Schrödinger transmission through one-dimensional complex potentials

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We prove that for a symmetric complex potential, the probabilities of quantal reflection and transmission of a particle are independent of the direction of incidence of the particle. However, the reflectivity for a nonsymmetric complex potential is found to be sensitive to the direction of incidence of the particle whether it is incident from left or right.

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Nucleus-nucleus elastic-scattering results in models with an added finite absorptive part to the short-ranged attractive nuclear interaction potential are better understood than results from models without such an absorptive term. The absorptive part of the potential is represented by a purely imaginary potential of the short-range type or the one that rapidly converges to zero. This absorptive potential is supposed to represent, in a crude way, the unknown (nonelastic) channels, which preferably remove some flux and reduce the elastically scattered flux. The relevant model, called the optical model [1], is found to be phenomenologically suitable for measuring the elastic scattering. In the optical model, the Hamiltonian is non-Hermitian and hence the condition for unitarity of the $S(E)$-matrix, i.e. $|S(E)|=1$, is replaced by the condition for pseudounitarity, i.e., $|S(E)|<1$.

A simpler but more intuitive model of optical scattering would, of course, be the Schrödinger transmission through a one-dimensional complex potential, $V_c(x)=V_r(x)-iV_i(x)$ [3]. For a stationary scattering state, the conservation of probability flux can be written as [1,3]

$$\frac{dJ}{dx}=-\frac{2}{\hbar}V_i(x)\Psi^*(x)\Psi(x), \tag{1}$$

where

$$J(x)=\frac{\hbar}{2im}\left(\Psi^*\ \frac{d\Psi(x)}{dx}-\Psi(x)\ \frac{d\Psi^*(x)}{dx}\right)$$

is the probability flux. Therefore, it is essential that $V_i(\pm\infty)=0$. In other words, $V_i(x)$ should either be of finite support or converging asymptotically. The probability of absorption, $A(E)$, is defined as

$$A(E)=\frac{J(-\infty)-J(\infty)}{J_{inc}}, \tag{2}$$

which becomes

$$A(E)=\frac{2m}{\hbar^2k}\int_{-\infty}^{\infty}V_i(x)|\Psi(x)|^2\ \ dx. \tag{3}$$

A negative value of A would indicate emission. However, a more interesting idea would be to connect the reflectivity, $R(E)$, and the transmitivity, $T(E)$. Let us insert $\Psi_{-\infty}=\exp(ikx)+r\exp(-ikx)$, $\Psi_{\infty}=t\exp(ikx)$, and $\Psi_{\rm inc}=\exp(ikx)$ in the above equation. Noting that $R(E)=r^*r$ and $T(E)=t^*t$, we obtain

$$R(E)+T(E)=1-A(E)\neq 1. \tag{4}$$

Thus, the condition for unitarity on the reflection $R(E)$ and the transmission $T(E)$ probabilities, i.e., $R(E)+T(E)=1$, is replaced by pseudounitarity, as given in Eq. (4). Interesting applications of transmission through complex one-dimensional potentials are found in Refs. [2–7]. Analogously, on can see a lot of interesting results and applications [9] of the propagation of (s-polarized) electromagnetic waves in a planer-stratified dielectric medium.

When the scattering potential is purely real, the reflection and transmission probabilities R and T are told to be independent of the direction (left or right) of incidence of the particle impinging on the potential. These are supposed to be interesting consequences of invariance of the Hamiltonian under time reversal [8,10]; see also pp. 34–37 and 249–251 in Ref. [9]. For a time-independent Hamiltonian, time-reversal invariance means that the Hamiltonian should not change when i is changed to $-i$. For a real potential, the said invariance is automatic.

In this paper, when the complex potential is symmetric, we prove that $T_{\rm left}(E)=T_{\rm right}(E)$ and $R_{\rm left}(E)=R_{\rm right}(E)$, and when it is nonsymmetric, we prove that $T_{\rm left}(E)=T_{\rm right}(E)$ but $R_{\rm left}(E)\neq R_{\rm right}(E)$. In both the cases, it is the pseudounitarity condition Eq. (4), instead of the unitarity, which holds. Here, the subscripts left/right indicate the direction of incidence of the particle impinging on the potential. In the above-mentioned physical situations, it is the loss of flux (rather than the creation of flux) that needs to be accounted for and hence one is more often concerned with the complex potentials that are nonpositive (absorptive) or nonnegative (emissive). However, the results mentioned above will also hold even for imaginary potentials that change sign in the domain of $x\in[-\infty,\infty]$.

It is worth noting that for the propagation of electromagnetic waves through a stratified medium, one generally speaks about transmission (reflection) from one side of the medium to the other, and the invariance of the reflectivity means $R_{1,2}=R_{2,1}$ if the medium is nonabsorbing (see pp.160 and 161 of Ref. [9]). In this regard, one of the very important points to be made is that the symmetry of the medium, even

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if it has imaginary components, determines the left and the right invariance of the reflectivities. For instance, if N denotes the nonabsorbing medium and A denotes the absorbing medium, the left and the right reflectivities of a (symmetric) stratification, $NAAN$, will be the same. But the left and the right reflectivities for the nonsymmetric stratifications—$NAA'N$, $NAAN'$, $NANA$, $NAA'N'$, and $NAN'A'$—will be different. However, the left and the right transmittivities will be the same for all these (symmetric and nonsymmetric) stratifications.

The time-independent Schrödinger equation for a general complex potential for the incident particle of mass m can be written as

$$\frac{d^2\Psi(x)}{dx^2}+\frac{2m}{\hbar^2}[E-V_r(x)+iV_i(x)]\Psi(x)=0, \tag{5}$$

where $V_i(\pm\infty)=0$. It may be emphasized that $V_i(x)$ need not be only positive or only negative in the entire domain of $x\in[-\infty,\infty]$. For practical purposes, let us assume that both $V_r(x)$ and $V_i(x)$ are appreciable only in $[-D,D]$, where D is some large asymptotic distance.

Case I : Left incidence

In this case, we can assume the wave function $\Psi(x)$ as

$$\Psi(x<-D)=\mathcal{A}^L\exp[ik(x+D)]+\mathcal{B}^L\exp[-ik(x+D)], \tag{6a}$$

$$\Psi(-D\leqslant x\leqslant D)=\alpha u(x)+\beta v(x), \tag{6b}$$

where $u(x)$ and $v(x)$ are two linearly independent solutions of Eq. (5). They are such that $u(0)=1$, $u'(0)=0$; $v(0)=0$, $v'(0)=1$. Here, the prime indicates first derivative with respect to x. Next, we have

$$\Psi(x>D)=\mathcal{C}^L\exp[ik(x-D)]. \tag{6c}$$

As usual, by matching the wave function and its derivative at $x=\pm D$, we will get the scattering amplitudes as $r_{\rm left}(E)=\mathcal{B}^L/\mathcal{A}^L$ and $t_{\rm left}(E)=\mathcal{C}^L/\mathcal{A}^L$. Let us use the subscript "1" to denote the value of the function at $x=-D$ and "2" for the value of the same at $x=D$. Then we obtain

$$r_{\rm left}(E)=-\frac{[u_2'v_1'-u_1'v_2']+ik[v_2\ u_1'+u_1\ v_2']-ik[u_2\ v_1'+v_1\ u_2']+k^2[u_1\ v_2-u_2\ v_1]}{[u_2'\ v_1'-u_1'\ v_2']+ik[v_2\ u_1'-u_1\ v_2']-ik[u_2\ v_1'-v_1\ u_2']-k^2[u_1\ v_2-u_2\ v_1]}, \tag{7a}$$

$$t_{\rm left}(E)=\frac{-2ik[u_2\ v_2'-v_2\ u_2']}{[u_2'v_1'-u_1'\ v_2']+ik[v_2\ u_1'-u_1\ v_2']-ik[u_2\ v_1'-v_1\ u_2']-k^2[u_1\ v_2-u_2\ v_1]}. \tag{7b}$$

The Wronskian function, $W_2=[u_2v_2'-v_2u_2']$, which is the constant of the process, equals unity. The chosen large distance D is arbitrary since the potentials in Eq. (5) are not actually of finite support. However, this is done so as to extend the utility of Eqs. (7a) and (7b), in general, for the potentials that have a fairly rapid asymptotic convergence. Thus, by including D in the boundary conditions (6a) and (6c), we have gotten rid of the arbitrary phase factors, such as $\exp(\pm 2ikD)$, which might have otherwise appeared (Eqs. [25] and [26] in Ref. [9]) in the scattering amplitudes (7).

Case II: Right incidence

For the right incidence of the particle, we choose

$$\Psi(x<-D)=\mathcal{C}^R\exp[-ik(x+D)], \tag{8a}$$

$$\Psi(-D\leqslant x\leqslant D)=\gamma u(x)+\delta v(x), \tag{8b}$$

$$\Psi(x>D)=\mathcal{A}^R\exp[-ik(x-D)]+\mathcal{B}^R\exp[ik(x-D)]. \tag{8c}$$

As above, we obtain the scattering amplitudes $r_{\rm right}(E)=\mathcal{B}^R/\mathcal{A}^R$ and $t_{\rm right}(E)=\mathcal{C}^R/\mathcal{A}^R$ as

$$r_{\rm right}(E)=-\frac{[u_2'v_1'-u_1'\ v_2']+ik[v_1\ u_2'+u_2\ v_1']-ik[u_1\ v_2'+u_1'\ v_2]+k^2[u_1\ v_2-u_2\ v_1]}{[u_2'\ v_1'-u_1'\ v_2']+ik[v_1\ u_2'-u_2\ v_1']-ik[u_1\ v_2'-u_1'\ v_2]-k^2[u_1\ v_2-u_2\ v_1]}, \tag{9a}$$

$$t_{\rm right}(E)=\frac{-2ik[u_1\ v_1'-u_1'\ v_1]}{[u_2'v_1'-u_1'\ v_2']+ik[v_1\ u_2'-u_2\ v_1']-ik[u_1\ v_2'-u_1'\ v_2]-k^2[u_1\ v_2-u_2\ v_1]}. \tag{9b}$$

The Wronskian function $W_1=[u_1v_1'-u_1'v_1]=1$. Notice that the transmission amplitudes in Eqs. (7b) and (9b) are identical. Thus, the insensitivity of both the transmission amplitude and the transmission coefficient to the direction of incidence is proved,

$$t_{\text{left}}(E)=t_{\text{right}}(E),\quad T_{\text{left}}(E)=T_{\text{right}}(E), \tag{10}$$

irrespective of whether the potential is symmetric, nonsymmetric, real, of complex. For real potentials, this result is known as a consequence of time-reversal symmetry of the Hamiltonian [8]. The result in Eq. (10) is consistent with one of the results of Abeles [11], which has also been discussed in Chap. 12-6 of Ref. [9].

When the potential is symmetric (real or complex), the function $u(x)$ is of even parity and $v(x)$ is of odd parity. Consequently, we have $u_2=u_1$, $v_2=-v_1$, $u_2'=-u_1'$, $v_2'=v_1'$. The second and third terms in the numerator of Eq. (7a) cancel each other as the square brackets in them become Wronskian functions. Similarly, the second and third terms in the numerator of Eq. (9a) cancel each other. Finally, we get both $r_{\text{left}}(E)=r_{\text{right}}(E)$ and

$$R_{\text{left}}(E)=R_{\text{right}}(E) \tag{11}$$

for symmetric potentials irrespective fact whether they are real or complex. It must be noted that when the potential is complex, the time-reversal symmetry of the Hamiltonian is broken.

Next, when the potential is real but nonsymmetric, all the square bracketed terms in Eqs. (7) and (9) are real and we have $r_{\text{left}}(E)\neq r_{\text{right}}(E)$, a result that is known [10]. However, the modulus of Eqs. (7a) and Eqs. (9a) will be identical, justifying once again Eq. (11) for the invariance of the reflectivity with respect to the direction of incidence.

The case of the complex potential is the most interesting in that all the square bracketed terms will essentially be complex quantities. As a serious consequence of this simple fact, the modulus of Eqs. (7a) and (9a) will always be unequal. Eventually, both the reflection amplitude and the reflectivity of the nonsymmetric complex potential will be sensitive to the direction of incidence whether it is left or right, i.e.,

$$R_{\text{left}}(E)\neq R_{\text{right}} \quad \text{if} \quad V_c(-x)\neq V_c(x). \tag{12}$$

There is another way of stating the above-mentioned results: the left reflectivity of the complex potential, $V_c(x)$, is equivalent to the right reflectivity of $V_c(-x)$ but the left and the right reflectivities of $V_c(x)$ will be different if $V_c(-x)\neq V_c(x)$.

Schrödinger transmission from one-dimensional potentials is a well-researched fundamental topic in several branches of physics. Let us ask whether the expositions presented here in Eqs. (10), (11), and (12) have been expected if not stated or proved earlier. In the literature [8], using the time-reversal invariance of the Hamilitonian (real potential), one shows that $T_{\text{left}}(E)=T_{\text{right}}(E)$, and since time-reversal invariance ensures unitarity, i.e., $R_{\text{left}}(E)+T_{\text{left}}(E)=1=R_{\text{right}}(E)+T_{\text{right}}(E)$, the invariance, $R_{\text{left}}(E)=R_{\text{right}}(E)$, follows automatically. On the other hand, for complex potentials the time-reversal symmetry is broken and the unitarity is withdrawn. The essential invariance $T_{\text{left}}(E)=T_{\text{right}}(E)$ in Eq. (10) only leaves the question of dependence of reflectivity on the direction of incidence open for investigations. The transmission is a two-sided ("symmetric") process, i.e., the particle to be transmitted enters from one side of the potential and exits from an other side to carry away the combined effect of the potential on both sides, hence its invariance [Eq. (10)] could be quite intuitive. On the other hand, reflection is a one-sided ("asymmetric") process and thus the variance of the left and right reflectivity may not be counterintuitive. The experimental evidence of the directionally asymmetric production of β particles from a polarized nucleus is known to have required a parity-violating part in the interaction Hamiltonian. Given this well-known phenomenon, the dependence of the reflectivity on the direction of incidence [Eq. (12)] for the nonsymmetric (parity-violating) complex potential may not be unexpected.

To summarize, both the scattering amplitudes and the scattering coefficients have been found to be independent of the direction of incidence of the particle when the potential is symmetric. The reflection amplitudes for a nonsymmetric potential have been found to be sensitive to the direction of incidence. These two results hold irrespective of whether the potential is real or complex. It is surprising to note that the time-reversal symmetry of the Hamiltonian does not have any role to play with regard to the (in) sensitivity of the scattering amplitudes to the direction of incidence. Instead, it is the parity (symmetry/nonsymmetry) of the Hamiltonian that determines the sensitivity of the scattering amplitudes to the direction of incidence of the particle. The time-reversal symmetry does, however, play a role in determining whether the Schrödinger transmission entails the unitarity or the pseudounitarity [Eq. (4)]. We would like to remark that the presently proved result of the dependence of the reflectivity on the direction of incidence for a nonsymmetric complex potential could be crucial to unraveling the physical truth behind the (in)sensitivity of the scattering co-efficients to the direction of incidence.

Analogously, for the propagation of electromagnetic waves through a stratified dielectric medium, the present study implies that the equivalence of the left and the right reflectivities of a stratification does not, as usual (e.g., see the first line on p. 161 of Ref. [9]), necessarily mean that the medium is nonabsorbing. It may also mean that the medium is absorbing but symmetric. Moreover, in general, if the reflectivity measurements display a difference in the left and the right reflectivities, the present study predicts that such a dielectric medium must be nonsymmetric and absorptive/emissive.

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