Vortex energy and vortex bending for a rotating Bose-Einstein condensate

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For a Bose-Einstein condensate placed in a rotating trap, we give a simplified expression of the Gross-Pitaevskii energy in the Thomas Fermi regime, which only depends on the number and shape of the vortex lines. Then we check numerically, that when there is one vortex line, our simplified expression leads to solutions with a bent vortex for a range of rotational velocities and trap parameters that are consistent with the experiments.

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I. INTRODUCTION

Since the experimental achievement of Bose-Einstein condensates in confined alkali-metal gases in 1995, there has been a huge experimental and theoretical interest in these systems [1–10]. The study of vortices is one of the key issues. Two different groups have obtained vortices experimentally, the JILA group [4] and the ENS group [7,8]. In the ENS experiment, a laser beam is imposed on the magnetic trap holding the atoms to create a harmonic anisotropic rotating potential. For sufficiently large angular velocities, vortices are detected in the system. Experimentally, the ENS group [8] has observed that when the vortex is nucleated, the contrast is not 100%, which means that the vortex line is not straight, but bending. Numerical computations solving the Gross-Pitaevskii equation [11,12] have shown that there is a range of velocities for which the vortex line is indeed bending. The aim of this paper is to justify these observations theoretically in the Thomas-Fermi regime. We define an asymptotic parameter that is small in the Thomas-Fermi regime and approximate the Gross-Pitaevskii energy to obtain a simpler form of the energy that only depends on the shape of the vortex lines. Then we check numerically that our characterization leads to a solution with a bent vortex for a range of values of the rotational velocity that are consistent with the ones obtained numerically [11]. Let us point out that Svidzinsky and Fetter [13] have studied the dynamics of a vortex line depending on its curvature. For a vortex velocity equal to zero, the equation obtained in [13] is the same as the equation corresponding to the minimum of our approximate energy, though the formulation in [13] was not derived from energy considerations. Moreover, their analysis is only valid for a single vortex line.

The Gross-Pitaevskii energy provides a very good description of Bose-Einstein condensates: it is assumed that the N particles of the gas are condensed in the same state for which the wave function ϕ minimizes the Gross-Pitaevskii energy. In the ENS experiment, a laser is applied to the trap that makes it rotate. By introducing a rotating frame at the angular velocity $\tilde{\mathbf{\Omega}}=\tilde{\Omega}\mathbf{e}_z$, the trapping potential becomes time independent, and the wave function ϕ minimizes the energy

$$\mathcal{E}_{3D}(\phi)=\int \frac{\hbar^2}{2m}|\nabla\phi|^2+\hbar\tilde{\mathbf{\Omega}}\cdot(i\phi,\nabla\phi\times\mathbf{x}) + \frac{m}{2}\sum_{\alpha}\ \omega_\alpha^2 r_\alpha^2|\phi|^2+\frac{N}{2}g_{3D}|\phi|^4, \tag{1.1}$$

under the constraint $\int|\phi|^2=1$. Here, for any complex quantities u and v and their complex conjugates $\bar{u}$ and $\bar{v}$, $(u,v)=(u\bar{v}+\bar{u}v)/2$.

We want to nondimensionalize the energy in order to get a parameter that is small in the Thomas-Fermi regime. This framework of study has been developed by one of the authors in [14], except that [14] was a two-dimensional study for a condensate confined in the z axis. We define the characteristic length $d=(\hbar/m\omega_x)^{1/2}$ and assume $\omega_y=\alpha\omega_x$, $\omega_z=\beta\omega_x$. We set

$$\varepsilon^2\sqrt{\varepsilon}=\frac{\hbar^2 d}{2Ngm}=\frac{d}{4\pi Na},$$

where $g_{3D}=4\pi\hbar^2 a/m$. For numerical applications, we are going to use the experimental values of the ENS group [8,11], $m=1.445\times10^{-26}$ kg, $a=5.8\times10^{-11}$ m, $N=1.4\times10^5$ and $\omega_x=1094\ \mathrm{s}^{-1}$ with $\alpha=1.06$, $\beta=0.067$. We find that $\varepsilon=0.0174$, thus, ε is small, which will be our asymptotic regime. We rescale the distance by $R=d/\sqrt{\varepsilon}$ and define $u(\mathbf{r})=R^{3/2}\phi(\mathbf{x})$ where $\mathbf{x}=R\mathbf{r}$ and we set $\Omega=\tilde{\Omega}/\varepsilon\omega_x$. The velocity Ω is chosen such that $\Omega<1/\varepsilon$, that is the trapping potential is stronger than the inertial potential. The energy can be rewritten as

$$E_{3D}(u)=\int \frac{1}{2}|\nabla u|^2+\mathbf{\Omega}\cdot(iu,\nabla u\times\mathbf{r}) + \frac{1}{2\varepsilon^2}(x^2+\alpha^2y^2+\beta^2z^2)|u|^2+\frac{1}{4\varepsilon^2}|u|^4. \tag{1.2}$$

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Due to the constraint $\int|u|^2=1$, we can add to E_{3D} any multiple of $\int|u|^2$ so that it is equivalent to minimize

$$\int \frac{1}{2}|\nabla u|^2+\boldsymbol{\Omega}\cdot(iu,\nabla u\times\mathbf{r})+\frac{1}{4\varepsilon^2}|u|^4-\frac{1}{2\varepsilon^2}\rho_{\mathrm{TF}}(\mathbf{r})|u|^2,$$

where $\rho_{\mathrm{TF}}(\mathbf{r})=\rho_0-(x^2+\alpha^2y^2+\beta^2z^2)$ for some constant ρ_0 to be determined. Let $\mathcal{D}$ be the ellipse $\{\rho_{\mathrm{TF}}>0\}=\{x^2+\alpha^2y^2+\beta^2z^2<\rho_0\}$. We impose the following constraint on ρ_{TF}:

$$\int_{\mathcal{D}}\rho_{\mathrm{TF}}(\mathbf{r})=1. \tag{1.3}$$

Indeed, ρ_{TF} is the Thomas-Fermi approximation of u, that is, for small ε, the minimizer satisfies that $|u|^2$ is close to ρ_{TF} so that the constraint is satisfied automatically by u if we impose Eq. (1.3). Equation (1.3) leads to

$$\rho_0^{5/2}=15\alpha\beta/8\pi. \tag{1.4}$$

To study the problem analytically, it is reasonable to minimize the energy over the domain $\mathcal{D}$ with zero boundary data for u. Indeed, when $\rho_{\mathrm{TF}}\leqslant 0$, the energy is convex so that the minimizer u goes to zero exponentially at infinity (see the numerical observation in [6] and the analysis on the behavior near the boundary of $\mathcal{D}$ as well as the decay at infinity of the order parameter in [15,16]). We consider the problem

$$\min E_\varepsilon(u)\ \text{ subject to} \int_{\mathcal{D}}|u|^2=1 \quad \text{and } u=0 \text{ on } \partial\mathcal{D} \quad (P)$$

where

$$E_\varepsilon(u)=\int_{\mathcal{D}}\frac{1}{2}|\nabla u|^2+\boldsymbol{\Omega}\cdot(iu,\nabla u\times\mathbf{r})+\frac{1}{4\varepsilon^2}[\rho_{\mathrm{TF}}(\mathbf{r})-|u|^2]^2. \tag{1.5}$$

Note that a critical point u of E_ε is a solution of

$$-\Delta u+2i(\boldsymbol{\Omega}\times\mathbf{r})\cdot\nabla u=\frac{1}{\varepsilon^2}u(\rho_{\mathrm{TF}}-|u|^2)+\mu_\varepsilon u \text{ in } \mathcal{D}, \tag{1.6}$$

with $u=0$ on $\partial\mathcal{D}$ and μ_ε is the Lagrange multiplier. The specific choice of ρ_0 in Eq. (1.4) will imply that the term $\mu_\varepsilon u$ is negligible in front of $\rho_{\mathrm{TF}}u/\varepsilon^2$.

We have set the framework of the study of our energy. In Sec. II, we will make an asymptotic expansion of the energy taking into account that ε is small (but $|\log\varepsilon|$ is not big). Then, in Sec. III, we will check that our approximate energy yields a solution that is consistent with the numerical and experimental observations.

II. ASYMPTOTIC EXPANSION OF THE ENERGY

Our aim is to decouple the energy into three terms: a part coming from the profile of the solution without vortices, a vortex contribution, and a term due to rotation.

A. The solution without vortices

First, we are interested in the profile of solutions so that we will study solutions without vortices. Thus, we consider functions of the form $\eta=fe^{iS}$, f is real and does not vanish in the interior of $\mathcal{D}$. We consider first minimizing E_ε over such functions without imposing the constraint that the norm is one, that is, f and S minimize

$$\mathcal{E}_\varepsilon(f,S)=\int_{\mathcal{D}}\frac{1}{2}|\nabla f|^2+\frac{1}{4\varepsilon^2}(\rho_{\mathrm{TF}}-f^2)^2+\frac{1}{2}\int f^2|\nabla S-\boldsymbol{\Omega}\times\mathbf{r}|^2-f^2\Omega^2r^2, \tag{2.1}$$

where $\mathbf{r}=x\mathbf{e}_x+y\mathbf{e}_y$. We have $f_\varepsilon=0$ on $\partial\mathcal{D}$ and

$$-\Delta f_\varepsilon+f_\varepsilon\nabla S_\varepsilon(\nabla S_\varepsilon-2\boldsymbol{\Omega}\times\mathbf{r})=\frac{1}{\varepsilon^2}f_\varepsilon(\rho_{\mathrm{TF}}-f_\varepsilon^2) \text{ in } \mathcal{D}, \tag{2.2}$$

$$\operatorname{div}[f_\varepsilon^2(\nabla S_\varepsilon-\boldsymbol{\Omega}\times\mathbf{r})]=0. \tag{2.3}$$

The continuity Eq. (2.3) implies that there exists $\boldsymbol{\Xi}_\varepsilon$ such that

$$f_\varepsilon^2(\nabla S_\varepsilon-\boldsymbol{\Omega}\times\mathbf{r})=\Omega\operatorname{curl}\boldsymbol{\Xi}_\varepsilon. \tag{2.4}$$

One can think of $\boldsymbol{\Xi}_\varepsilon$ as the equivalent of a stream function in the case of fluid vortices. So, $\boldsymbol{\Xi}_\varepsilon$ is the unique solution of

$$\operatorname{curl}\left(\frac{1}{f_\varepsilon^2}\operatorname{curl}\boldsymbol{\Xi}_\varepsilon\right)=-2 \text{ in } \mathcal{D}, \quad \boldsymbol{\Xi}_\varepsilon=0 \text{ on } \partial\mathcal{D}. \tag{2.5}$$

In the special case where the cross section of $\mathcal{D}$ is a disc, the minimum of Eq. (2.1) is reached for $\nabla S=0$, but this is not the case if the cross section is an ellipse and there is a nontrivial solution of Eq. (2.3). When ε is small, since the ellipticity of the cross section is small, the zero-order approximation of f_ε^2 is ρ_{TF} and the function $\boldsymbol{\Xi}_\varepsilon$ given by Eqs. (2.4) or (2.5) is approximated by the unique solution $\boldsymbol{\Xi}$ of

$$\operatorname{curl}\left(\frac{1}{\rho_{\mathrm{TF}}}\operatorname{curl}\boldsymbol{\Xi}\right)=-2 \text{ in } \mathcal{D}, \quad \boldsymbol{\Xi}=0 \text{ on } \partial\mathcal{D}. \tag{2.6}$$

One can easily get that

$$\boldsymbol{\Xi}(x,y)=-\rho_{\mathrm{TF}}^2(x,y)/(2+2\alpha^2)\mathbf{e}_z. \tag{2.7}$$

Using Eq. (2.4), we can define S_0, the limit of S_ε, to be the solution of $\rho_{\mathrm{TF}}(\nabla S_0-\boldsymbol{\Omega}\times\mathbf{r})=\Omega\operatorname{curl}\boldsymbol{\Xi}$ with zero value at the origin. We have $S_0=C\Omega xy$ with $C=(\alpha^2-1)/(\alpha^2+1)$. We see that S_0 vanishes when $\alpha=1$ that is when the cross section is a disc. This computation is consistent with the one in [9], though it is derived in a different way.

B. Decoupling the energy

Let $\eta_\varepsilon=f_\varepsilon e^{iS_\varepsilon}$ be the vortex free minimizer of E_ε discussed previously without imposing the constraint on the norm of u. Let u_ε be a configuration that will minimize E_ε and let $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$. Since η_ε satisfies the Gross Pitaevskii Eqs. (2.2)–(2.3), we have,

$$\int_{\mathcal{D}} (|v_\varepsilon|^2-1)\left(-\frac{1}{2}\Delta f_\varepsilon^2 - \frac{1}{\varepsilon^2} f_\varepsilon^2(\rho_{\mathrm{TF}}-f_\varepsilon^2) + |\nabla f_\varepsilon e^{iS_\varepsilon}|^2 - 2 f_\varepsilon^2 [\nabla S_\varepsilon \cdot \mathbf{\Omega}\times\mathbf{r}]\right)=0.$$

This trick was introduced in [17] and leads to the following decoupling of the energy $E_\varepsilon(u_\varepsilon)$:

$$E_\varepsilon(u_\varepsilon)=E_\varepsilon(\eta_\varepsilon)+G_{\eta_\varepsilon}(v_\varepsilon)+I_{\eta_\varepsilon}(v_\varepsilon), \qquad (2.8)$$

where

$$G_{\eta_\varepsilon}(v_\varepsilon)=\int_{\mathcal{D}}\frac{1}{2}|\eta_\varepsilon|^2|\nabla v_\varepsilon|^2+\frac{|\eta_\varepsilon|^4}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2,$$

is the energy of vortices and

$$I_{\eta_\varepsilon}(v_\varepsilon)=\int_{\mathcal{D}}|\eta_\varepsilon|^2(\nabla S_\varepsilon - \mathbf{\Omega}\times\mathbf{r})\times(iv_\varepsilon,\nabla v_\varepsilon),$$

is the angular momentum of vortices. The first term in the energy is independent of the solution u_ε, so we have to compute the next two, and find for which configuration u_ε the minimum is achieved. We assume that the solution u_ε has a vortex line along γ, that is u_ε vanishes along γ with a winding number equal to one. Then we expect that the vortex core is of size $\lambda\varepsilon$, that is, away from a tube of size $\lambda\varepsilon$, $|v_\varepsilon|$ is very close to one and only the phase of v_ε is of influence. In the vortex core, the profile of v_ε is given by the cubic NLS equation. We have to determine λ, which is a matching parameter.

Our aim is to estimate the energy of u_ε depending on γ. We use that at zero order $|\eta_\varepsilon|^2$ is approximated by ρ_{TF} when ε is small so that we can approximate G_{η_ε} by $G_{\sqrt{\rho}_{\mathrm{TF}}}=G_\varepsilon$ and I_{η_ε} by $I_{\sqrt{\rho}_{\mathrm{TF}}}=I_\varepsilon$.

C. Estimate of $G_\varepsilon(v_\varepsilon)$

We want to estimate

$$G_\varepsilon(v_\varepsilon)=\int_{\mathcal{D}}\frac{1}{2}|\rho_{\mathrm{TF}}||\nabla v_\varepsilon|^2+\frac{|\rho_{\mathrm{TF}}|^2}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

The mathematical techniques to approximate G_ε have been introduced in [18] in dimension two and in [19] in dimension three, when ε is very small. The problem here is that $\varepsilon = 0.0174$, so that $|\log\varepsilon|$ is not large and there will be additional terms in the asymptotic expansion.

We define

$$T_{\lambda\varepsilon}=\{x\in\mathcal{D} \text{ such that } d(x,\gamma)\leqslant\lambda\varepsilon\}, \qquad (2.9)$$

where d is the distance, and assume that $\lambda\varepsilon$ is small, λ being our matching parameter to be fixed later on. Then we split G_ε into two integrals: one in $T_{\lambda\varepsilon}$, the energy of the vortex core and the other in $\mathcal{D}\backslash T_{\lambda\varepsilon}$, the energy away from the vortex core.

1. Estimate near the vortex core

We are going to estimate G_ε in $T_{\lambda\varepsilon}$. At each point $\gamma(t)$ of γ, we define $\Pi^{-1}(\gamma(t))$ to be the plane orthogonal to γ at $\gamma(t)$. Since $\lambda\varepsilon$ is small, we assume that ρ_{TF} is constant in $\Pi^{-1}(\gamma(t))\cap T_{\lambda\varepsilon}$ and we call the value $\rho_t=\rho_{\mathrm{TF}}(\gamma(t))$. We want to compute

$$\int_{T_{\lambda\varepsilon}}\frac{1}{2}\rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2+\frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 \simeq \int_\gamma \frac{\rho_t}{2}\int_{\Pi^{-1}(\gamma(t))\cap T_{\lambda\varepsilon}}|\nabla v_\varepsilon|^2+\frac{\rho_t}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

This computation is valid as long as $k\lambda\varepsilon$ is small, where k is the curvature of γ. The zero-order approximation of v_ε is given by $u_1(r\sqrt{\rho_t}/\varepsilon)$, where $u_1(r,\theta)=f_1(r)e^{i\theta}$ is the solution with a single zero at the origin of the cubic NLS equation

$$\Delta u + u(1-|u|^2)=0 \text{ in } \mathbb{R}^2.$$

Thus,

$$\begin{aligned}\int_{\Pi^{-1}(\gamma(t))\cap T_{\lambda\varepsilon}}|\nabla v_\varepsilon|^2+\frac{\rho_t}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 &\simeq \int_{B_{\lambda\varepsilon}}\left|\nabla\left(f_1\left(r\sqrt{\frac{\rho_{\mathrm{TF}}}{\varepsilon^2}}\right)e^{i\theta}\right)\right|^2 + \frac{\rho_t}{2\varepsilon^2}\left(1-f_1^2\left(r\sqrt{\frac{\rho_{\mathrm{TF}}}{\varepsilon^2}}\right)\right)^2\\ &= \int_{B_{\lambda\sqrt{\rho_t}}}|\nabla u_1|^2+\frac{1}{2}(1-|u_1|^2)^2\\ &\simeq c_* + 2\pi\ln(\lambda\sqrt{\rho_t}),\end{aligned} \qquad (2.10)$$

where

$$c_* = \int_{\mathbb{R}^2} f_1'^2 + \frac{1}{2}(1-f_1^2)^2 + \int_{\mathbb{R}^2\backslash B_1}\frac{f_1^2-1}{r^2}+\int_{B_1}\frac{f_1^2}{r^2}.$$

The last line of Eq. (2.10) would be an equality if the first two integrals in the expression of c_* were taken in $B_{\lambda\sqrt{\rho_t}}$ instead of $\mathbb{R}^2$. This approximation is correct if $\lambda\sqrt{\rho_t}$ is large (in fact, bigger than three is enough).

The final estimate of this section is

$$G_\varepsilon(v_\varepsilon)_{|T_{\lambda\varepsilon}} \simeq \int_\gamma \rho_{\mathrm{TF}}\left(\frac{c_*}{2}+\pi\log(\lambda\sqrt{\rho_{\mathrm{TF}}})\right)dl. \qquad (2.11)$$

2. Estimate away from the vortex core

We are going to estimate G_ε in $\mathcal{D}\backslash T_{\lambda\varepsilon}$. In this region $|v_\varepsilon|\simeq 1$, and we have seen that $\lambda\sqrt{\rho_t}$ is large, so that only the kinetic energy of the phase has a contribution.

$$\int_{\mathcal{D}\backslash T_{\lambda\varepsilon}}\frac{1}{2}\rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2+\frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2$$

$$\simeq\int_{\mathcal{D}\backslash T_{\lambda\varepsilon}}\frac{1}{2}\rho_{\mathrm{TF}}|\nabla\phi_\varepsilon|^2,$$

where ϕ_ε is the phase of v_ε. Of course, ϕ_ε is not defined everywhere, but we use the analogy with fluid vortices for a line vortex in an otherwise in-viscid incompressible fluid. We let $\boldsymbol{\Psi}$ be a stream function that is div $\boldsymbol{\Psi}=0$ and

$$\mathrm{curl}\,\boldsymbol{\Psi}=\rho_{\mathrm{TF}}\nabla\phi.$$

Then, $\boldsymbol{\Psi}$ is the unique solution of

$$\mathrm{curl}\left(\frac{1}{\rho_{\mathrm{TF}}}\mathrm{curl}\,\boldsymbol{\Psi}\right)=2\pi\boldsymbol{\delta}_\gamma,\quad \boldsymbol{\Psi}=0\ \text{ on }\ \partial\mathcal{D},\tag{2.12}$$

where $\boldsymbol{\delta}_\gamma$ is the vectorial Dirac measure along γ, that is for a vectorial test function $\mathbf{w}$,

$$\langle\boldsymbol{\delta}_\gamma,\mathbf{w}\rangle=\int_\gamma \mathbf{w}\times dl,$$

while δ_γ is the Dirac measure along γ. Thus,

$$\int_{\mathcal{D}\backslash T_{\lambda\varepsilon}}\frac{1}{2}\rho_{\mathrm{TF}}|\nabla\phi_\varepsilon|^2=\int_{\mathcal{D}\backslash T_{\lambda\varepsilon}}\frac{1}{2\rho_{\mathrm{TF}}}|\mathrm{curl}\,\boldsymbol{\Psi}|^2$$

$$=-\frac{1}{2}\int_{\partial T_{\lambda\varepsilon}}\boldsymbol{\Psi}\cdot\nabla\phi_\varepsilon\times\nu,$$

where ν is the outward unit normal to the tube $T_{\lambda\varepsilon}$. We will see that $\boldsymbol{\Psi}$ is almost constant at a distance $\lambda\varepsilon$ from γ and we call this value $\boldsymbol{\Psi}_{\lambda\varepsilon}(\gamma)$. Since the vortex line has a winding number 2π,

$$\int_{\mathcal{D}\backslash T_{\lambda\varepsilon}}\frac{1}{2}\rho_{\mathrm{TF}}|\nabla\phi_\varepsilon|^2\simeq\pi\int_\gamma\boldsymbol{\Psi}_{\lambda\varepsilon}(\gamma)\times dl.$$

We have to compute $\boldsymbol{\Psi}$ on $\partial T_{\lambda\varepsilon}$. The computation is inspired by the paper of Svidzinsky and Fetter [13]. It follows from Eq. (2.12) that $\boldsymbol{\Psi}$ satisfies

$$-\Delta\boldsymbol{\Psi}-\frac{\nabla\rho_{\mathrm{TF}}}{\rho_{\mathrm{TF}}}\times\mathrm{curl}\,\boldsymbol{\Psi}=2\pi\rho_{\mathrm{TF}}\boldsymbol{\delta}_\gamma.$$

Let $x_0\in\gamma$. We denote by $\mathbf{e}_3=\dot{\gamma}(x_0)$ and $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ an orthogonal base in local coordinates. Then, $\boldsymbol{\Psi}$ has coordinates ψ_i in $\mathbf{e}_i$ and the variations of ψ_3 are the only ones of influence in the equation for $\boldsymbol{\Psi}$, since we want to compute $\boldsymbol{\Psi}\times dl$. In the equation for $\boldsymbol{\Psi}$, we neglect the terms in $\nabla\psi_1$ and $\nabla\psi_2$ in front of $\nabla\psi_3$ and we get

$$-\Delta\psi_3+\frac{\nabla\overline{\rho_{\mathrm{TF}}}}{\overline{\rho_{\mathrm{TF}}}}\times\nabla\psi_3=2\pi\rho_{\mathrm{TF}}\delta_\gamma,\tag{2.13}$$

where $\overline{\rho_{\mathrm{TF}}}(x^1,x^2)=\rho_{\mathrm{TF}}(x^1,x^2,x_0^3)$. Let $\xi=\psi_3/\sqrt{\overline{\rho_{\mathrm{TF}}}}$. Then it follows from Eq. (2.13) that ξ satisfies

$$-\Delta\xi+\mu\xi=2\pi\sqrt{\rho_{\mathrm{TF}}}\delta_\gamma,\tag{2.14}$$

where

$$\mu=\sqrt{\overline{\rho_{\mathrm{TF}}}}\Delta\frac{1}{\sqrt{\overline{\rho_{\mathrm{TF}}}}}=\sqrt{\rho_{\mathrm{TF}}}\Delta_\perp\frac{1}{\sqrt{\rho_{\mathrm{TF}}}}.\tag{2.15}$$

Here, $\Delta_\perp$ is the Laplacian in the plane perpendicular to $\mathbf{e}_3=\dot{\gamma}(x_0)$. If the cross section of the condensate $\mathcal{D}$ is a disc, one can compute μ. We denote by θ the angle of $\mathbf{e}_3$ that is $\mathbf{e}_3=\mathbf{e}_r\cos\theta+\mathbf{e}_z\sin\theta$ and (r,z) are the coordinates of x_0 in the original frame. Then,

$$\mu=\frac{(1+\sin^2\theta)+\beta^2\cos^2\theta}{\rho_{\mathrm{TF}}}+\frac{3(r\sin\theta-\beta^2 z\cos\theta)^2}{\rho_{\mathrm{TF}}^2}.\tag{2.16}$$

Note that $\mu>0$. In fact, our numerical computations even yield $\mu>7$. Our aim is now to give an approximate expression for ξ. We locally approximate the curve γ near the point x_0 by the parabola $x=kz^2/2$, where k is the curvature of γ at x_0. This is where we use the same ideas as in [13]. Note that in our approximations, we are only taking into account the shape of γ close to x_0. The justification for this relies on the fact that $\mu>7$ as our numerics show. Indeed, if we solve

$$-\Delta X+\mu X=f,$$

where f is supported at a distance d of x_0. Then using the Green function, we find that

$$|X|\leqslant\frac{e^{-\sqrt{\mu}d}}{4\pi\mu^3 d}.$$

In particular, for $d=0.1$, this gives an error less than 10^{-3}. This is to be compared to the Euler constant and our approximation is reasonable. We rewrite Eq. (2.14) in local coordinates to get

$$-\Delta_\perp\xi+k\partial_{x_1}\xi+\mu\xi=2\pi\sqrt{\rho_{\mathrm{TF}}(x_0)}\,\delta_{\mathbf{e}_3},$$

where $\delta_{\mathbf{e}_3}$ is the Dirac mass supported along the line $\mathbf{e}_3$ and $\mathbf{e}_1$ is the normal to the vortex line γ. Thus,

$$-\Delta\left(e^{-kx_1/2}\xi\right)+\left[\left(\frac{k}{2}\right)^2+\mu\right]\left(e^{-kx_1/2}\xi\right)=2\pi\sqrt{\rho_{\mathrm{TF}}(x_0)}\,\delta_{\mathbf{e}_3}.\tag{2.17}$$

The solution of this equation is

$$\sqrt{\rho_{\mathrm{TF}}(x_0)}K_0\left(\sqrt{\mu+\frac{k^2}{4}}\,\mathscr{d}(x,\gamma)\right),$$

where K_0 is a modified Bessel function. In particular, $K_0(x) \simeq -\ln(e^{C_0}x/2)$ for small x where $C_0 \simeq 0.577$ is the Euler constant. Hence, we deduce

$$\mathbf{\Psi}(x) \simeq -\rho_{\mathrm{TF}} \ln\left(\frac{e^{C_0}}{2}\sqrt{\mu+\frac{k^2}{4}}\mathscr{d}(x,\gamma)\right)\dot{\boldsymbol{\gamma}}. \tag{2.18}$$

Thus, we conclude by the estimate for $G_\varepsilon(v_\varepsilon)$ in $\mathcal{D}\backslash T_{\lambda\varepsilon}$

$$G_\varepsilon(v_\varepsilon)_{|\mathcal{D}\backslash T_{\lambda\varepsilon}} \simeq -\pi\int_\gamma \rho_{\mathrm{TF}} \ln\left(\frac{e^{C_0}}{2}\sqrt{\mu+\frac{k^2}{4}}\lambda\varepsilon\right)dl. \tag{2.19}$$

Here, we have used that $\lambda\varepsilon$ is sufficiently small. In the previous section, we needed $\lambda\sqrt{\rho_t}$ large. The existence of λ is justified by the fact that $\sqrt{\rho_{\mathrm{TF}}}/\varepsilon$ is much bigger than one, except very close to the boundary. But in this region, the contribution of the energy is negligible.

D. Estimate of $I_\varepsilon(v_\varepsilon)$

We want to estimate

$$I_\varepsilon(v_\varepsilon) = \int_{\mathcal{D}} \rho_{\mathrm{TF}}(\nabla S_\varepsilon - \mathbf{\Omega}\times\mathbf{r})\times(iv_\varepsilon, \mathbf{\nabla} v_\varepsilon). \tag{2.20}$$

Recall that the unique solution of Eq. (2.5) satisfies $\rho_{\mathrm{TF}}(\mathbf{\nabla} S_\varepsilon - \mathbf{\Omega}\times\mathbf{r}) = \Omega \,\mathrm{curl}\,\mathbf{\Xi}_\varepsilon$. Hence we integrate by part in Eq. (2.20) to get

$$I_\varepsilon(v_\varepsilon) = \Omega\int_{\mathcal{D}} \mathbf{\Xi}_\varepsilon \times \mathrm{curl}(iv_\varepsilon, \mathbf{\nabla} v_\varepsilon).$$

Let ϕ_ε be the phase of v_ε. Since v_ε is tending to one everywhere except on the vortex line, then $(iv_\varepsilon, \mathbf{\nabla} v_\varepsilon) \sim \mathbf{\nabla}\phi_\varepsilon$, hence, we can approximate $\mathrm{curl}(iv_\varepsilon, \mathbf{\nabla} v_\varepsilon)$ by $2\pi\boldsymbol{\delta}_\gamma$. We use the value of $\mathbf{\Xi}$ given by Eq. (2.7) and the fact that $\dot{\gamma}(t)\times\mathbf{e}_z = dz$ to get

$$I_\varepsilon(v_\varepsilon) \simeq -\frac{\Omega\pi}{(1+\alpha^2)}\int_\gamma \rho_{\mathrm{TF}}^2\,dz. \tag{2.21}$$

E. Final estimate for the energy

We use Eqs. (2.8)–(2.11)–(2.21) to derive the energy of a solution with a vortex line. Indeed, the energy of any solution minus the energy of a solution without vortex is roughly the vortex contribution in the sense

$$E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon) \simeq \mathcal{E}_\gamma. \tag{2.22}$$

We find that the vortex contribution $\mathcal{E}_\gamma$ is

$$\mathcal{E}_\gamma = \int_\gamma \rho_{\mathrm{TF}}\left(\frac{c_*}{2} + \pi\ln\left(\frac{2}{\varepsilon e^{C_0}}\sqrt{\frac{\rho_{\mathrm{TF}}}{\mu+\frac{k^2}{4}}}\right)\right)dl - \frac{\Omega\pi}{(1+\alpha^2)}\int_\gamma \rho_{\mathrm{TF}}^2\,dz. \tag{2.23}$$

Hence, if the right-hand-side of Eq. (2.23) is negative, it means that it is energetically favorable to have vortices. Note that in the first integral of $\mathcal{E}_\gamma$, we have $dl = |\dot{\gamma}(z)|dz$, whereas in the second one, we have dz.

Our approximations rely on the fact that the ellipticity of the cross section is weak and on the fact that ε is sufficiently small. The first hypothesis was used to get that the first-order approximation of $|u_\varepsilon|$ is ρ_{TF}, and to compute μ in Eq. (2.15), whereas corrections are needed if the ellipticity is strong. The second hypothesis was used to get the decoupling of the energy, the existence of the matching parameter λ and the fact that $k\lambda\varepsilon$ is small. In the case where ε is of order 10^{-2}, to get an error on the energy of 10^{-2} and the existence of λ, we need $\rho_{\mathrm{TF}} > 10^{-3}$, that is we are computing the energy away from a small distance of the boundary, but the amount we miss is negligible.

If the vortex line is straight, our computation yields

$$\frac{\rho_0^{3/2}}{\beta}\left[\frac{2}{3}\left(\frac{c_*}{2} + \pi\ln\left(\sqrt{\frac{2}{\varepsilon e^{C_0}}}\right)\right) + \frac{2\pi}{3}\ln\rho_0 + \pi\left(\frac{-10}{9} + \frac{4}{3}\log 2\right) - \Omega\frac{8\pi\rho_0}{15(1+\alpha^2)}\right]. \tag{2.24}$$

This gives a critical angular velocity Ω_1 for which a straight vortex has a lower energy than a vortex-free solution. With our experimental data, it yields $\Omega_1 \sim 22.45$, that is $\tilde{\Omega}_1/\omega_x \sim 0.39$. We are going to see in the numerical section that there is a range of value of Ω less than O_1 for which a bent vortex has a negative energy, in particular, a less energy than a straight vortex.

F. Case of several vortices

Let us assume that the solution u_ε has n vortices along the lines γ_i, $1 \leqslant i \leqslant n$. We want to estimate the energy in this case. For each γ_i, we define $T_{i,\lambda\varepsilon}$ as in Eq. (2.9).

One can check that the estimates (2.21) and (2.11), respectively, for $I_\varepsilon(v_\varepsilon)$ and for $G_\varepsilon(v_\varepsilon)$ close to each vortex core, are unchanged if the integral along γ is replaced by the sum of the integrals along γ_i. The only difference is for the estimate away from the vortex cores where we have to take into account the interaction between the vortex lines. Let us denote $\mathcal{D}_n = \mathcal{D}\backslash\cup_j T_{j,\lambda\varepsilon}$. We still have

$$G_\varepsilon(v_\varepsilon)_{|\mathcal{D}_n} \simeq \int_{\mathcal{D}_n}\frac{1}{2\rho_{\mathrm{TF}}}|\mathrm{curl}\,\mathbf{\Psi}|^2, \tag{2.25}$$

where $\mathbf{\Psi} = \Sigma_i \mathbf{\Psi}_i$ and $\mathbf{\Psi}_i$ solves Eq. (2.12) with γ_i instead of γ. Thus, we need to estimate

$$\sum_i \int_{\mathcal{D}_n}\frac{1}{2\rho_{\mathrm{TF}}}|\mathrm{curl}\,\mathbf{\Psi}_i|^2 + \sum_{i\neq k}\int_{\mathcal{D}_n}\frac{1}{2\rho_{\mathrm{TF}}}\mathrm{curl}\,\mathbf{\Psi}_k \times \mathrm{curl}\,\mathbf{\Psi}_i. \tag{2.26}$$

The first integral is estimated as in Sec. II C 2 by

$$\sum_i -\pi\int_{\gamma_i}\rho_{\mathrm{TF}}\ln\left(\frac{e^{C_0}}{2}\sqrt{\mu+\frac{k^2}{4}}\lambda\varepsilon\right)dl. \tag{2.27}$$

As for the second integral in Eq. (2.26), we integrate it by part to get

$$\pi\sum_{i\neq k}\int_{\gamma_i}\boldsymbol{\Psi}_k\times dl. \tag{2.28}$$

The computation of $\boldsymbol{\Psi}_k(x)$ from Sec. C 2 is still valid and we have $\boldsymbol{\Psi}_k(x)\simeq-\rho_{\rm TF}K_0[\sqrt{\mu+k^2/4}\mathscr{d}(x,\gamma_k)]\dot{\gamma}$. This yields the contribution of n vortex lines [to be compared with Eq. (2.23) for 1 vortex]

$$\mathcal{E}_n=\sum_i\int_{\gamma_i}\rho_{\rm TF}\left[\frac{c_*}{2}+\pi\ln\left(\frac{2}{\varepsilon e^{C_0}}\sqrt{\frac{\rho_{\rm TF}}{\mu+\frac{k^2}{4}}}\right)\right]dl$$
$$-\frac{\Omega\pi}{(1+\alpha^2)}\int_{\gamma_i}\rho_{\rm TF}^2\,dz$$
$$-\pi\sum_{i\neq k}\int_{\gamma_i}\rho_{\rm TF}K_0\left[\sqrt{\mu+\frac{k^2}{4}}\mathscr{d}(x,\gamma_k)\right]dl, \tag{2.29}$$

where K_0 is a modified Bessel function. The extra term in the energy models the interaction between vortex lines. Note that the curves are going to interact only in the region where they are close to one another.

III. COMPARISON WITH THE NUMERICS

We are interested in the shape of the vortex line that minimizes Eq. (2.23) according to the value of Ω. We write $\gamma(t)=(r(t),z(t))$ and we assume that the vortex line is in the plane (y,z). We will denote $\rho_{\rm TF}(t)=\rho_0-\alpha^2r^2(t)-\beta^2z^2(t)$ and we define

$$C(t)=\frac{c_*}{2}+\pi\ln\left(\frac{2}{\varepsilon e^{C_0}}\sqrt{\frac{\rho_0-\alpha^2r^2(t)-\beta^2z^2(t)}{\mu+k^2/4}}\right).$$

Since Eq. (2.23) does not depend on the parametrization $\boldsymbol{\gamma}(t)$, we choose a special parametrization on the curve such that

$$C^2(t)\rho_{\rm TF}^2(t)[\dot{r}^2(t)+\dot{z}^2(t)]=1. \tag{3.1}$$

Then, it is equivalent to minimize

$$\int_\gamma C^2(t)\rho_{\rm TF}^2(t)[\dot{r}^2(t)+\dot{z}^2(t)]dt-\frac{\Omega\pi}{(1+\alpha^2)}\int_\gamma\rho_{\rm TF}^2(t)\dot{z}(t)dt, \tag{3.2}$$

under the constraint (3.1). In our computations below, we will proceed to a minimization of Eq. (3.2) releasing the constraint Eq. (3.1). Indeed, computations show that Eq. (3.1) is true from $t=0$ to t^* where the shape of the vortex is determined. Under the assumption that μ and the curvature do not vary too much along the curve, we derive an equation for the minimum γ

$$\frac{d}{dt}(C^2\rho_{\rm TF}^2\dot{r})=-\frac{2\alpha^2r(t)}{\rho_{\rm TF}(t)}+\frac{2\alpha^2\Omega}{(1+\alpha^2)}\rho_{\rm TF}r(t)\dot{z}(t),$$

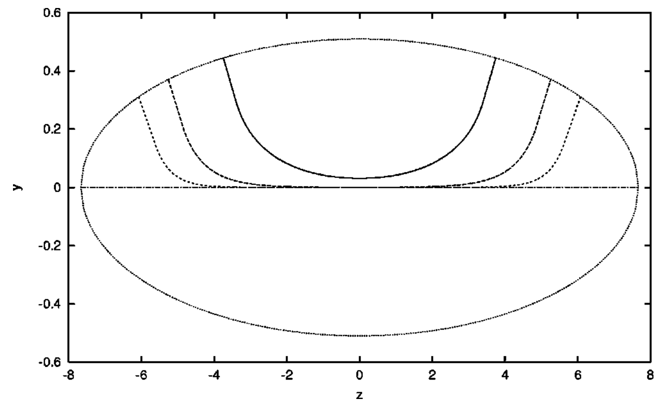

FIG. 1. The vortex line for various values of Ω in the z-y plane: $\Omega=21.8$ (straight line), $\Omega=25.8$ (dotted line), $\Omega=33.1$ (dashed line).

$$\frac{d}{dt}(C^2\rho_{\rm TF}^2\dot{z})=-\frac{2\beta^2z(t)}{\rho_{\rm TF}(t)}-\frac{2\alpha^2\Omega}{(1+\alpha^2)}\rho_{\rm TF}r(t)\dot{r}(t).$$

Thus, we solve this system with initial conditions $r(0)=r_0$, $\dot{r}_0=0$, $z(0)=0$, $C(0)\rho_{\rm TF}(0)\dot{z}(0)=1$.

We let r_0 vary in order to find the minimizing solution. We have drawn the vortex line for the minimizing solution for some values of Ω in Fig. 1. We find that indeed the vortex line is bending for a range of Ω. The bent vortex starts to exist near the boundary of the ellipse, that is $y=\sqrt{\rho_0}/\alpha$, $z=0$ for $\Omega_0=21.2$, that is $\tilde{\Omega}_0/\omega_x=0.368$. As Ω increases, the value of r_0 decreases: $r_0=0.03$ for $\Omega=21.8$, $r_0=2.9\times10^{-4}$ for $\Omega=25.8$, $r_0=10^{-6}$ for $\Omega=33.1$. As Ω increases, r_0 becomes smaller, the bent vortex gets very close to the straight vortex. The shape of the vortex lines are similar to those obtained in [12] using the full Gross Pitaevskii energy.

We plot the energy of the straight vortex line and the bent vortex vs Ω in Fig. 2. One can observe that for $\Omega_c=21.8$, that is $\tilde{\Omega}_c/\omega_x=0.38$ in the initial units, the energy of the bent vortex starts to be negative (that is below the energy of a solution without vortex), while the energy of a straight vortex line is positive. For $\Omega=33.1$, the energy of the bent vortex and of a straight vortex line become equal. These results are consistent with the ones in [11]. They obtain the same value of Ω_c for which the bent vortex has a negative energy.

Let us point out that the bent vortex is a minimizer even if the cross section is a disc. Nevertheless, when ε is fixed, if β gets too big, the straight vortex becomes the minimizer, which is the case for $\beta=1$. Our analysis could give the critical value of β above which the vortex line should be straight.

We believe that our analysis justifies why in the conditions of the ENS experiment, when a vortex is nucleated, the contrast is not 100%: indeed, a bent vortex has a lower energy than a straight vortex. Nevertheless, the velocity of nucleation in the experiment is higher than our critical angular velocity Ω_c: we compute the thermodynamical critical velocity whereas the velocity of nucleation is likely to be closer to the velocity where the vortex-free solution loses

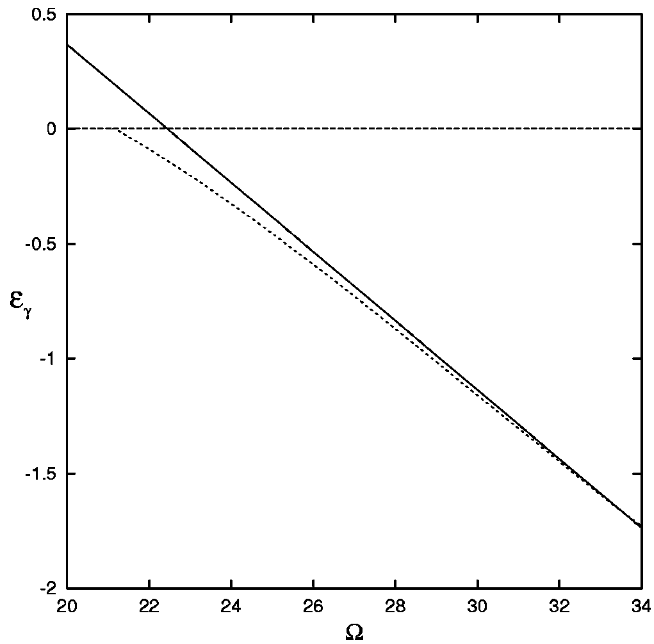

FIG. 2. The energy vs Ω curves for the solution with a straight vortex (solid line) and a bent vortex (dotted line).

local stability. Once the first vortex is obtained experimentally, if Ω is decreased, the bent vortex is likely to exist down to Ω_0.

When there are several vortices, it would be interesting to find numerically the shape of the vortex lines which minimize Eq. (2.29) but we believe that our simplified energy is a good description of the experiments [8] and the numerics [12] and we hope that it can be easier to handle than the full Gross Pitaevskii energy.

IV. CONCLUSION

We have obtained a simplified expression (2.23) of the energy of a minimizing solution of the Gross Pitaevskii energy with a vortex line $\boldsymbol{\gamma}$ and Eq. (2.29) for n vortex lines $\boldsymbol{\gamma}_i$. This expression depends on the shape of the vortex line. It has a term coming from the energy of vortices and another one due to the angular momentum of vortices. This has allowed us to draw the vortex line for the minimizing solution and compute its energy. We have seen that there is a range of rotational velocities for which a bent vortex line has a lower energy than a straight vortex and a vortex-free solution. These computations on the simplified expression of the energy are in agreement with the computations on the full energy [11,12].

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