# Mixedness in the Bell violation versus entanglement of formation

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Recently, Munro, Nemoto, and White (*The Bell Inequality: A Measure of Entanglement?*, quant-ph/0102119) tried to indicate that the reason behind a state $\rho$ having a higher amount of entanglement (as quantified by the entanglement of formation) than a state $\rho'$, but producing the same amount of Bell violation, is due to the fact that the amount of mixedness (as quantified by the linearized entropy) in $\rho$ is higher than that in $\rho'$. We counter their argument with examples. We extend these considerations to the von Neumann entropy. Our results suggest that the reason as to why equal amount of Bell violation requires different amounts of entanglement cannot, at least, be explained by mixedness alone.

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Werner [1] (see also Popescu [2]) first demonstrated the existence of states that are entangled but do not violate any Bell-type inequality [3,4]. However, there exist classes of states (pure states, mixture of two Bell states), which violate Bell inequality whenever they are entangled [5,6]. This implies that to produce an equal amount of Bell violation, some states require one to have more entanglement (with respect to some measure) than others. It would be interesting to find out what property of the first state requires it to have more entanglement to produce the same Bell violation. Recently, Munro *et al.* [7] have tried to indicate that this anomalous property of the first state is due to its being more *mixed* than the second, where they took the linearized entropy [8] as the measure of mixedness.

As in [7], we use the entanglement of formation as our measure of entanglement. For a state $\rho$ of two qubits, its entanglement of formation $\mathcal{E}(\rho)$ is given by [9]

$$\mathcal{E}(\rho)=h\left(\frac{1+\sqrt{1-\tau}}{2}\right)$$

with

$$h(x)=-x\log_2 x-(1-x)\log_2(1-x).$$

The tangle $\tau$ [10] is given by

$$\tau(\rho)=[\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}]^2,$$

the $\lambda_i$'s being the square root of eigen values of $\rho\tilde{\rho}$, in decreasing order, where

$$\tilde{\rho}=(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y),$$

the complex conjugation being taken in the standard product basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ of two qubits. Note that $E$ is monotonically increasing, ranging from 0 to 1 as $\tau$ increases from 0 to 1 and hence, like Munro *et al.* [7], we take $\tau$ as our measure of entanglement.

The maximum amount of Bell violation ($B$) of a state $\rho$ of two qubits is given by [6]

$$B(\rho)=2\sqrt{M(\rho)},$$

where $M(\rho)$ is the sum of the two larger eigenvalues of $T_\rho T_\rho^\dagger$, $T_\rho$ being the $3\times3$ matrix whose $(m,n)$ element is

$$t_{mn}=\mathrm{tr}(\rho\sigma_n\otimes\sigma_m).$$

The $\sigma$'s are the Pauli matrices.

The linearized entropy [8]

$$S_L(\rho)=\tfrac{4}{3}[1-tr(\rho^2)]$$

is taken as the measure of mixedness.

Munro *et al.* [7] proposed that given two two-qubit states $\rho$ and $\rho'$ with

$$B(\rho)=B(\rho'),$$

but

$$\tau(\rho)>\tau(\rho'),$$

would imply

$$S_L(\rho)>S_L(\rho').$$

To support this proposal, it was shown that it holds for any combination of states from the following three classes of states:

(1) The class of all pure states

$$\rho_{pure}=P[a|00\rangle+b|11\rangle],$$

with $a$, $b\geqslant 0$, and $a^2+b^2=1$.

(2) The class of all Werner states [1]

$$\rho_{werner}=xP[\Phi^+]+\frac{1-x}{4}I_2\otimes I_2,$$

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with $0 \leqslant x \leqslant 1$ and $\Phi^{+} = (1/\sqrt{2})(|00\rangle + |11\rangle)$.

(3) the class of all maximally entangled mixed states [11]

$$\rho_{mems} = \tfrac{1}{2}[2g(\gamma)+\gamma]P[\Phi^{+}] + \tfrac{1}{2}[2g(\gamma)-\gamma]P[\Phi^{-}] + [1 - 2g(\gamma)]P[|01\rangle\langle 01|,$$

with $g(\gamma)=1/3$ for $0<\gamma<2/3$ and $g(\gamma)=\gamma/2$ for $2/3 \leqslant \gamma \leqslant 1$, and $\Phi^{\pm} = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$.

However, consider the class of all mixtures of two Bell states

$$\rho_2 = wP[\Phi^{+}] + (1-w)P[\Phi^{-}],$$

with $0<w<1$. $\rho_2$ is entangled whenever $w \neq \frac{1}{2}$, and for that entire region, $\rho_2$ is Bell violating [6]. For this class, it is easy to show that

$$B = 2\sqrt{1+\tau}.$$

But the corresponding curve for pure states $\rho_{pure}$ is also given by [7]

$$B = 2\sqrt{1+\tau}.$$

We see that for any fixed Bell violation, the corresponding $\rho_2$ has its tangle equal to that for the corresponding pure state. But the mixedness of $\rho_2$ is obviously *larger* than that of the pure state (as the mixedness is always zero for pure states).

Next, consider the following class of mixtures of *three* Bell states

$$\rho_3 = w_1 P[\Phi^{+}] + w_2 P[\Phi^{-}] + w_3 P[\Psi^{+}],$$

with $1 \geqslant w_1 \geqslant w_2 \geqslant w_3 \geqslant 0$, $\Sigma_i w_i = 1$, and $\Psi^{+} = (1/\sqrt{2})(|01\rangle + |10\rangle)$. We take $w_1 > 1/2$ so that $\rho_3$ is entangled [12].

For $\rho_3$, we have (as $w_1 \geqslant w_2 \geqslant w_3$)

$$B(\rho_3) = 2\sqrt{2 - 4w_2(1-w_2) - 4w_3(1-w_3)},$$

$$\tau(\rho_3) = 1 - 4w_1(1-w_1),$$

$$S_L(\rho_3) = \tfrac{4}{3}\{w_1(1-w_1) + w_2(1-w_2) + w_3(1-w_3)\}.$$

Let

$$\rho_3' = w_1' P[\Phi^{+}] + w_2' P[\Phi^{-}] + w_3' P[\Psi^{+}],$$

with $1 \geqslant w_1' \geqslant w_2' \geqslant w_3' \geqslant 0$, $\Sigma_i w_i' = 1$, $w_1' > 1/2$ be such that

$$B(\rho_3) = B(\rho_3'),$$

which gives

$$w_2(1-w_2) + w_3(1-w_3) = w_2'(1-w_2') + w_3'(1-w_3').$$

Now, if

$$\tau(\rho_3) > \tau(\rho_3'),$$

we have

$$w_1(1-w_1) < w_1'(1-w_1')$$

so that

$$w_1(1-w_1) + w_2(1-w_2) + w_3(1-w_3) < w_1'(1-w_1') + w_2'(1-w_2') + w_3'(1-w_3'),$$

that is

$$S_L(\rho_3) < S_L(\rho_3').$$

Thus, for a fixed Bell violation, the order of $S_L$ for $\rho_3$ and $\rho_3'$ is *always* reversed with respect to the order of their $\tau$'s. That is, the indication of [7], referred to earlier, is *always* violated for any two states from the class of mixtures of *three* Bell states.

One can now feel that if the *entanglement of formation of two states are equal*, it could imply some order between the amount of Bell violation and mixedness of the two states; but even that is not true.

For our first example, if

$$\tau(\rho_2) = \tau(\rho_{pure}),$$

then

$$B(\rho_2) = B(\rho_{pure}),$$

but

$$S_L(\rho_2) > S_L(\rho_{pure}).$$

On the other hand, for our second example, if

$$\tau(\rho_3) = \tau(\rho_3'),$$

then

$$B(\rho_3) > B(\rho_3')$$

implies

$$S_L(\rho_3) < S_L(\rho_3').$$

In Ref. [7], the linearized entropy was the only measure of mixedness that was considered. However, the von Neumann entropy [13]

$$S(\rho) = -\mathrm{tr}(\rho \log_4 \rho),$$

of a state $\rho$ of two qubits, is a more physical measure of mixedness than the linearized entropy. We have taken the logarithm to the base four to normalize the von Neumann entropy of the maximally mixed state $(1/2)I_2 \otimes (1/2)I_2$ to unity as it is for the linearized entropy. One may now feel that the conjecture under discussion may turn out to be true if we change our measure of mixedness from linearized entropy to von Neumann entropy. Yet both the von Neumann entropy and the linearized entropy are convex functions, attaining their maximum for the same state $(1/2)I_2 \otimes (1/2)I_2$ and each of them are symmetric about this maximum. Thus,

$$S_L(\rho) > S_L(\rho')$$

would imply

$$S(\rho) > S(\rho')$$

and vice versa. Thus, all our considerations with linearized entropy as the measure of mixedness would carry over to the von Neumann entropy as the measure of mixedness.

Our results emphasize that the reason as to why an equal amount of Bell violation requires different amounts of entanglement cannot, at least, be explained by mixedness alone.

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