0$, we obtain that

$$\gamma^k_m\leqslant\delta^k_m, \tag{C3}$$

where

$$\delta^k_m=j(j+1)-\left(m-\frac{k}{2}\right)\left(m-\frac{k}{2}+1\right)-\left(\frac{k}{2}\right)^2. \tag{C4}$$

We will also use the inequality between the geometric and arithmetic means to estimate

$$a^k_m a^k_{m+1}\leqslant\frac{1}{2}[(a^k_m)^2+(a^k_{m+1})^2]. \tag{C5}$$

We may now use the inequalities given in Eqs. (C3) and (C5) to find an upper bound on the second term in Eq. (63a). A simple rearrangement of the terms yields

$$f^k(\mathbf{a}^k)\leqslant\sum_{m=l_k}^{u_k}m(m-k)(a^k_m)^2+\frac{1}{2}\sum_{m=l_k}^{u_k-1}\delta^k_m[(a^k_m)^2+(a^k_{m+1})^2]$$
$$=\sum_{m=l_k}^{u_k}m(m-k)(a^k_m)^2+\frac{1}{2}\delta^k_{l_k}(a^k_{l_k})^2+\frac{1}{2}\delta^k_{u_k-1}(a^k_{u_k})^2$$
$$+\sum_{m=l_k+1}^{u_k-1}\frac{1}{2}(\delta^k_{m-1}+\delta^k_m)(a^k_m)^2. \tag{C6}$$

We may now add to the right-hand side two terms of the form $\delta^k_{l_k-1}(a^k_{l_k})^2/2$ and $\delta^k_{u_k}(a^k_{u_k})^2/2$, which are nonnegative as can be easily checked, and use the identity

$$\frac{1}{2}(\delta^k_{m-1}+\delta^k_m)=j(j+1)-m(m-k)-\frac{k^2}{2}. \tag{C7}$$

This finally yields

$$f^k(\mathbf{a}^k)\leqslant\sum_{m=l_k}^{u_k}m(m-k)(a^k_m)^2$$
$$+\sum_{m=l_k}^{u_k}\left(j(j+1)-m(m-k)-\frac{k^2}{2}\right)(a^k_m)^2$$
$$=\left(j(j+1)-\frac{k^2}{2}\right)\sum_{m=l_k}^{u_k}(a^k_m)^2. \tag{C8}$$

If the vector $\mathbf{a}^k$ is normalized to unity, i.e., $\Sigma^{u_k}_{m=l_k}(a^k_m)^2=1$, we obtain the bound given in Eq. (66).

We note that for $k=0$ the derived bound is tight as it can be achieved for $\mathbf{a}^{k=0}=(1/\sqrt{2j+1},\ldots,1/\sqrt{2j+1})$.

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