Quantum systems subject to the action of classical stochastic fields

Adrián A. Budini*
Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Cep 21945-970, Rio de Janeiro, Brazil
(Received 2 April 2001; published 17 October 2001)

In this paper we investigate the averaged dynamics of quantum systems under the influence of classical stochastic fields. This influence is modeled using a stochastic Hamiltonian evolution for the system-density matrix. From the averaged dynamics, a general characterization of the short-time decoherence behavior is obtained. General applicable short-time perturbation expansions for the input-output fidelity and its generalizations for mixed states are developed. The master equations of the systems that can be worked out without a perturbative expansion, i.e., any system subject to dispersive noise and the quantum harmonic oscillator subject to amplitude noise, are extensively analyzed. In both cases all non-Markovian features are worked out in an exact way. We apply these cases to the study of heating of trapped ions.

DOI: 10.1103/PhysRevA.64.052110 PACS number(s): 03.65.Ta, 03.65.Yz, 42.50.Vk, 03.67.−a

I. INTRODUCTION

The study of open quantum systems [1–4] plays an important role in various fields of physics. The central problem in the study of these systems consists in characterizing the effects originated by the interaction with the external world. In general, it is desirable to avoid this perturbative action, which works as an "eraser" of quantum properties of any initial state of the system.

The most natural and unavoidable external influence is due to thermal reservoirs. This subject was largely investigated and characterized for different systems and situations. Another type of influence is due to stochastic fields generated in the experimental device that constitutes and monitors the system of interest.

The notable experimental advances of the last decade have allowed to monitor directly these types of influences in different systems. For example, numerous theoretical and experimental investigations were devoted to quantum electrodynamic cavities [5–8], trapped ions [9–17], laser systems, etc. The high experimental accuracy and sensitivity achieved, has permitted to test different hypotheses and theoretical approaches that describe the environment-system interaction. For this reason, it is highly desirable to obtain exact evolutions and solutions of the different models.

In this paper we develop a systematic study of the influence of classical stochastic fields on quantum systems. This influence will be modeled by the introduction of a stochastic Hamiltonian in the evolution of the system density matrix. Thus, the statistical properties of the system will be obtained from the averaged density matrix

$$\rho(t)=\langle\rho_{st}(t)\rangle, \tag{1}$$

where $\langle\cdots\rangle$ means average over the stochastic realizations of the force fields.

It is well known, that a stochastic field that exchanges energy with a quantum system, induces—in a long-time regime—an averaged dynamics that is equivalent to a reservoir at infinite temperature [18,19], i.e., it continuously feeds energy in the system. Therefore, this model will give correct results in a time regime previous to the time in which the effects of the stochastic fields are saturated by the influence of the thermal environment.

In general, the origin of these stochastic fields is due to many uncorrelated random sources. Therefore, after invoking the central-limit theorem, we will assume that the stochastic fields are Gaussian noises. In this manner, all the statistical field properties can be specified by providing their intensity and their characteristic correlation time. In any real problem, the field noise spectrum is not flat, which implies correlation times different from zero. In consequence, the evolution of $\rho(t)$ will be non-Markovian.

The paper is organized as follows. In Sec. II we define the stochastic dynamics and obtain the exact averaged density matrix evolution. Furthermore, we sketch the procedure to develop perturbative expansions. These points can be obtained from a previous work on non-Markovian dissipative stochastic wave vectors [20]. Nevertheless, a stochastic Hamiltonian dynamics is a much more simple case, therefore, here we reobtain it in order to clarify the later calculations. In Sec. III, we study the short-time decoherence dynamics induced by the noises. As a measure of the decoherence processes, we use the input-output fidelity and its generalizations for mixed states. General applicable short-time perturbation expansions are developed. In Sec. IV we present the cases whose averaged evolutions can be obtained in an exact way, i.e., any system subject to dispersive noise and the quantum harmonic oscillator subject to amplitude noise. Special attention is paid to the latter case. The exact time-convolutionless evolution for the density matrix and a closed expression for the Wigner function are obtained. In Sec. V we apply the results of the previous section to the study of heating of trapped ions. The short- and long-time regimes are studied. The population's behavior for different initial conditions and the problem of noise-induced decoherence are worked out. The case of an ion subject to dispersive noise is briefly analyzed. In Sec. VI we give the conclusions.

II. DENSITY-MATRIX DESCRIPTION

In this section we define the stochastic dynamics and obtain the master equation for the averaged-density matrix.

*Email address: adrian@if.ufrj.br

A. Stochastic evolution

A stochastic Hamiltonian dynamics is characterized by the evolution

$$i\hbar \frac{d}{dt}\rho_{st}(t)=[H+\lambda\tilde{H}(t),\rho_{st}(t)], \tag{2}$$

where H describes the free evolution of the system and $\tilde{H}(t)$ represents a stochastic Hamiltonian given by

$$\tilde{H}(t)=\sum_{\alpha=1}^{n} l_\alpha(t)V_\alpha, \tag{3}$$

where V_α are operators acting on the Hilbert space of the system and $l_\alpha(t)$ are complex Gaussian stochastic processes with zero mean value, whose correlations are defined by

$$\chi_{\alpha\beta}(t,t_1)\equiv\langle\langle l_\alpha^*(t)l_\beta(t_1)\rangle\rangle. \tag{4}$$

We remark that due to the Hermiticity of $\tilde{H}(t)$, it follows that if $l_\alpha(t)$ were complex numbers there would be an index α' such that $l_{\alpha'}(t)$ is the complex conjugate of $l_\alpha(t)$ ($V_{\alpha'}=V_\alpha^\dagger$), i.e., $l_{\alpha'}(t)=l_\alpha^*(t)$, otherwise $\tilde{H}(t)$ would not be Hermitian. Note that in the particular case where an operator V_α is Hermitian, the noise $l_\alpha(t)$ ought to be real.

B. Master equation

Now we wish to find the exact evolution of the averaged-density matrix,

$$i\hbar\frac{d}{dt}\langle\rho_{st}(t)\rangle=\langle[H+\lambda\tilde{H}(t),\rho_{st}(t)]\rangle. \tag{5}$$

Here we have to take the average of a linear stochastic multiplicative equation. Fortunately, this kind of average can be obtained exactly by applying Novikov's theorem [21,22]. This theorem gives an exact result for the mean value of the product of a Gaussian noise and any functional of that noise. Using the fact that $\rho_{st}(t)$ is a functional of all noises $l_\alpha(t)$ and that $\tilde{H}(t)$ is Hermitian, after applying Novikov's theorem, we obtain [20]

$$\begin{aligned}\langle\tilde{H}(t)\rho_{st}(t)\rangle&=\int_0^t dt_1\chi_{\alpha\beta}(t,t_1)V_\alpha^\dagger\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}\right\rangle,\\ \langle\rho_{st}(t)\tilde{H}(t)\rangle&=\int_0^t dt_1\chi_{\alpha\beta}(t,t_1)\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}\right\rangle V_\alpha^\dagger.\end{aligned} \tag{6}$$

Here and from now on, we assume the convention of addition over repeated indices. Inserting expressions (6) into Eq. (5), it follows that

$$i\hbar\frac{d}{dt}\langle\rho_{st}(t)\rangle=[H,\langle\rho_{st}(t)\rangle]+\lambda\int_0^t dt_1\chi_{\alpha\beta}(t,t_1)\times\left[V_\alpha^\dagger,\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}\right\rangle\right]. \tag{7}$$

To go ahead with this result, we now need to know the expression for the "response function" $\delta\rho_{st}(t)/\delta l_\beta(t_1)$. We obtain this object by following Refs. [21,22]. After integrating Eq. (2) formally, we obtain $i\hbar\rho_{st}(t)=\rho_{st}(0)+\int_0^t du[H+\lambda\tilde{H}(u),\rho_{st}(u)]$ and taking a functional derivative with respect to $l_\beta(t_1)$, we get ($t>t_1$)

$$i\hbar\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}=\left\{\lambda[V_\beta,\rho_{st}(t_1)]+\int_{t_1}^t du\left[H+\lambda\tilde{H}(u),\frac{\delta\rho_{st}(u)}{\delta l_\beta(t_1)}\right]\right\}, \tag{8}$$

where we have assumed that at the initial time, the system and the noises are uncorrelated. Now, if this expression is differentiated with respect to t, an equation for the variational derivative of the stochastic matrix is obtained

$$i\hbar\frac{d}{dt}\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}=\left[H+\lambda\tilde{H}(t),\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}\right]. \tag{9}$$

The initial condition of this equation follows from the first term of the right-hand side of Eq. (8). Note that the variational derivative follows the same evolution as the stochastic density matrix. Therefore we can write

$$\frac{\delta\rho_{st}(t)}{\delta l_\beta(t_1)}=-i\frac{\lambda}{\hbar}G_{st}(t,t_1)[V_\beta,\rho_{st}(t_1)]G_{st}^\dagger(t,t_1), \tag{10}$$

where the propagator $G_{st}(t,t_1)$ is given by

$$G_{st}(t,t_1)=\left\lceil\exp-\frac{i}{\hbar}\int_{t_1}^t du(H+\lambda\tilde{H}(u))\right\rceil. \tag{11}$$

Here $\lceil\cdots\rceil$ indicates time ordering. Performing the average of expression (10) over the noise realizations, we note that $\rho_{st}(t_1)\to\rho(t_1)$. This can be seen by using the identity $G_{st}(t_1,t_1)=I$. Inserting the average of Eq. (10) in Eq. (7), the exact evolution of the density matrix follows:

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}[H,\rho(t)]-\left(\frac{\lambda}{\hbar}\right)^2\int_0^t dt_1\chi_{\alpha\beta}(t,t_1)\times[V_\alpha^\dagger,\langle G_{st}(t,t_1)[V_\beta,\rho(t_1)]G_{st}^\dagger(t,t_1)\rangle]. \tag{12}$$

As was expected, this equation is nonlocal in time.

C. Time-convolutionless master equation

Here we will obtain an equivalent exact evolution that is local in time. This goal can be achieved by expressing the stochastic matrix at an intermediate time as

$$\rho_{st}(t_1)=G_{st}^\dagger(t,t_1)\rho_{st}(t)G_{st}(t,t_1). \tag{13}$$

Here we have used the fact that at all times the stochastic evolution is unitary, thus $G_{st}^{-1}(t,t_1)=G_{st}^\dagger(t,t_1)$. From this expression, we can write the functional derivative (10) as

$$\frac{\delta \rho_{st}(t)}{\delta l_{\beta}(t_1)} = -i \frac{\lambda}{\hbar}[G_{st}(t,t_1)V_{\beta}G_{st}^{\dagger}(t,t_1),\rho_{st}(t)]. \tag{14}$$

Finally, introducing this expression into Eq. (7), we obtain

$$\frac{d}{dt}\langle \rho_{st}(t)\rangle = -\frac{i}{\hbar}[H,\langle \rho_{st}(t)\rangle] - \left(\frac{\lambda}{\hbar}\right)^2 \int_0^t dt_1 \chi_{\alpha\beta}(t,t_1) \times [V_{\alpha}^{\dagger},\langle [G_{st}(t,t_1)V_{\beta}G_{st}^{\dagger}(t,t_1),\rho_{st}(t)]\rangle]. \tag{15}$$

This expression gives us the desired time-convolutionless evolution. It will be the starting point of the rest of our paper.

D. Perturbative expansions

In general, the average in expression (15) cannot be obtained without appealing to a perturbative scheme. Since it is desirable to work out the free evolution in an exact way, the starting point is the density matrix evolution in the interaction picture,

$$\frac{d}{dt}\langle \hat{\rho}_{st}(t)\rangle = -\left(\frac{\lambda}{\hbar}\right)^2 \int_0^t dt_1 \chi_{\alpha\beta}(t,t_1) \times [V_{\alpha}^{\dagger}(t),\langle [\hat{G}_{st}(t,t_1)V_{\beta}(t_1)\hat{G}_{st}^{\dagger}(t,t_1),\hat{\rho}_{st}(t)]\rangle]. \tag{16}$$

Here the hat symbol indicates the explicitly time-dependent objects. The difficulty with this expression consists in obtaining the average of the product of two functionals of a set of Gaussian noises: the stochastic matrix $\hat{\rho}_{st}(t)$ and the channel propagators $\hat{G}_{st}(t,t_1)V_{\beta}(t_1)\hat{G}_{st}^{\dagger}(t,t_1)$. This average can be worked out in an perturbative way, using a generalization of Novikov's theorem. The details of this procedure can be found in Ref. [20]. Here, we only show the second-order approximation,

$$\frac{d}{dt}\hat{\rho}(t) \simeq -\left(\frac{\lambda}{\hbar}\right)^2 \int_0^t dt_1 \chi_{\alpha\beta}(t,t_1)[V_{\alpha}^{\dagger}(t),[V_{\beta}(t_1),\hat{\rho}(t)]]. \tag{17}$$

In order to obtain this result, in Eq. (16) we have only retained the zeroth-order contribution of the stochastic dynamics, $\hat{G}_{st}(t,t_1)\simeq I$, which imply the discarding of terms of order $\varphi(\lambda/\hbar)^4$. Using the Jacobi identity, it is possible to rewrite this evolution as a time-dependent Kossakowsky-Lindblad generator with an effective Hamiltonian [19].

III. SHORT-TIME DYNAMICS AND FIDELITIES

An important topic in the theory of open quantum systems is the short-time dynamics induced by the environment. Many quantities have been introduced to characterize this problem [23–29]. Here, in a first step, we are interested in characterizing the physical magnitudes that control the decoherence processes at short times. Therefore, we will assume that initially the system is in a pure state $\rho(0)=|\Psi\rangle\langle\Psi|$, and as a measure of the decoherence processes we will use the input-output fidelity [27]

$$F(t) = \mathrm{Tr}[\hat{\rho}(t)\rho(0)] = \langle \Psi|\hat{\rho}(t)|\Psi\rangle. \tag{18}$$

In the short-time regime, the fidelity can be expanded as

$$F(t) = 1 - \frac{\omega_0 t}{\Gamma_1} - \left(\frac{\omega_0 t}{\Gamma_2}\right)^2 - \cdots, \tag{19}$$

where in order to gain insight into the problem, we have introduced the quantity ω_0, which represents a characteristic frequency of the system. The constants Γ can be obtained from $1/(\Gamma_n)^n = n!\,\mathrm{Tr}[(d^n/dt^n)\hat{\rho}(t)|_{t=0}\rho(0)]$, where the derivatives of the density matrix follow from the master equations obtained in the previous section (see Appendix). Up to second order, we obtain

$$\frac{1}{\Gamma_1} = 0 \tag{20}$$

and

$$\frac{1}{\Gamma_2} = \lambda \frac{\sqrt{\langle\langle \tilde{H}\rangle\rangle_{\Psi}}}{\hbar\omega_0}, \tag{21}$$

where we have defined

$$\langle\langle \tilde{H}\rangle\rangle_{\Psi} = \chi_{\alpha\beta}[\langle V_{\alpha}^{\dagger}V_{\beta}\rangle_{\Psi} - \langle V_{\alpha}^{\dagger}\rangle_{\Psi}\langle V_{\beta}\rangle_{\Psi}]. \tag{22}$$

Here $\langle V\rangle_{\Psi} \equiv \langle \Psi|V|\Psi\rangle$ is an operator average over the initial quantum state and $\chi_{\alpha\beta} \equiv \chi_{\alpha\beta}(t,t_1)|_{t=t_1=0}$.

Equation (20) shows in general that the decoherence processes evolve at least quadratically in time for short times. On the other hand, note that the expression for the second fidelity damping rate Γ_2 is independent of the correlation times of the noises. It expresses that the fidelity loss in a characteristic period of time $2\pi\omega_0^{-1}$ is given by the initial stochastic Hamiltonian variance $\langle\langle \tilde{H}\rangle\rangle_{\Psi}$, measured in units of the characteristic system energy $\hbar\omega_0$. The variance is over the noises and over the initial quantum state. In this way, the environment influence can be attenuated by reducing the noise intensities, or by selecting preferential states that reduce the quantum operator averages [25].

We remark that the same conclusions are obtained by studying the short-time behavior of the idempotency defect $\delta(t) = 1 - \mathrm{Tr}[\rho^2(t)]$ [23].

A. Fidelities for mixed states

In the context of quantum information theory [26–29], the necessity of ''measuring'' the decoherence processes when a system begins in a mixed state naturally arises. In this case, the definition (18) is not useful and other quantities have been proposed. Here, in a second step, we will examine the short-time behavior of these generalized fidelities, considering that the quantum system evolves under the action of classical stochastic fields.

The entanglement fidelity [26–28] is defined by

$$F_e(t)=\mathrm{Tr}[\hat{\rho}_{rs}(t)\rho_{rs}(0)]=1-\frac{\omega_0 t}{\Gamma_{1e}}-\left(\frac{\omega_0 t}{\Gamma_{2e}}\right)^2-\cdots. \tag{23}$$

Here $\rho_{rs}(0)=|\Psi_{rs}\rangle\langle\Psi_{rs}|$, where the state $|\Psi_{rs}\rangle$ is a purification of the initial system density matrix, i.e., $\mathrm{Tr}_r[|\Psi_{rs}\rangle\langle\Psi_{rs}|]=\rho(0)$, where s denotes the original system and r denotes an ancillary system. The matrix $\hat{\rho}_{rs}(t)$ is the result of applying to $\rho_{rs}(0)$, the original propagator extended to the ancillary Hilbert space with the identity. The constants Γ_e can be obtained in a similar way as in the previous case. Up to second order, we obtain

$$\frac{1}{\Gamma_{1e}}=0 \tag{24}$$

and

$$\frac{1}{\Gamma_{2e}}=\lambda\,\frac{\sqrt{\langle\langle\widetilde{H}\rangle\rangle_{\Psi_{rs}}}}{\hbar\,\omega_0}, \tag{25}$$

where $\langle\langle\widetilde{H}\rangle\rangle_{\Psi_{rs}}$ is the same as in Eq. (22), but now the operator averages are given by $\langle V\rangle_{\Psi_{rs}}\equiv\mathrm{Tr}[\rho(0)V]$, i.e., an average over the initial mixed state.

Expressing the initial state as an arbitrary mixture of pure states

$$\rho(0)=\sum_i p_i|\Psi_i\rangle\langle\Psi_i|, \tag{26}$$

where $\Sigma_i p_i=1$, the averaged fidelity [26–28] is defined by

$$F_a(t)=\sum_i p_i F(|\psi_i\rangle)=1-\frac{\omega_0 t}{\Gamma_{1a}}-\left(\frac{\omega_0 t}{\Gamma_{2a}}\right)^2-\cdots, \tag{27}$$

where $F(|\Psi_i\rangle)$ indicates the input-output fidelity for the pure state $|\Psi_i\rangle$. The characteristic constants Γ_a are

$$\frac{1}{\Gamma_{1a}}=0 \tag{28}$$

and

$$\frac{1}{\Gamma_{2a}}=\lambda\,\frac{\sqrt{\sum_i p_i\langle\langle\widetilde{H}\rangle\rangle_{\Psi_i}}}{\hbar\,\omega_0}, \tag{29}$$

where $\langle\langle\widetilde{H}\rangle\rangle_{\Psi_i}$ is the same as in expression (22), but performing the operator averages with the state $|\Psi_i\rangle$.

Finally, we mention a natural generalization of the definition (18) analyzed by Jozsa in Ref. [29]. In this generalization, the fidelity between two mixed states ρ_b and ρ_a is defined by

$$F(\rho_b,\rho_a)=\{\mathrm{Tr}[(\sqrt{\rho_a}\rho_b\sqrt{\rho_a})^{1/2}]\}^2. \tag{30}$$

As in the previous generalizations, it is possible to expand

$$F(\hat{\rho}(t),\rho(0))=1-\frac{\omega_0 t}{\Gamma_{1m}}-\left(\frac{\omega_0 t}{\Gamma_{2m}}\right)^2-\cdots. \tag{31}$$

In this case, the derivation of the damping rates is more complicated. In the Appendix, we have developed a general first-order perturbation theory, which allows us to treat this problem. We get

$$\frac{1}{\Gamma_{1m}}=0 \tag{32}$$

and

$$\frac{1}{\Gamma_{2m}}=\lambda\,\frac{\sqrt{\sum_i P_i\langle\langle\widetilde{H}\rangle\rangle'_{\Phi_i}}}{\hbar\,\omega_0}. \tag{33}$$

The probabilities P_i and the states $|\Phi_i\rangle$ are, respectively, the eigenvalues and eigenvectors of the initial density matrix

$$\rho(0)|\Phi_i\rangle=P_i|\Phi_i\rangle. \tag{34}$$

Unlike the previous cases, the average $\langle\langle\widetilde{H}\rangle\rangle'_{\Phi_i}$ is given by

$$\langle\langle\widetilde{H}\rangle\rangle'_{\Phi_i}=\chi_{\alpha\beta}[\langle V_\alpha^\dagger V_\beta\rangle_{\Phi_i}-\langle V_\alpha^\dagger\mathcal{D}V_\beta\rangle_{\Phi_i}], \tag{35}$$

where the operator $\mathcal{D}$ is

$$\mathcal{D}=\sum_{\{P_j\neq 0\}}|\Phi_j\rangle\langle\Phi_j|. \tag{36}$$

Here the sum is extended only to the eigenvectors $|\Phi_j\rangle$ of $\rho(0)$ with nonzero eigenvalues. Furthermore, the expansion (31) is valid only under the condition $t^2\ll\{P_i\}$. Note that $1/\Gamma_{2m}$ vanish when the initial density matrix has support over the whole Hilbert space. This property is not a quantum one. In fact, it will be also present if we apply the perturbation theory of the Appendix to the classical definition of fidelity between two probability distributions [26] (for example, Eq. (30) when $[\rho_b,\rho_a]=0$).

From the previous result, we realize that the difference between the short-time dynamics of the different generalized fidelities is given by the kind of quantum average performed over the operators that define the stochastic Hamiltonian. In fact, in all cases, the influence of the noises only appear through their amplitudes without any information about their correlation times. Furthermore, the inequality relations that one can find between the different damping rates coincide with the ones obtained in Refs. [26–28].

IV. EXACT NONPERTURBATIVE EVOLUTIONS

In this section we will work out some cases whose averaged evolutions can be treated to all orders in the interaction. This possibility is highly dependent on the properties of the noise correlations and on the commutation relations of the different operators that describe the problem. Fortunately, there are situations of practical interest, where the average term in the master equation (15) can be obtained in an exact and easy way.

A. White noises

When the stochastic fields are characterized by uncorrelated white fluctuations,

$$\chi_{\alpha\beta}(t,t_1)=\delta_{\alpha\beta}(t-t_1), \tag{37}$$

the average in Eq. (15) can be trivially calculated, giving rise to a standard Lindblad generator,

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}[H,\rho(t)]-\left(\frac{\lambda}{\hbar}\right)^2([V_\alpha^\dagger,\rho(t)V_\alpha]+[V_\alpha^\dagger\rho(t),V_\alpha]). \tag{38}$$

In obtaining this result, we have used the fact that $G_{st}(t,t)=I$. Note that in this correlation model, any physical information must be introduced in an *ad hoc* manner through the intensity of the noises.

B. Dispersive noise

Here we analyze a second example that can be worked out in an exact, non-perturbative way. We will assume that the stochastic Hamiltonian has only one term, $\tilde{H}(t)=l(t)V$, and that it does not induce energy excitations in the system,

$$[H,V]=0. \tag{39}$$

This condition implies that the operator V commutes with the stochastic propagator G_{st}, and as a consequence the variational derivative of the stochastic density matrix [Eq. (14)] does not depend explicitly on the noise. This fact allows us to perform the averaging in a trivial way, resulting in

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}[H,\rho(t)]-\left(\frac{\lambda}{\hbar}\right)^2\left(\int_0^t ds\,\chi(t,s)\right)[V,[V,\rho(t)]], \tag{40}$$

where $\chi(t,s)$ is the noise correlation.

If we assume white noise and the operator V proportional to the Hamiltonian H, the master equation (40) reduces to Milburn's model of intrinsic decoherence [30]. On the other hand, dispersive noise can be used to represent laser-intensity fluctuations in a trapped ion [14].

C. Quantum harmonic oscillator subject to color-amplitude noise

Now we will obtain the density-matrix evolution that corresponds to a quantum harmonic oscillator subject to the influence of "amplitude noise." This means that the field fluctuations induce energy excitations in the system. The Hamiltonians describing this situation are

$$H=\hbar\omega_0 a^\dagger a,\quad \tilde{H}(t)=i\hbar[u(t)a^\dagger-u^*(t)a], \tag{41}$$

where ω_0 represents the natural frequency of the oscillator, $a^\dagger(a)$ represents the creation (annihilation) operator, and $u(t)$ is a general complex Gaussian stochastic field.

In what follows we will apply the results of Sec. II. First, as in Ref. [16], we note that the stochastic propagator corresponding to the evolution (41) can be written, in the interaction representation, as

$$\hat{G}_{st}(t,t_1)=\exp[i\phi(t,t_1)]D[v(t,t_1)], \tag{42}$$

where $D[v]=\exp[va^\dagger-va]$ is the displacement operator, $\phi(t,t_1)$ is a phase factor (which turns out to be important for the current problem), and the amplitude $v(t,t_1)$ is given by the formula

$$v(t,t_1)=\lambda\int_{t_1}^t dt'\,u(t')\exp(i\omega_0 t'). \tag{43}$$

Using the properties

$$D[v]aD^\dagger[v]=a-v,\quad D[v]a^\dagger D^\dagger[v]=a^\dagger-v^*, \tag{44}$$

from Eq. (14), it is possible to obtain

$$\begin{aligned}\frac{\delta\hat{\rho}_{st}(t)}{\delta u(t_1)}&=\lambda[a^\dagger(t_1),\hat{\rho}_{st}(t)],\\ \frac{\delta\hat{\rho}_{st}(t)}{\delta u^*(t_1)}&=-\lambda[a(t_1),\hat{\rho}_{st}(t)],\end{aligned} \tag{45}$$

where the terms proportional to v and v^* have canceled out, due to the commutator operation. Now, the average of the variational derivatives can be performed in a trivial way giving rise, after a little algebra, to the exact evolution,

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt}=&\frac{1}{2}\eta(t)([a,\hat{\rho}(t)a^\dagger]+[a\hat{\rho}(t),a^\dagger])\\ &+\frac{1}{2}\eta(t)([a^\dagger,\hat{\rho}(t)a]+[a^\dagger\hat{\rho}(t),a])\\ &+\xi(t)([a^\dagger,\hat{\rho}(t)a^\dagger]+[a^\dagger\hat{\rho}(t),a^\dagger])\\ &+\xi^*(t)([a,\hat{\rho}(t)a]+[a\hat{\rho}(t),a]),\end{aligned} \tag{46}$$

where we have defined

$$\eta(t)\equiv\lambda^2\int_0^t dt_1\{\langle\langle u^*(t)u(t_1)\rangle\rangle\exp[-i\omega_0(t-t_1)]+\text{c.c.}\},$$

$$\xi(t)\equiv-\lambda^2\int_0^t dt_1\langle\langle u(t)u(t_1)\rangle\rangle\exp[i\omega_0(t+t_1)]. \tag{47}$$

We remark that, to our knowledge, these expressions were not previously reported. This Lindblad-like generator gives the exact non-Markovian evolution of an harmonic oscillator subjected to color Gaussian fluctuations. Notice that this evolution is the same that would be obtained in a second-order approximation [Eq. (17)], implying that all higher-order terms give a total null contribution.

1. Wigner function

An alternative and equivalent description to that given by a density matrix, is that obtained with the use of phase-space distributions. These methods are of considerable practical interest, since they allow one to deal with functions rather than operators.

The Wigner function is defined by [31,32]

$$W(\alpha,\alpha^*)=\frac{1}{\pi}\int d^2z\,\exp[-z\alpha^*+\alpha z^*]\chi_w(z,z^*), \tag{48}$$

where the characteristic function $\chi_w(z,z^*)$ is given by

$$\chi_w(z,z^*)=\mathrm{Tr}[\rho\,\exp(za^\dagger-z^*a)]. \tag{49}$$

Using Eq. (46), it is possible to obtain the exact evolution of the characteristic function,

$$\frac{\partial}{\partial x}\chi_w(z,z^*,t)=-\{|z|^2\eta(t)+(z^*)^2\xi(t)+z^2\xi^*(t)\}\chi_w(z,t), \tag{50}$$

which can be immediately integrated as

$$\chi_w(z,z^*,t)=\exp\left[-\int_0^t dt'\{|z|^2\eta(t')+(z^*)^2\xi(t')+z^2\xi^*(t')\}\right]\chi_w(z,z^*,0). \tag{51}$$

In this way, specifying the correlation properties of the stochastic field $u(t)$ and the initial condition of the system $\chi_w(z,z^*,0)$, it is possible to obtain the exact Wigner function by quadrature. Alternatively, from Eqs. (48) and (50), it is possible to obtain the temporal evolution of the Wigner function,

$$\frac{\partial}{\partial t}W(\alpha,\alpha^*,t)=\eta(t)\frac{\partial^2W(\alpha,\alpha^*,t)}{\partial\alpha\partial\alpha^*}-\xi(t)\frac{\partial^2W(\alpha,\alpha^*,t)}{\partial^2\alpha}-\xi^*(t)\frac{\partial^2W(\alpha,\alpha^*,t)}{\partial^2\alpha^*}. \tag{52}$$

Notice that this dynamics is governed by diffusive processes with time-dependent diffusion coefficients.

2. Rotating-wave approximation and analytical signals

A common description of a field is in terms of analytical signals [33]. This description is equivalent to applying the rotating-wave approximation (RWA), which consists in discarding antiresonant terms. In the present problem this corresponds to assuming $\langle\langle u(t)u(t_1)\rangle\rangle=0$, and in consequence

$$\xi(t)=0. \tag{53}$$

Under this condition, and proposing the following time dependence of the density matrix,

$$\hat\rho(t)\equiv\hat\rho(\tau(t)), \tag{54}$$

where the dimensionless parameter $\tau(t)$ is given by

$$\tau(t)\equiv\int_0^t dt'\,\eta(t'), \tag{55}$$

the evolution of the density matrix can be written as

$$\frac{d\hat\rho(\tau)}{d\tau}=\frac{1}{2}([a,\hat\rho(\tau)a^\dagger]+[a\hat\rho(\tau),a^\dagger]+[a^\dagger,\hat\rho(\tau)a]+[a^\dagger\hat\rho(\tau),a]). \tag{56}$$

Thus, we realize that under the condition (53), all the non-Markovian effects are described by a time renormalization given by $\tau(t)$. We remark that Eq. (56) provides an extremely simple result, which in the RWA is exact at all orders in the interaction. Note that in the parameter τ, this evolution is Markovian and coincides with that obtained in Ref. [15].

V. HEATING OF TRAPPED IONS

The theory of the heating of the vibrational modes of a trapped ion due to the presence of stochastic electrical fields has been explored in many works [15–17]. Here, unlike these works, we will apply the results of Sec. IV, which allow us to work all non-Markovian effects in an exact way.

The interaction of a single ion of mass M interacting with a classical stochastic electric field $E(t)$ can be described by the Hamiltonians (41) with $u(t)=ieE(t)/\sqrt{2M\hbar\omega_0}$, e being the ion charge [16]. Furthermore, for simplicity we will assume that $E(t)$ has the properties of a stationary analytical signal, with correlation $\langle\langle E^*(t)E(t_1)\rangle\rangle=\langle|E|^2\rangle\exp(-|t-t_1|/T)$, where $\langle|E|^2\rangle$ represents the rms electric-field strength. In this manner, we have $\langle\langle u(t)u(t_1)\rangle\rangle=0$ and

$$\langle\langle u^*(t)u(t_1)\rangle\rangle=\frac{e^2}{2M\hbar\omega_0}\langle|E|^2\rangle\exp(-|t-t_1|/T). \tag{57}$$

From the definitions of $\eta(t)$ [Eq. (47)] and $\tau(t)$ [Eq. (55)] it is possible to obtain

$$\tau(t)=\frac{\lambda^2T}{\tau_1}\left[\exp\left(-\frac{t}{T}\right)\cos(\omega_0t+2\phi)-\cos(2\phi)+\frac{t}{T}\right], \tag{58}$$

where we have defined

$$\frac{1}{\tau_1}=\frac{e^2\langle|E|^2\rangle}{M\hbar\omega_0}\frac{T}{1+\omega_0^2T^2},\quad \tan\phi=\omega_0T. \tag{59}$$

The dimensionless parameter $\tau(t)$ contains all the information about the non-Markovian effects. Thus, it is of interest to study its behavior for short and long times. In the short-time limit we obtain

$$\tau_s(t)=\lim_{t\to0}\tau(t)\simeq\lambda^2\left(\frac{t}{\tau_d}\right)^2, \tag{60}$$

where the "characteristic decoherence time" τ_d is

$$\tau_d = \sqrt{\frac{2M\hbar\omega_0}{e^2\langle |E|^2\rangle}}. \tag{61}$$

This last expression has a clear physical interpretation: the term $\Delta P = \sqrt{2M\hbar\omega_0}$ is the momentum quantum of the system; the term $\Delta F = \sqrt{e^2\langle |E|^2\rangle}$ represents the square root of the rms force acting on the system. Therefore, a Newton-like law is satisfied: $(\Delta P/\tau_d) = \Delta F$, expressing that τ_d is the average time over which the fluctuations create, or destroy, an excitation quantum in the system. This averaged time is independent of the correlation time T.

In the long-time regime $(t \gg \omega_0, T)$, we obtain

$$\tau_l(t) = \lim_{t\to\infty} \tau(t) \simeq \lambda^2 \frac{1}{\tau_1}(t - t_0), \tag{62}$$

where $t_0 = T(1-\omega_0^2 T^2)/(1+\omega_0^2 T^2)$. The time t_0 represents a usual slippage present in any non-Markovian evolution. The time τ_1 defines the characteristic long-time scale. In terms of the characteristic decoherence time, it can be written as

$$\tau_1 = \left[\frac{2T}{1+\omega_0^2 T^2}\right]^{-1} \tau_d^2. \tag{63}$$

Fixing the time τ_d, this expression reaches a minimal value when $\omega_0 T = 1$. Thus, the worse situation, where the "heating influence" of the stochastic field—in the long-time regime—is maximized, occurs when the maximum of the noise spectrum is centered at the natural frequency of the ion trap.

Now we will analyze some properties that are of special interest in the theory of ion trapping.

A. Fidelity damping rates

In the RWA, from Eq. (21) it is possible to obtain

$$\frac{1}{\Gamma_2} = \frac{\lambda}{\omega_0 \tau_d}[1 + 2(\langle a^\dagger a\rangle_\Psi - \langle a^\dagger\rangle_\Psi \langle a\rangle_\Psi)]. \tag{64}$$

From this expression it is easy to see that, in the short-time regime, the least affected states are the coherent ones. Note that in this time scale, the coherence loss of these states is only originated by the "vacuum" contribution. Any other state will have a faster fidelity loss.

We remark that it is possible to realize that the states least affected *at any time* are the coherent ones. This follows after considering that the Wigner function is governed by purely diffusive processes [see Eq. (52) under the RWA].

B. Mean excitation number

A central quantity of interest is the mean excitation number $\langle n\rangle = \mathrm{Tr}[\rho a^\dagger a]$. From Eq. (56) it follows that

$$\frac{d\langle n(\tau)\rangle}{d\tau} = 1, \tag{65}$$

which can be immediately integrated, giving

$$\langle n(t)\rangle = \tau(t) + \langle n(0)\rangle. \tag{66}$$

From this result, we realize that the dimensionless parameter $\tau(t)$ gives the mean excitation number when the trapped ion starts in its ground state. In fact, the expression (66) and the previous analysis of $\tau(t)$ for short and long times, coincide exactly with those obtained by James for the heating of the ground state [16]. Furthermore, in a Markovian approximation, Eq. (66) reduces to the expressions obtained in Refs. [15,17].

C. Populations

In this section we will obtain the evolution of the trapped-ion populations, which are given by the diagonal elements of the density matrix, $P_n = \langle n|\hat{\rho}|n\rangle$. The time evolution obtained from Eq. (56) is

$$\frac{dP_n(\tau)}{d\tau} = [(n+1)P_{n+1}(\tau) + nP_{n-1}(\tau) - (2n+1)P_n(\tau)]. \tag{67}$$

We remark that this evolution was obtained by Lamoreaux [17] in a Markovian approximation. In contrast, here we are working all non-Markovian effects in an hidden way through the parameter $\tau(t)$. Furthermore, we will solve exactly this infinite set of coupled equations by introducing the characteristic function [34]

$$Q(x,\tau) = \sum_{n=0}^{\infty} (1-x)^n P_n(\tau). \tag{68}$$

This function allows us to obtain the populations by differentiation

$$P_n(\tau) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial^n x} Q(x,\tau)\bigg|_{x=1}. \tag{69}$$

From Eqs. (67)–(69), the evolution of the characteristic function is given by

$$\frac{\partial Q(x,\tau)}{\partial\tau} = -\left\{ xQ(x,\tau) + x^2 \frac{\partial Q(x,\tau)}{\partial x}\right\}. \tag{70}$$

This equation can be solved using the method of characteristics. We get

$$Q(x,\tau) = \frac{1}{x} g\left(\tau + \frac{1}{x}\right), \tag{71}$$

where $g(x)$ is an arbitrary function that must be determined by the initial condition: $g(1/x) = xQ(x,0)$. After a simple manipulation result

$$Q(x,\tau) = \frac{1}{1+\tau x} Q\left(\frac{x}{1+\tau x}, 0\right). \tag{72}$$

In the next, we will analyze the behavior of the populations for different initial conditions of interest.

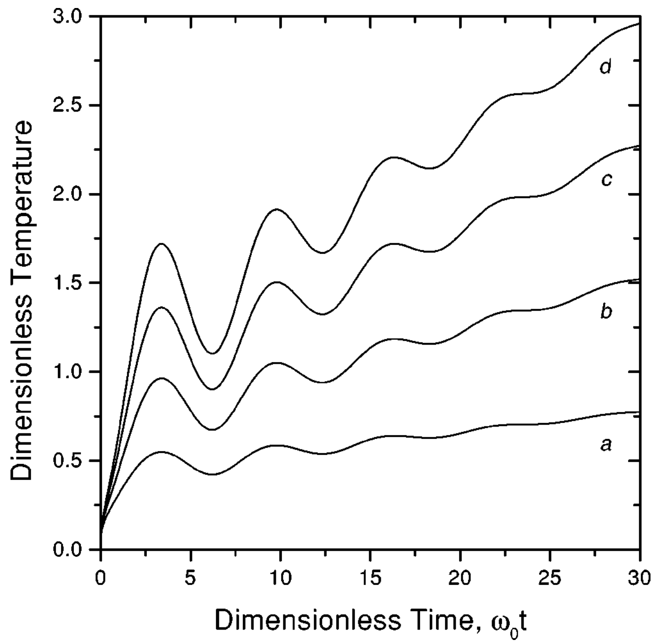

FIG. 1. Dimensionless temperature $\beta^{-1}/\hbar\omega_0$ as a function of time for an ion that starts in its ground state. The parameters used were as follows: $\lambda=1$, curve *a*, $\omega_0 T=10$, $\omega_0\tau_1=105$; curve *b*, $\omega_0 T=10$, $\omega_0\tau_1=37$; curve *c*, $\omega_0 T=10$, $\omega_0\tau_1=22$; curve *d*, $\omega_0 T=10$, $\omega_0\tau_1=16$.

1. Thermal state

If at the beginning the trapped ion is in a thermal state, the initial populations are given by

$$P_n(0)=\frac{\overline{n}^n}{(\overline{n}+1)^{n+1}},\quad \overline{n}=\frac{1}{\exp(\beta\hbar\omega_0)-1} \tag{73}$$

and the initial characteristic function is

$$Q(x,0)=\frac{1}{1+\overline{n}x}. \tag{74}$$

From Eq. (72) it follows that the evolved characteristic function is

$$Q(x,t)=\frac{1}{1+[\tau(t)+\overline{n}]x}. \tag{75}$$

Comparing the two previous expressions, we realize that at any time the populations correspond to that of a thermal state of the system with $\langle n(t)\rangle$ quanta. This fact permits us to define the temperature of the system at all instants of the nonequilibrium evolution, which is given by

$$\beta(t)=\frac{1}{\hbar\omega_0}\ln\left[\frac{1}{\langle n(t)\rangle}+1\right]. \tag{76}$$

In the context of ion trapping, only very low temperatures are of interest. In particular, we can affirm that when the trapped ion starts in its ground state, all the populations are occupied following a thermal distribution, with the temperature given by expression (76).

In Fig. 1, we have plotted the time-dependence of the dimensionless temperature $\beta^{-1}/\hbar\omega_0$ when the ion starts in its ground state. The oscillatory behavior is a purely non-Markovian effect.

2. Fock state

Now we assume that the trapped ion begins in a number state $|n_o\rangle$,

$$P_n(0)=\delta_{n,n_o}. \tag{77}$$

The initial generating function is given by

$$Q(x,0)=(1-x)^{n_o}, \tag{78}$$

and its temporal evolution is

$$Q(x,t)=\frac{1}{1+\tau(t)x}\left[1-\frac{x}{1+\tau(t)x}\right]^{n_o}. \tag{79}$$

Consistently, we note that in the case $n_o=0$, the distribution of the populations reduces to that of a thermal state. In general, performing an nth derivative, it is possible to obtain a general closed expression for the n population. Of special interest is the short-time behavior. We obtain

$$P_{n_o}(t)\simeq 1-(1+2n_o)\tau_s(t),$$

$$P_n(t)\simeq\frac{n_>!}{n_<!(|n-n_o|)!}[\tau_s(t)]^{|n-n_o|}, \tag{80}$$

where $n_>$ ($n_<$) is the maximum (minimum) of the pair (n,n_o), and $\tau_s(t)$ is given by Eq. (60). The first result expresses that the initial population can be abandoned either by creating or destroying any one of the original n_o quanta, in addition to the possibility of a vaccuum excitation. These processes are incoherent, giving rise to the factor $(1+2n_o)$. The second expression can be interpreted in a similar way: an arbitrary state $|n\rangle$ can be populated after creating, or destructing, $|n-n_o|$ quanta excitations; the number of ways in which these processes can be realized is given by the combinatorial of $n_>$ and $n_<$. The number of ways follows after considering the indistinguishability of the n final excitation quanta.

In Fig. 2, we have plotted the populations of the states $n=0,1,3,4$, when the ion starts in its first excited state $n=1$. The short-time behavior, for populations different from the initial condition, have a polynomial increase with time. The population revivals are a consequence of the non-Markovian character of the evolution.

It is interesting to note that it is possible to characterize an intermediate-time regime. As an example, we study the population of the ground state, which is given by

$$P_0(t)=\frac{[\tau(t)]^{n_o}}{[1+\tau(t)]^{n_o+1}}. \tag{81}$$

This expression starts from zero with a polynomial growth and reaches its maximum value when $\tau_{\max}(t)=n_o$, with $P_0(\tau_{\max})=[n_o]^{n_o}/[1+n_o]^{n_o+1}$. Later on, it follows a monotic

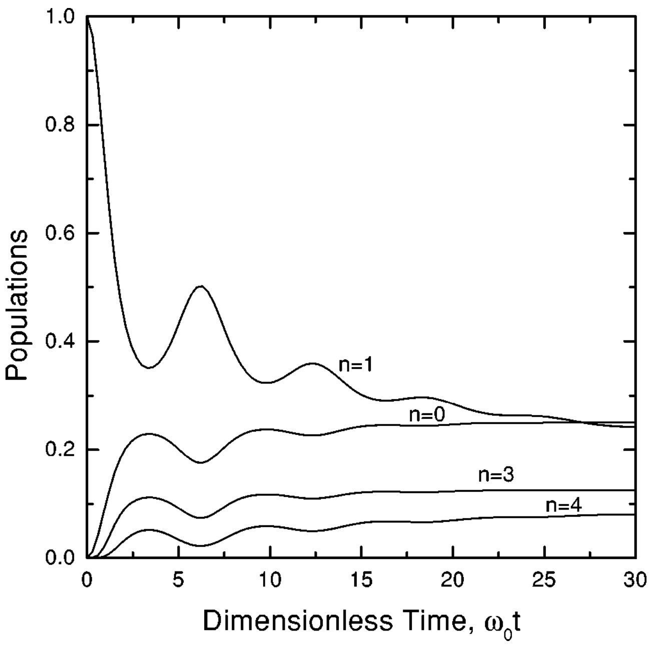

FIG. 2. Population behavior as a function of time for Fock states with $n=0,1,3,4$ for an ion starting in its first excited state. The parameters used were as follows: $\lambda=1$, $\omega_0 T=10$, $\omega_0\tau_1=37$.

decrease. This behavior is characteristic of all populations different from the initial condition, and comes from the structure of the master equation (56). Thus, each state $|n\rangle$ has a characteristic time where it is occupied most probably. In addition to this behavior, we have to consider the revivals originated from the nonzero correlation time of the field. Nevertheless, in general they have disappeared on this intermediate time scale.

3. Coherent state

Finally we give the exact analytical results when the initial state is coherent, i.e.,

$$P_n(0)=\frac{\bar{n}^n}{n!}\exp(-\bar{n}). \tag{82}$$

The initial generating function is

$$Q(x,0)=\exp(-\bar{n}x), \tag{83}$$

and its time dependence is given by

$$Q(x,t)=\frac{1}{1+\tau(t)x}\exp\left[-\bar{n}\frac{x}{1+\tau(t)x}\right]. \tag{84}$$

Again, if $\bar{n}=0$, the generating function reduces to that of a thermal state.

D. Noise-induced decoherence

As in the case of thermal reservoirs, the influence of a classical stochastic force on a quantum system leads to the destruction of the quantum-mechanical nature of an initial state. Here we will study the decoherence phenomena by means of the Wigner function. As an initial condition we choose a superposition of two coherent states with opposite amplitudes (a ‘‘Schrödinger-cat’’ state),

$$\rho(0)=\frac{1}{\mathcal{N}}(|\alpha_0\rangle+|-\alpha_0\rangle)(\langle\alpha_0|+\langle-\alpha_0|), \tag{85}$$

where $\mathcal{N}=2[1+\exp(-2|\alpha_0|^2)]$ is a normalization constant. Using Eqs. (48) and (51), in the RWA, the explicit expression for the Wigner function is

$$W(\alpha,\alpha^*,t)=\frac{1}{\mathcal{N}K(t)}\left[\exp\left\{-\frac{|\alpha-\alpha_0|^2}{K(t)}\right\}+\exp\left\{-\frac{|\alpha+\alpha_0|^2}{K(t)}\right\}+2F(\alpha,\alpha^*,t)\right], \tag{86}$$

where the fringe function $F(\alpha,\alpha^*,t)$ is given by

$$F(\alpha,\alpha^*,t)=e^{-2|\alpha_0|^2}\exp\left\{-\frac{|\alpha|^2-|\alpha_0|^2}{K(t)}\right\}\times\cos\left[2\frac{\mathrm{Im}(\alpha\alpha_0^*)}{K(t)}\right] \tag{87}$$

and the width function is

$$K(t)=\frac{1}{2}+\tau(t). \tag{88}$$

From expression (86), we explicitly see that the initial Wigner function consists of two Gaussians centered around $\pm\alpha_0$ and interference fringes in between. In contrast to the results obtained for a thermal bath [35], the center of the Gaussians does not move as time evolves. The function $K(t)$ governs both the width of the Gaussians and the change in fringe visibility. From the limiting behaviors of the dimensionless parameter $\tau(t)$, Eqs. (60) and (62), we realize that at short times, the Wigner function is characterized by a superdiffusive behavior, converging to a diffusive behavior in the long-time limit. This behavior can be classified as a ‘‘weak non-Markovian effect’’ [36].

In Fig. 3 we have plotted the fringe function in the origin of the complex plane $F(0,0,t)$ for a cat state with $|\alpha_0|^2=2$. We realize that increasing the correlation time T, the phenomena of recoherence occurs, i.e., oscillations in the decoherence process. Note that the effect of increasing the correlation time T is to narrow the recurrence peaks.

E. Dispersive noise

Finally, we will analyze a trapped ion under the influence of dispersive noise. In this case, the stochastic Hamiltonian must not induce excitations in the energy system. As an example, we choose $\tilde{H}(t)=l(t)N$, where $N=a^\dagger a$ is the number operator. Working in the interaction picture, from Eq. (40) the exact density-matrix evolution reads

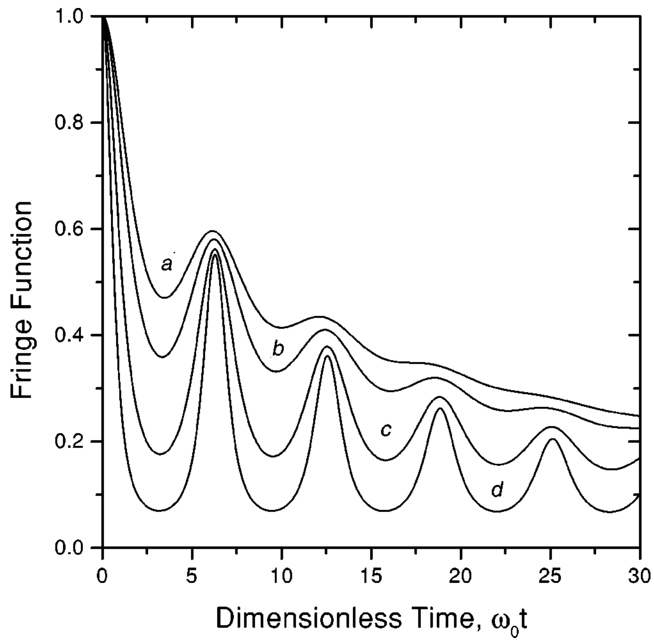

FIG. 3. Fringe function in the origin of the complex plane as a function of time. The parameters used were as follows: $\lambda=1$, $|\alpha_0|^2=2$, curve a, $\omega_0 T=8$, $\omega_0\tau_1=141$; curve b, $\omega_0 T=12$, $\omega_0\tau_1=141$; curve c, $\omega_0 T=27$, $\omega_0\tau_1=141$; curve d, $\omega_0 T=70$, $\omega_0\tau_1=141$.

$$\frac{d}{dt}\hat{\rho}(t)=-\left(\frac{\lambda}{\hbar}\right)^2\left[\int_0^t ds\chi(t,s)\right][N,[N,\hat{\rho}(t)]]. \tag{89}$$

In the Fock-state basis, the density-matrix elements evolve as

$$\frac{d}{dt}\hat{\rho}_{nm}(t)=-\left(\frac{\lambda}{\hbar}\right)^2\left[\int_0^t ds\chi(t,s)\right](n-m)^2\hat{\rho}_{nm}(t), \tag{90}$$

with solution

$$\hat{\rho}_{nm}(t)=\exp[-\gamma(t)(n-m)^2]\hat{\rho}_{nm}(0). \tag{91}$$

Here, the decay function $\gamma(t)$ is given by

$$\gamma(t)=\left(\frac{\lambda}{\hbar}\right)^2\int_0^t dt'\int_0^{t'} ds\chi(t',s). \tag{92}$$

As in the previous case, all non-Markovian effects are introduced through a time scaling given here by $\gamma(t)$. Proposing the noise correlation $\chi(t,s)=\langle E_D^2\rangle\exp(-|t-s|/T_D)$, where $\langle E_D^2\rangle$ is the strength of the "stochastic dispersive energy," the time scaling results in

$$\gamma(t)=\left(\frac{\lambda}{\hbar}\right)^2\langle E_D^2\rangle T_D\left[t-T_D\left\{1-\exp\left(-\frac{t}{T_D}\right)\right\}\right]. \tag{93}$$

From the asymptotic behavior of this function, it is possible to conclude that the short-time regime depends quadratically on time and is independent of the correlation time, T_D. In the long-time regime, the dispersive effect can be attenuated by reducing the product $\langle E_D^2\rangle T_D$.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have studied—with the density-matrix formalism—the averaged dynamics of quantum systems subject to the influence of classical stochastic forces. From the averaged dynamics, we have constructed general applicable short-time perturbative expansions for some fidelities, such as the input-output fidelity, the entanglement fidelity, and the averaged fidelity. The evolution of any system subject to dispersive noise was obtained in an exact way. The paradigmatic case of the harmonic oscillator under the action of amplitude noise is another example that was worked out at all orders.

From these cases and from the analysis of the short-time fidelities behavior, we can give some general characteristics of the averaged dynamics. The short-time dynamics—associated with decoherence—depend quadratically in time and is characterized by the average time needed by the fluctuations to induce an excitation quantum in the system. The characteristic decoherence time is independent of the noise-correlation time and is given in terms of the rms fluctuation of the force field. On the other hand, in the long-time regime, for the case of amplitude noise—associated with heating—we have argued that the influence of the stochastic field can be attenuated by reducing the noise intensity or by detuning the noise spectrum with respect to the natural frequency of the system. On the contrary, the dispersive effect can be attenuated reducing both the intensity and the noise-correlation time.

In the study of non-Markovian effects, we have showed that this feature is introduced in the evolutions through a time scaling. In the case of amplitude noise, the scaling gives rise to revivals in the averaged dynamics. This fact was seen in the population behavior and in the behavior of the decoherence phenomena. On the other hand, the scaling for dispersive noises only introduces a slow down of all the irreversible processes.

As an application of our results, we have studied the problem of heating of trapped ions. The previous exact results for the heating of the ground state [37] were generalized for arbitrary initial conditions. We have proved that an initial thermal state maintains this property during the whole non-equilibrium evolution, allowing us to define temperature at any time. We remark that this theoretical result agrees with the measurements reported by Wineland's group about ions heating with "natural reservoirs" [11]. The short-time behavior for initial Fock conditions was clearly interpreted from the indistinguishability of the elementary incoherent excitations. Furthermore the noise-induced decoherence problem was studied by means of the Wigner function. This function has a diffusivelike behavior, in which the interference terms exhibit the phenomena of recoherence. This behavior is a direct consequence of the non-Markovian evolution.

The results concerning the heating of trapped ions can be straightforwardly applied to quantum electrodynamic cavities, or to any system that can be described by a harmonic approximation. In general, any other system must be worked out in a perturbative way. Finally, we hope that the present results can be useful for identifying different sources of decoherence and heating in different systems.

ACKNOWLEDGMENTS

I am grateful to R. L. de Matos Filho and N. Zagury for stimulating conversation, to O. Mensio for discussing the possibility of defining temperature in nonequilibrium NMR systems, and to B. Rohwedder for a critical reading of this manuscript. This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

APPENDIX: FIRST-ORDER PERTURBATION THEORY FOR THE MIXED-STATE FIDELITY

In this appendix we will develop a first-order perturbation theory for the fidelity

$$F(\rho_b,\rho_a)=\{\mathrm{Tr}[(\sqrt{\rho_a}\rho_b\sqrt{\rho_a})^{1/2}]\}^2. \quad \text{(A1)}$$

We will assume that one of the states can be expressed as a power series around the other,

$$\rho_b=\rho_a-\varepsilon\sigma^{(1)}-\varepsilon^2\sigma^{(2)}-\cdots. \quad \text{(A2)}$$

Here the symbol ε denotes the expansion parameter and $\mathrm{Tr}[\sigma^{(m)}]=0$ for all m. In a similar way, we will assume that

$$F(\rho_b,\rho_a)=1-\varepsilon F^{(1)}-\varepsilon^2F^{(2)}-\cdots. \quad \text{(A3)}$$

Inserting Eq. (A2) in Eq. (A1), we get

$$F(\rho_b,\rho_a)=\{\mathrm{Tr}[M^{1/2}]\}^2, \quad \text{(A4)}$$

where the matrix M is given by

$$M=\rho_a^2-\varepsilon\sqrt{\rho_a}\sigma^{(1)}\sqrt{\rho_a}-\varepsilon^2\sqrt{\rho_a}\sigma^{(2)}\sqrt{\rho_a}-\cdots. \quad \text{(A5)}$$

Now, in order to perform the trace operation in Eq. (A4), we need to find the eigenvalues and eigenvectors of the matrix M

$$M=\sum_j P_j(\varepsilon)|\Phi_j(\varepsilon)\rangle\langle\Phi_j(\varepsilon)|. \quad \text{(A6)}$$

From the usual stationary perturbation theory, up to first order, we get

$$P_j(\varepsilon)=(P_j)^2-\varepsilon P_j\langle\Phi_j|\sigma^{(1)}|\Phi_j\rangle+\vartheta(\varepsilon^2) \quad \text{(A7)}$$

and

$$|\Phi_j(\varepsilon)\rangle=|\Phi_j\rangle+\varepsilon|\Phi_j^{(1)}\rangle+\vartheta(\varepsilon^2),$$

where $|\Phi_j^{(1)}\rangle$ is the first-order correction to the eigenvectors and P_j ($|\Phi_j\rangle$) are the eigenvalues (eigenvectors) of the matrix ρ_a,

$$\rho_a|\Phi_j\rangle=P_j|\Phi_j\rangle. \quad \text{(A8)}$$

In this manner, inserting Eq. (A6) in Eq. (A4) results in

$$F(\rho_b,\rho_a)=\left\{\sum_j[(P_j)^2-\varepsilon P_j\langle\Phi_j|\sigma^{(1)}|\Phi_j\rangle+\vartheta(\varepsilon^2)]^{1/2}\right\}^2. \quad \text{(A9)}$$

Maintaining in this expression the first-order contribution, we arrive at

$$F(\rho_b,\rho_a)=1-\varepsilon\sum_j\langle\Phi_j|\sigma^{(1)}|\Phi_j\rangle-\vartheta(\varepsilon^2). \quad \text{(A10)}$$

This equation gives us the desired first-order expansion, whose validity is subjected to the condition $\varepsilon\ll\{P_j\}$. It is important to note that the sum over states is restricted to the eigenvectors of ρ_a with nonzero eigenvalues, i.e., in general, the sum is not the trace operation. In fact, only when the matrix ρ_a has support over the whole Hilbert space, the sum is the trace operation and therefore the first-order contribution cancels.

Now we will apply this result to calculate perturbatively $F(\hat{\rho}(t),\rho(0))$, where the evolution of $\hat{\rho}(t)$ is given by the master equation obtained is Sec. II. First, we expand $\hat{\rho}(t)$ as

$$\hat{\rho}(t)=\hat{\rho}(0)+\frac{d}{dt}\hat{\rho}(t)\bigg|_{t=0}t+\frac{1}{2!}\frac{d^2}{dt^2}\hat{\rho}(t)\bigg|_{t=0}t^2+\cdots. \quad \text{(A11)}$$

Up to second order, the more direct way to obtain the density-matrix derivatives is from the master equation (17). We get

$$\frac{d}{dt}\hat{\rho}(t)\bigg|_{t=0}=0 \quad \text{(A12)}$$

and

$$\frac{d^2}{dt^2}\hat{\rho}(t)\bigg|_{t=0}=-\left(\frac{\lambda}{\hbar}\right)^2\chi_{\alpha\beta}[V_\alpha^\dagger,[V_\beta,\hat{\rho}(0)]]. \quad \text{(A13)}$$

Therefore we can identify $\varepsilon\leftrightarrow t^2$ and $\sigma^{(1)}\leftrightarrow-(1/2)(d^2/dt^2)\hat{\rho}(t)|_{t=0}$. Using the Hermiticity of the stochastic Hamiltonian $\widetilde{H}(t)$, the expressions of Sec. III follow.

[1] F. Haake, *Statistical Treatment of Open Systems by Generalized Master Equations* (Springer, Berlin, 1973).

[2] C. W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).

[3] H. J. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics Vol. m18 (Springer, Berlin, 1993).

[4] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1998).

[5] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).

[6] L. Davidovich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. A **53**, 1295 (1996).
[7] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **77**, 4887 (1996).
[8] J. M. Raimond, M. Brune, and S. Haroche, Phys. Rev. Lett. **79**, 1964 (1997).
[9] C. J. Myatt, B. E. King, Q. A. Turchette, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, Nature (London) **403**, 269 (2000).
[10] Q. A. Turchette, D. Kielpinski, B. E. King, D. Leibfried, D. M. Meekhof, C. J. Myatt, M. A. Rowe, C. A. Sackett, C. S. Wood, W. M. Itano, C. Monroe, and D. J. Wineland, Phys. Rev. A **61**, 063418 (2000).
[11] Q. A. Turchette, C. J. Myatt, B. E. King, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, Phys. Rev. A **62**, 053807 (2000).
[12] C. Di Fidio and W. Vogel, Phys. Rev. A **62**, 031802(R) (2000).
[13] M. Murao and P. L. Knight, Phys. Rev. A **58**, 663 (1998).
[14] S. Schneider and G. J. Milburn, Phys. Rev. A **57**, 3748 (1998).
[15] S. Schneider and G. J. Milburn, Phys. Rev. A **59**, 3766 (1999).
[16] D. F. V. James, Phys. Rev. Lett. **81**, 317 (1998).
[17] S. K. Lamoreaux, Phys. Rev. A **56**, 4970 (1997).
[18] A. Abragam, *The Principles of Nuclear Magnetism* (Oxford University Press, Oxford, 1961).
[19] A. A. Budini, A. K. Chattah, and M. O. Cáceres, J. Phys. A **32**, 631 (1999).
[20] A. A. Budini, Phys. Rev. A **63**, 012106 (2001).
[21] A. Novikov, Zh. Eksp. Teor. Fiz. **47**, 1915 (1964) [Sov. Phys. JETP **20**, 1290 (1965)].
[22] (a) K. Linderberg, B. J. West, and J. Masoliver, in *Noise in Nonlinear Dynamical Systems*, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, 1989), Vol. 1.; (b) J. M. Sancho and M. San Miguel *ibid.*; and (c) P. Hänggi, *ibid.*
[23] W. H. Zurek, S. Habib, and J. P. Paz, Phys. Rev. Lett. **70**, 1187 (1993).
[24] J. H. Kim, M. C. Nemes, A. F. R. de Toledo Piza, and H. E. Borges, Phys. Rev. Lett. **77**, 207 (1996).
[25] J. C. Retamal and N. Zagury, Phys. Rev. A **63**, 032106 (2001).
[26] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[27] B. Schumacher, Phys. Rev. A **54**, 2614 (1996).
[28] Lu-Ming Duan and Guang-Can Guo, Phys. Rev. A **56**, 4466 (1997).
[29] R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).
[30] G. J. Milburn, Phys. Rev. A **44**, 5401 (1991); H. Moya-Cessa, V. Buzek, M. S. Kim, and P. L. Knight, *ibid.* **48**, 3900 (1993).
[31] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[32] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[33] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995).
[34] M. Orszag, *Quantum Optics* (Springer, Berlin, 2000).
[35] M. S. Kim and V. Buzek, Phys. Rev. A **46**, 4239 (1992).
[36] M. O. Cáceres, Phys. Rev. E **60**, 5208 (1999).
[37] In Ref. [16], the mean excitation number and the fidelity of the ground state were calculated maintaining nonresonant contributions. The expression for the first quantity is exactly the same as from our calculations, because the nonresonant terms do not contribute to this quantity. Furthermore, the fidelity of the ground state coincides with our calculation after erasing the antiresonant terms, from which it is possible to see that the nonresonant contributions are small.