Optimal nonuniversally covariant cloning

G. M. D'Ariano and P. Lo Presti
Theoretical Quantum Optics Group, Università degli Studi di Pavia and INFM Unità di Pavia, Via A. Bassi 6, I-27100 Pavia, Italy

(Received 29 January 2001; published 12 September 2001)

We consider nonuniversal cloning maps, namely, cloning transformations that are covariant under a proper subgroup **G** of the universal unitary group $U(d)$, where d is the dimension of the Hilbert space $\mathcal{H}$ of the system to be cloned. We give a general method for optimizing cloning for any cost function. Examples of applications are given for phase-covariant cloning (cloning of equatorial qubits and qutrits) and for the Weyl-Heisenberg group (cloning of "continuous variables").

DOI: 10.1103/PhysRevA.64.042308 PACS number(s): 03.67.−a

I. INTRODUCTION

The impossibility of perfectly cloning an unknown input state is a typical quantum feature [1]; nonetheless, in the laws of quantum mechanics there is enough room either to systematically produce approximate copies [2] or to make perfect copies of orthogonal states [3] or of nonorthogonal ones with a nonunit probability [4]. These possibilities have been studied in several works [5–7].

Recently, quantum cloning has entered the realm of experimental physics [8,9]. Moreover, it has become interesting from a practical point of view, since it can be used to speed up some quantum computations [10] or to perform some quantum measurements [11,12]. All these tasks require a spreading of the quantum information contained in a system into a larger system, and quantum cloning is a way to achieve such a spreading.

In this paper we will see how any "spreading" corresponds to a particular completely positive (CP) map. By exploiting the correspondence between CP maps and positive operators on the tensor product of the output and input spaces [13], we can parametrize all the possible spreading transformations. Then we focus on covariant CP maps, showing that quantum cloning is a particular case of permutation covariance. By means of Schur's lemmas we completely characterize the positive operators corresponding to quantum cloning transformations. By the same technique, we characterize **G**-covariant cloning transformations, where **G** is any single-copy covariance group.

The parametrization of CP maps, and in particular of cloning and covariant cloning, stands at the base of any further optimization. In fact, quantum cloning can be used to perform some tasks on the copies, and, depending on what these copies will be used for, one defines a "goodness" criterion for the cloning process and optimizes accordingly.

The paper is organized as follows. In Sec. II, we briefly describe a quantum cloning transformation and its relation to CP maps. Section III is devoted to the description of CP maps in terms of positive operators, while in Sec. IV we treat the case of covariant CP maps, giving their parametrization with suitable covariant positive operators. In Sec. V we use the previously explained techniques to deal with cloning optimization, focusing on the covariant case.

II. CLONING TRANSFORMATIONS

In a quantum cloning transformation, the input state $\rho \in \mathcal{L}(\mathcal{H})$ is processed in order to produce N output clones [throughout the paper $\mathcal{L}(\mathcal{H})$ will denote the vector space of linear bounded operators on the Hilbert space $\mathcal{H}$]. This requires a "spreading" of ρ into the joint state $\rho' \in \mathcal{L}(\mathcal{H}^{\otimes N})$ of N identical quantum systems. The most general setup for such purpose is the following. Initially, ρ is encoded in a quantum system S_1, while $N-1$ equivalent systems S_i, $i=2,\ldots,N$, are prepared in a fixed state $|\omega\rangle_{(N-1)}$. An auxiliary system E is provided in a state $|e\rangle$, in order to make the whole system isolated. A unitary transformation U acts on the overall state producing the output

$$\Lambda = U\,\rho\otimes(|\omega\rangle\langle\omega|)_{(N-1)}\otimes|e\rangle\langle e|\,U^{\dagger}. \tag{1}$$

By taking the partial trace of Λ on the auxiliary system, we get the joint state ρ' of the N output systems S_i. This state will eventually support the clones. Upon calculating the trace with respect to a chosen basis $\{|j\rangle_E\}$ for $\mathcal{H}_E$ one has

$$\rho' = \sum_{j=1}^{\dim \mathcal{H}_E} {}_E\langle j|\Lambda|j\rangle_E = \sum_{j=1}^{\dim \mathcal{H}_E} A_j \rho A_j^{\dagger} \doteq \mathcal{E}(\rho), \tag{2}$$

where $A_j = {}_E\langle j|U|\omega\rangle_{(N-1)}|e\rangle_E$.

The map $\rho \rightarrow \mathcal{E}(\rho)$ in Eq. (2) is a completely positive and trace-preserving linear map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}^{\otimes N})$. Trace-preserving CP maps generally describe the evolution of open quantum systems. To understand the general features of quantum cloning and for the sake of optimization, it is convenient to treat these maps at an abstract level: a realization theorem guarantees that any CP map can be achieved as a unitary transformation on an extended Hilbert space [16,17], similarly to Eq. (1). CP maps will be briefly reviewed in the next section.

III. CP MAPS AND POSITIVE OPERATORS

A linear map $\mathcal{E}:\mathcal{L}(\mathcal{H})\rightarrow\mathcal{L}(\mathcal{K})$ is completely positive if its trivial extension $\mathcal{E}\otimes\mathcal{I}_{\mathcal{H}'}$ to $\mathcal{L}(\mathcal{H}\otimes\mathcal{H}')$ is positive, for any $\mathcal{H}'$ [$\mathcal{I}_{\mathcal{H}'}$ denoting the trivial map on $\mathcal{L}(\mathcal{H}')$].

Here we recall a convenient notation [14]. Fixing two orthonormal bases $\{|i\rangle_1\}$ and $\{|j\rangle_2\}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, any vector $|\Psi\rangle\rangle \in \mathcal{H}_1\otimes\mathcal{H}_2$ can be written as

$$|\Psi\rangle\rangle=\sum_{ij} c_{ij}|i\rangle_1|j\rangle_2\doteq|C\rangle\rangle, \tag{3}$$

where $C=\Sigma_{ij}c_{ij}|i\rangle_1\,{}_2\langle j|\in\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)$ is a linear bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. The following relations can easily be verified:

$$A\otimes B|C\rangle\rangle=|ACB^T\rangle\rangle, \tag{4}$$

$$\mathrm{Tr}_{\mathcal{H}_2}[|A\rangle\rangle\langle\langle B|]=AB^\dagger\in\mathcal{L}(\mathcal{H}_1). \tag{5}$$

For every CP map $\mathcal{E}:\mathcal{L}(\mathcal{H})\rightarrow\mathcal{L}(\mathcal{K})$ we define the positive operator $R_\mathcal{E}$ in $\mathcal{L}(\mathcal{K}\otimes\mathcal{H})$

$$R_\mathcal{E}\doteq\mathcal{E}\otimes\mathcal{I}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|), \tag{6}$$

where $\mathcal{I}$ denotes the identical map over the extension space $\mathcal{H}$, and for the vector $|\mathbb{1}\rangle\rangle\in\mathcal{H}\otimes\mathcal{H}$ we used the notation (3) for $\Psi=\mathbb{1}$, the identity matrix with respect to a fixed basis on $\mathcal{H}$. The action of $\mathcal{E}$ on $\rho\in\mathcal{L}(\mathcal{H})$ can be expressed as

$$\mathcal{E}(\rho)=\mathrm{Tr}_\mathcal{H}[\mathbb{1}\otimes\rho^T R_\mathcal{E}], \tag{7}$$

where the transposition $\rho\rightarrow\rho^T$ is performed with respect to the same fixed basis. In fact, substituting Eq. (6) in Eq. (7), one has

$$\mathrm{Tr}_\mathcal{H}[\mathbb{1}\otimes\rho^T R_\mathcal{E}]=\mathrm{Tr}_\mathcal{H}[\mathbb{1}\otimes\rho^T\mathcal{E}\otimes\mathcal{I}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)].$$

Then it is possible to take the factor $\mathbb{1}\otimes\rho^T$ inside the CP map $\mathcal{E}\otimes\mathcal{I}$, since they act independently on different spaces. By applying Eq. (4), one obtains

$$\mathrm{Tr}_\mathcal{H}[\mathbb{1}\otimes\rho^T R_\mathcal{E}]=\mathrm{Tr}_\mathcal{H}[\mathcal{E}\otimes\mathcal{I}(|\rho\rangle\rangle\langle\langle\mathbb{1}|)],$$

and thus, commuting the partial trace with $\mathcal{E}\otimes\mathcal{I}$ and using Eq. (5), one finally gets Eq. (7).

The operator $R_\mathcal{E}$ is the only one for which Eq. (7) holds true. In fact, suppose $R_\mathcal{E}$ and R' give the same CP map $\mathcal{E}$ by means of Eq. (7); then

$$\mathrm{Tr}_\mathcal{H}[\mathbb{1}\otimes\rho^T(R_\mathcal{E}-R')]=0\in\mathcal{L}(\mathcal{K}),\quad\forall\rho\in\mathcal{L}(\mathcal{H}).$$

Since an operator $O\in\mathcal{L}(\mathcal{H}\otimes\mathcal{K})$ is null if $\langle v|O|v\rangle=0\in\mathcal{L}(\mathcal{K})$ for all $|v\rangle\in\mathcal{H}$, it follows that $R_\mathcal{E}=R'$. Thus, the correspondence from CP maps to positive operators is "into."

Since $R_\mathcal{E}$ is positive, it can be written as

$$R_\mathcal{E}=\sum_i |A_i\rangle\rangle\langle\langle A_i|, \tag{8}$$

where there are many different choices of the vectors $|A_i\rangle\rangle$, which are not necessarily eigenvectors of $R_\mathcal{E}$, and generally are not normalized. Substituting this relation in Eq. (7) and remembering that $A_i\in\mathcal{L}(\mathcal{H},\mathcal{K})$, we find

$$\mathcal{E}(\rho)=\sum_i \mathrm{Tr}_\mathcal{H}[\mathbb{1}\otimes\rho^T|A_i\rangle\rangle\langle\langle A_i|]=\sum_i A_i\rho A_i^\dagger, \tag{9}$$

thus recovering the result that any CP map admits different Kraus's decompositions [16], depending on the choice of the vectors $|A_i\rangle\rangle$ in Eq. (8).

Clearly, Eq. (8) holds for any positive operator R on $\mathcal{K}\otimes\mathcal{H}$. The map defined by R through Eq. (7) is completely positive, since it can be expressed in the form of Eq. (9), which trivially gives a CP map. Thus the correspondence from CP maps to operators is also "onto."

Concluding, Eq. (7) defines a one-to-one correspondence between CP maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ and positive operators on $\mathcal{K}\otimes\mathcal{H}$. By exploiting this correspondence, properties of $\mathcal{E}$ can be translated into properties of $R_\mathcal{E}$. For example, the trace-preserving condition for $\mathcal{E}$,

$$\mathrm{Tr}_\mathcal{K}[\mathcal{E}(\rho)]=1=\mathrm{Tr}_\mathcal{H}[\rho^T\mathrm{Tr}_\mathcal{K}[R_\mathcal{E}]],$$

for all $\rho\in\mathcal{L}(\mathcal{H})$ such that $\mathrm{Tr}[\rho]=1$, becomes

$$\mathrm{Tr}_\mathcal{K}[R_\mathcal{E}]=\mathbb{1}\in\mathcal{L}(\mathcal{H}). \tag{10}$$

In the following, it will be useful to consider the dual map $\mathcal{E}^\vee$ of a CP map $\mathcal{E}$, namely, the transformation in the Heisenberg picture versus the Schrödinger picture map $\rho\rightarrow\mathcal{E}(\rho)$. The dual map $\mathcal{E}^\vee$ is defined by the identity

$$\mathrm{Tr}[\rho\mathcal{E}^\vee(O)]=\mathrm{Tr}[\mathcal{E}(\rho)O], \tag{11}$$

which must be valid for all operators $O\in\mathcal{L}(\mathcal{K})$. In terms of the operator $R_\mathcal{E}$ one has

$$\mathcal{E}^\vee(O)=\mathrm{Tr}_\mathcal{K}[O\otimes\mathbb{1}R_\mathcal{E}^{T_\mathcal{H}}], \tag{12}$$

where $T_\mathcal{H}$ denotes partial transposition on the Hilbert space $\mathcal{H}$ only [15].

In the next section, the correspondence $\mathcal{E}\leftrightarrow R_\mathcal{E}$ will be applied to the covariance condition for a CP map, which turns out to be the key idea for ring deal with cloning and covariant cloning.

IV. COVARIANT CP MAPS

To talk about covariance, we must first give some important definitions (see, for example, Ref. [18]). A unitary (projective) representation U of the group $\mathbf{G}$ on $\mathcal{H}$ is a homomorphism associating any element $g\in\mathbf{G}$ with a unitary transformation $U_g\in\mathcal{L}(\mathcal{H})$ in such a way that the composition law of the group is preserved under the correspondence, i.e.,

$$U_{g_1}U_{g_2}=\omega(g_1,g_2)U_{g_1g_2}, \tag{13}$$

where $|\omega(g_1,g_2)|=1$ are the so called cocycles, and they satisfy the following restrictions:

$$\omega(g_1g_2,g_3)\omega(g_1,g_2)=\omega(g_1,g_2g_3)\omega(g_2,g_3),$$
$$\omega(g,g^{-1})=\omega(g,e)=1. \tag{14}$$

A unitary representation is *irreducible* (UIR) if there are no proper subspaces left invariant by the action of all its elements. Two $\mathbf{G}$ representations U on $\mathcal{H}$ and V on $\mathcal{K}$ are equivalent if there exists an isomorphism (a one-to-one and norm-preserving linear correspondence) $I:\mathcal{H}\rightarrow\mathcal{K}$, such that $IU_g=V_gI$ for any $g\in\mathbf{G}$.

The most important result for a UIR is the so called Schur's lemma: let U on $\mathcal{H}$ and V on $\mathcal{K}$ be irreducible unitary $\mathbf{G}$ representations, and let $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ satisfy

$$B\,U_g=V_g\,B,\quad \forall g\in\mathbf{G}. \tag{15}$$

If U and V are equivalent then B is proportional to the isomorphism I connecting them, otherwise B is null.

Now we will use these tools to deal with covariant CP maps. Let $\mathcal{E}:\mathcal{L}(\mathcal{H})\rightarrow\mathcal{L}(\mathcal{K})$ be a CP map, and let $\mathbf{G}$ be a group with unitary representations U and V on $\mathcal{H}$, and $\mathcal{K}$, respectively. $\mathcal{E}$ is $\mathbf{G}$ covariant with respect to U and V if

$$\mathcal{E}(U_g\rho U_g^\dagger)=V_g\mathcal{E}(\rho)V_g^\dagger, \tag{16}$$

for any $\rho\in\mathcal{L}(\mathcal{H})$ and $g\in\mathbf{G}$.

By means of Eq. (7), the covariance condition becomes

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathcal{H}}[\mathbb{1}\otimes\rho^TR_{\mathcal{E}}]\equiv\mathrm{Tr}_{\mathcal{H}}[\mathbb{1}\otimes\rho^TV_g^\dagger\otimes U_g^TR_{\mathcal{E}}V_g\otimes U_g^*]. \tag{17}$$

From the uniqueness of the operator associated with a CP map, we conclude that $\mathcal{E}$ is $\mathbf{G}$ covariant if and only if

$$R_{\mathcal{E}}=V_g^\dagger\otimes U_g^TR_{\mathcal{E}}V_g\otimes U_g^*,\quad \forall g\in\mathbf{G}, \tag{18}$$

or equivalently

$$[R_{\mathcal{E}},V_g\otimes U_g^*]=0,\quad \forall g\in\mathbf{G}. \tag{19}$$

Thus, $\mathbf{G}$ covariance of a CP map $\mathcal{E}$ is equivalent to $\mathbf{G}$ invariance of the corresponding positive operator $R_{\mathcal{E}}$.

Group invariant operators

The $\mathbf{G}$ representation $W=\mathrm{diag}(V\otimes U^*)$ on $\mathcal{K}\otimes\mathcal{H}$, defined as $W_g=V_g\otimes U_g^*$, is generally reducible, i.e., the space can be decomposed into a direct sum of minimal invariant subspaces $\mathcal{M}_i$,

$$\mathcal{K}\otimes\mathcal{H}=\oplus_{i=1}\mathcal{M}_i, \tag{20}$$

each $\mathcal{M}_i$ supporting a unitary irreducible representation of the group. Given this decomposition one can look at any operator O on $\mathcal{K}\otimes\mathcal{H}$ as a set of operators O_j^i in $\mathcal{L}(\mathcal{M}_j,\mathcal{M}_i)$, so that $O=\Sigma_{ij}O_j^i$.

Due to irreducibility of the subspaces $\mathcal{M}_i$, W_g will be decomposed as follows:

$$(W_g)_j^i=\delta_{ij}T_g^j,$$

where T^j is the UIR supported by $\mathcal{M}_j$. Two UIR's T^i and T^j are equivalent, $i\sim j$, if they are connected by similarity, i.e., through an isomorphism $I_j^i\in\mathcal{L}(\mathcal{M}_j,\mathcal{M}_i)$ such that $T^j=(I_j^i)^{-1}T^iI_j^i$.

The invariance equation (18) becomes

$$R_j^iT_g^j=T_g^iR_j^i,\quad \forall g\in\mathbf{G},$$

so that, by Schur's lemmas, one finally has

$$R_j^i=c_{ij}I_j^i, \tag{21}$$

where, if $i\nsim j$, then $c_{ij}=0$, and, if $i\sim j$, c_{ij} can be different from zero.

Since equivalent representations are related by similarity, in any invariant subspace $\mathcal{M}_i$ one can choose the basis $\{|i,l\rangle,l=1,\ldots,\dim\mathcal{M}_i\}$ so that for $i\sim j$

$$\langle i,l|T^i|i,m\rangle=\langle j,l|T^j|j,m\rangle; \tag{22}$$

hence

$$I_j^i=\sum_l |i,l\rangle\langle j,l|\doteq\mathbb{1}_j^i, \tag{23}$$

and finally

$$R=\sum_{ij} c_{ij}\mathbb{1}_j^i. \tag{24}$$

In order to have a positive R, the matrix c_{ij} must be positive, since taking $|\psi\rangle\rangle=\Sigma_i\Sigma_{l=1}^{\dim\mathcal{M}_i}\psi_{il}|i,l\rangle$ one has

$$\langle\langle\psi|R|\psi\rangle\rangle=\sum_{ij}\sum_{l=1}^{\dim\mathcal{M}_i}\psi_{li}^*c_{ij}\psi_{lj}.$$

Recalling that $c_{ij}=0$ if $i\nsim j$, and reordering the indices of the representations by grouping the equivalent ones, the matrix c_{ij} assumes a block diagonal form, different blocks corresponding to inequivalent representations, each block including all representations equivalent to the same one. In this way, each block has dimension equal to the multiplicity of the representation. Positivity of R implies positivity of each block of matrix c_{ij}. This structure of c_{ij} is reflected on R by means of Eq. (24).

V. OPTIMAL COVARIANT CLONING

A cloning map is just a CP map $\mathcal{C}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}^{\otimes N})$ with the output copies invariant under the permutations of the N output spaces. This is equivalent to a particular covariance of the CP map $\mathcal{C}$ for the group of permutations $\mathbf{S}_N$, namely, it corresponds to the invariance of the positive operator $R_{\mathcal{C}}$ under the representation $W=\mathrm{diag}(V\otimes I)$, where V is the representation of $\mathbf{S}_N$ permuting the N identical output spaces, and I [corresponding to U in Eq. (16)] is the $\mathbf{S}_N$-trivial representation on the input space. One has

$$\mathcal{C}(\rho)=V_\pi\mathcal{C}(\rho)V_\pi^\dagger,\quad \forall\pi\in\mathbf{S}_N. \tag{25}$$

Notice that permutation covariance does not imply that the output state has support in the symmetric subspace of the output space $\mathcal{H}^{\otimes N}$.

As explained in the previous section, Eq. (25) determines a peculiar block structure for the operator $R_{\mathcal{C}}$ associated with the map $\mathcal{C}$. Such a structure is strictly related to the decomposition of $\mathcal{H}^{\otimes(N+1)}$ into invariant subspaces for $V_\pi\otimes\mathbb{1}$. Any possible cloner is described by an $R_{\mathcal{C}}$ with that structure and satisfying the trace-preserving condition of Eq. (10). In this way, one classifies all possible cloning maps through the decomposition into irreducibles of the $\mathbf{S}_N$ representation V on $\mathcal{H}^{\otimes N}$.

In addition to permutation invariance, in this paper we will consider covariance under a group of transformations **G**, with representation T on $\mathcal{H}$. This corresponds to the following identity:

$$\mathcal{C}(T_g\rho T_g^\dagger)=T_g^{\otimes N}\mathcal{C}(\rho)T_g^{\dagger\otimes N}. \tag{26}$$

One can choose a cost function $\Xi(R_\mathcal{C})$ related to the usage of the clones. Covariant cloning is suited to **G**-invariant cost functions, which satisfy

$$\Xi(R_\mathcal{C})=\Xi(T_g^{\otimes N}\otimes T_g^* R_\mathcal{C} T_g^{\dagger\otimes N}\otimes T_g^T). \tag{27}$$

A typical example of an invariant cost function arises when one is interested in cloning a restricted covariant family of states $\{T_g\rho T_g^\dagger\}$ given with a covariant *a priori* probability $dp(g)$, maximizing the average input-clone fidelity $F(R_\mathcal{C})$

$$\begin{aligned}-\Xi(R_\mathcal{C})&\equiv F(R_\mathcal{C})\\ &=\int_\mathbf{G} dp(g)\mathrm{Tr}[\mathbb{1}^{\otimes N-1}\otimes(T_g\rho T_g^\dagger)\otimes(T_g\rho T_g^\dagger)^T R_\mathcal{C}]\\ &=\mathrm{Tr}[\mathbb{1}^{\otimes N-1}\otimes\rho\otimes\rho^T\overline{R_\mathcal{C}}],\end{aligned} \tag{28}$$

where ρ is the seed of the covariant family, and $\overline{R_\mathcal{C}}$ is the covariant counterpart of $R_\mathcal{C}$ obtained by group averaging,

$$\overline{R_\mathcal{C}}=\int_\mathbf{G} dp(g)T_g^{\otimes N}\otimes T_g^*\, R_\mathcal{C} T_g^{\dagger\otimes N}\otimes T_g^T. \tag{29}$$

Clearly for a **G**-covariant cloning one has $R_\mathcal{C}=\overline{R_\mathcal{C}}$.

The optimal cloner minimizes $\Xi(R_\mathcal{C})$ vs $R_\mathcal{C}$ with the constraints for $R_\mathcal{C}$: (i) invariance under permutation and **G**; (ii) positivity; (iii) trace preservation. The constraint (i) leads to Eq. (24); constraint (ii) can be taken into account by writing c_{ij} of Eq. (24) via a Cholevsky decomposition (see, for example, Ref. [19]); constraint (iii) is imposed directly on the resulting parametrization of the operator $R_\mathcal{C}$.

VI. EXAMPLES

Phase-covariant qubit cloning

Here, we consider the problem of cloning a qubit in a U(1)-covariant fashion, where the group representation is given by

$$T_\phi=\exp\left[\frac{i}{2}\phi(\mathbb{1}-\sigma_z)\right]. \tag{30}$$

Since the cloning to two copies is already given in Ref. [20], whereas the general case for N copies is very complicated, here for simplicity we will consider the case of $N=3$ copies. We want to achieve the maximum fidelity between input and clones, when the input is an equatorial qubit

$$|\psi_\phi\rangle=T_\phi\frac{1}{\sqrt{2}}[|0\rangle+|1\rangle]=\frac{1}{\sqrt{2}}[|0\rangle+e^{i\phi}|1\rangle]. \tag{31}$$

In other terms, we want to maximize the average "equatorial" fidelity

$$F=\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\mathrm{Tr}[\mathbb{1}^{\otimes 2}\otimes|\psi_\phi\rangle\langle\psi_\phi|\mathcal{C}(|\psi_\phi\rangle\langle\psi_\phi|)], \tag{32}$$

which, by covariance, can be written as

$$F=\mathrm{Tr}[\mathbb{1}^{\otimes 2}\otimes|\psi_0\rangle\langle\psi_0|\otimes(|\psi_0\rangle\langle\psi_0|)^T R_\mathcal{C}]. \tag{33}$$

Since the equator is invariant even for spin flipping, here we will require the additional covariance with respect to the group $\mathbb{Z}_2$, with representation $\{\mathbb{1},\sigma_x\}$.

In order to satisfy all the covariance requirements, $R_\mathcal{C}$ must be invariant for permutations, phase shift, and spin flip, i.e., for products of any of the following unitary operators:

$$V_\pi\otimes\mathbb{1},\quad T_\phi^{\otimes 3}\otimes T_\phi^*,\quad \sigma_x^{\otimes 3}\otimes\sigma_x^*.$$

The Hilbert space $\mathcal{H}^{\otimes 3+1}$ can be decomposed into subspaces that are irreducible with respect to the joint action of U(1) and $\mathbf{S}_3$. In Table I, we list the irreducible subspaces with their basis, reporting in the columns 3 and 4 the kind of representation supported for U(1) and $\mathbf{S}_N$, respectively.

Referring to Table I, one has to group together the subspaces supporting equivalent representations for U(1) and for $\mathbf{S}_3$. This leads to the peculiar block structure for the matrix c_{ij} that we mentioned in Sec. IV. In this example, we find

TABLE I. $\mathcal{H}^{\otimes 3+1}$ decomposition into U(1)-$\mathbf{S}_3$ irreducibles. U(1) acts on each subspace as a phase shift $e^{in\phi}$, where $n\in\mathbb{Z}$ (column 3) labels inequivalent representations. $\mathbf{S}_3$ acts trivially (T) on one-dimensional subspaces, whereas on bidimensional ones it acts as the defining representation (D). Spin flipping connects subspaces (column 5).

Space	Unnormalized basis	U(1)	$\mathbf{S}_3$	Flipped
$\mathcal{M}_1$	$\|0001\rangle$	-1	T	$\mathcal{M}_5$
$\mathcal{M}_2$	$\|0000\rangle$	0	T	$\mathcal{M}_6$
$\mathcal{M}_3$	$\|1001\rangle+\|0101\rangle+\|0011\rangle$	0	T	$\mathcal{M}_7$
$\mathcal{M}_4$	$\|1001\rangle-\|0101\rangle$, $\frac{1}{2}\|1001\rangle+\frac{1}{2}\|0101\rangle-\|0011\rangle$	0	D	$\mathcal{M}_8$
$\mathcal{M}_5$	$\|1110\rangle$	3	T	$\mathcal{M}_1$
$\mathcal{M}_6$	$\|1111\rangle$	2	T	$\mathcal{M}_2$
$\mathcal{M}_7$	$\|0110\rangle+\|1010\rangle+\|1100\rangle$	2	T	$\mathcal{M}_3$
$\mathcal{M}_8$	$\|0110\rangle-\|1010\rangle$, $\frac{1}{2}\|0110\rangle+\frac{1}{2}\|1010\rangle-\|1100\rangle$	2	D	$\mathcal{M}_4$
$\mathcal{M}_9$	$\|1000\rangle+\|0100\rangle+\|0010\rangle$	1	T	$\mathcal{M}_{10}$
$\mathcal{M}_{10}$	$\|0111\rangle+\|1011\rangle+\|1101\rangle$	1	T	$\mathcal{M}_9$
$\mathcal{M}_{11}$	$\|1000\rangle-\|0100\rangle$, $\frac{1}{2}\|1000\rangle+\frac{1}{2}\|0100\rangle-\|0010\rangle$	1	D	$\mathcal{M}_{12}$
$\mathcal{M}_{12}$	$\|0111\rangle-\|1011\rangle$, $\frac{1}{2}\|0111\rangle+\frac{1}{2}\|1011\rangle-\|1101\rangle$	1	D	$\mathcal{M}_{11}$

TABLE II. Content of the blocks of the matrix c_{ij}, chosen in order to have $R_{\mathcal{C}}$ describing the most general CP map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}^{\otimes 3})$ that is covariant with respect to permutations, phase shift, and spin flip.

Blocks	Content
{1}, {5}	a
{4}, {8}	b
{2,3}, {6,7}	$c\mathbb{1}+\mathbf{v}\cdot\sigma$
{9,10}	$d\mathbb{1}+e\sigma_x$
{11,12}	$f\mathbb{1}+g\sigma_x$

that a phase- and flip-covariant cloning map is described though Eq. (24) by a matrix c_{ij} having the following positive diagonal blocks:

$$\{1\},\{2,3\},\{4\},\{5\},\{6,7\},\{8\},\{9,10\},\{11,12\}.$$

To ensure spin flipping covariance, the elements of c_{ij} connected by a flip must be equal, for example, $c_{23}=c_{67}$.

Finally, to fill the blocks of c_{ij} in the right way, we need the parameters $a,b,c,d,e,f,g\in\mathbb{R}^+$, and $\mathbf{v}\in\mathbb{R}^3$, where $d\geqslant e$, $f\geqslant g$, and $c\geqslant||\mathbf{v}||$. Table II explains how to employ them.

The parameters must satisfy another constraint given by the trace-preserving condition defined in Eq. (10). Within this parametrization it reads

$$a+2b+2c+d+2f=1. \tag{34}$$

Substituting this equation into the equatorial fidelity F defined in Eq. (33), one has

$$F=\frac{1}{2}+\frac{1}{3}(e-g)+\frac{\sqrt{3}}{3}v_x. \tag{35}$$

This quantity can be easily maximized by hand, taking into account the constraint given by Eq. (34) and the properties of the parameters. The maximum fidelity is $F=\frac{5}{6}$ and is achieved for $d=e=1$ and all the other parameters equal to zero. The value $F=\frac{5}{6}$ exceeds the bound given in Ref. [20] (see [23]). The optimal phase covariant cloning is thus described by the operator

$$R_{\mathcal{C}}^{opt}=|\Phi\rangle\rangle\langle\langle\Phi|, \tag{36}$$

where

$$|\Phi\rangle\rangle=\frac{1}{\sqrt{3}}[|1000\rangle+|0100\rangle+|0010\rangle+|0111\rangle+|1011\rangle+|1101\rangle].$$

The Kraus decomposition of the optimal cloner is $\mathcal{C}(\rho)=B\rho B^\dagger$, where

$$B=\frac{1}{\sqrt{3}}[|100\rangle\langle 0|+|010\rangle\langle 0|+|001\rangle\langle 0|+|011\rangle\langle 1|+|101\rangle\times\langle 1|+|110\rangle\langle 1|]. \tag{37}$$

The fidelity for the $1\rightarrow 2$ case is $\frac{1}{2}+\sqrt{\frac{1}{8}}$, as demonstrated in Ref. [20], and it is larger than the present $1\rightarrow 3$ value, since the ‘‘information’’ is spread into a smaller number of copies.

Notice that in the general case one could have many cloning maps attaining the same global maximum of a covariant cost function like F in Eq. (33). These maps can be covariant or, if not, they are mapped one into the other by the covariance group, whence for a continuous group they make a manifold of maps. However, the noncovariant clonings are averaged into a covariant cloning via the integral (29). Therefore, for a linear cost function, every optimal covariant cloning is just a convex combination of maps giving the best fidelity. In this particular example, the optimal covariant cloning is an extremal point of the convex set of all the cloning maps, either covariant or not, leading to a unique optimal cloning.

Qutrits double-phase covariant cloning

In this section, we show another example of the use of the techniques presented. Here, our target will be the construction of the $1\rightarrow 2$ qutrit cloning that gives the best average fidelity on the set of states of the form

$$|\psi_{\phi\vartheta}\rangle=\frac{1}{\sqrt{3}}[|0\rangle+e^{i\phi}|1\rangle+e^{i\vartheta}|2\rangle]. \tag{38}$$

According to what we said at the end of the last example, such an optimal cloning can be found among the ones that are covariant with respect to ϕ and ϑ phase rotations, and with respect to permutations of the basis $\{|0\rangle,|1\rangle,|2\rangle\}$, which are the equivalent of the spin-flip symmetry of the previous example. The above are all the symmetries of the set of input states and of the fidelity.

Our $R_{\mathcal{C}}$ will be a positive operator on $\mathcal{H}^{\otimes 2+1}$ being invariant for products of any of the following unitary transformations:

$$V\otimes\mathbb{1},\quad T_{\phi\vartheta}^{\otimes 2}\otimes T_{\phi\vartheta}^*,\quad U_\pi^{\otimes 2}\otimes U_\pi^*, \tag{39}$$

where V is the permutation of the two clone spaces (for two copies V is usually called a ‘‘swap’’), and

$$T_{\phi\vartheta}\doteq|0\rangle\langle 0|+e^{i\phi}|1\rangle\langle 1|+e^{i\vartheta}|2\rangle\langle 2|,$$

$$U_\pi|i\rangle\doteq|\pi(i)\rangle,\quad \forall\,\pi\in\mathbf{S}_3. \tag{40}$$

Remember that swap invariance makes $R_{\mathcal{C}}$ a $1\rightarrow 2$ cloning map. The entries of Table III correspond to all the phase- and swap-invariant subspaces. Since they are all unidimensional, each is labeled by its generating vector.

Let us define the operators O_i, $i=1,\ldots,5$, on $\mathcal{H}^{\otimes 2+1}$ having the following matrix elements with respect to the basis reported in Table IV:

TABLE III. $\mathcal{H}^{\otimes 2+1}$ decomposition into unidimensional invariant subspaces. The invariant subspaces are obtained by making the tensor products of any vector from the first column with either $|0\rangle$, $|1\rangle$, or $|2\rangle$: the corresponding cell in the table gives the full symmetry of the subspace. The first letter denotes the kind of action of the swap (symmetric-antisymmetric); the two numbers indicate the representation for ϕ and ϑ phase shifts, respectively. Subspaces having the same greek letter are connected by a permutation $U_\pi^{\otimes 2}\otimes U_\pi^*$ for some $\pi\in\mathbf{S}_3$.

$\otimes$	$\|0\rangle$	$\|1\rangle$	$\|2\rangle$
$\|00\rangle$	S, 0, 0, α	S, -1, 0, β	S, 0, -1, β
$\|11\rangle$	S, 2, 0, β	S, 1, 0, α	S, 2, -1, β
$\|22\rangle$	S, 0, 2, β	S, -1, 2, β	S, 0, 1, α
$\frac{1}{\sqrt{2}}[\|01\rangle+\|10\rangle]$	S, 1, 0, γ	S, 0, 0, γ	S, 1, -1, δ
$\frac{1}{\sqrt{2}}[\|02\rangle+\|20\rangle]$	S, 0, 1, γ	S, -1, 1, δ	S, 0, 0, γ
$\frac{1}{\sqrt{2}}[\|12\rangle+\|21\rangle]$	S, 1, 1, δ	S, 0, 1, γ	S, 1, 0, γ
$\frac{1}{\sqrt{2}}[\|01\rangle-\|10\rangle]$	A, 1, 0, γ	A, 0, 0, γ	A, 1, -1, δ
$\frac{1}{\sqrt{2}}[\|02\rangle-\|20\rangle]$	A, 0, 1, γ	A, -1, 1, δ	A, 0, 0, γ
$\frac{1}{\sqrt{2}}[\|12\rangle-\|21\rangle]$	A, 1, 1, δ	A, 0, 1, γ	A, 1, 0, γ

$$O_1\to\begin{pmatrix} a & d & d\\ d^* & b & c\\ d^* & c & b\end{pmatrix},\quad O_2\to\begin{pmatrix} e & f\\ f & e\end{pmatrix},$$

$$O_3\to g,\quad O_4\to h,\quad O_5\to i. \tag{41}$$

These five operators are clearly invariant with respect to swapping and phase shifts, as one can see by comparing their expressions with Table III. Sums of operators of the form of the O_i and of the form of the operators obtained from O_i with permutations, i.e., by acting on each O_i with $U_\pi^{\otimes 2}\otimes U_\pi^*$, are swap and phase invariant. One may note that by permutations O_5 generates $3!$ different operators, whereas the other ones generate only 3 different operators each, since they are invariant with respect to the transposition $|1\rangle\leftrightarrow|2\rangle$.

TABLE IV. Vector basis to which the matrix elements of the operators O_i are referred.

Operator	Ordered basis
O_1	$\|000\rangle, \frac{1}{\sqrt{2}}[\|011\rangle+\|101\rangle], \frac{1}{\sqrt{2}}[\|022\rangle+\|202\rangle]$
O_2	$\frac{1}{\sqrt{2}}[\|011\rangle-\|101\rangle], \frac{1}{\sqrt{2}}[\|022\rangle-\|202\rangle]$
O_3	$\frac{1}{\sqrt{2}}[\|210\rangle+\|120\rangle]$
O_4	$\frac{1}{\sqrt{2}}[\|210\rangle-\|120\rangle]$
O_5	$\|001\rangle$

Thanks to these observations, one realizes that these five operators give rise to five independent families of covariant cloning maps described by the invariant positive operators

$$R_{\mathcal{C}}^i=\sum_{\pi\in\mathbf{S}_3}U_\pi O_i U_\pi. \tag{42}$$

The positivity constraint for any family simply becomes $O_i\geqslant 0$, while the trace-preserving condition leads to $\mathrm{Tr}O_1=\mathrm{Tr}O_2=\mathrm{Tr}O_3=\mathrm{Tr}O_4=1/2$ and $\mathrm{Tr}O_5=1/4$.

Any other covariant cloning map can be written as a convex linear combination of these five kinds of map in a unique way. Since the average fidelity is linear in $R_{\mathcal{C}}$, we can look for the optimal maps among these five families separately. With a little algebra one finds $\max(F_2)=1/2$ and $F_3=F_4=F_5=1/3$, while $\max(F_1)=\frac{1}{12}(5+\sqrt{17})\simeq 0.76$. Thus the optimal covariant cloning map belongs to the $R_{\mathcal{C}}^1$ family; in particular, it is obtained for the following values of the parameters:

$$a=\frac{1}{4}\left(1-\frac{1}{\sqrt{17}}\right),\quad c=b=\frac{1}{8}\left(1+\frac{1}{\sqrt{17}}\right),$$

$$d=\sqrt{\frac{ab}{2}}=\frac{1}{2\sqrt{17}}, \tag{43}$$

which have been determined by maximizing the quantity

$$F_1=\frac{2}{3}[a+2b+c+2\sqrt{2}\,\mathrm{Re}(d)], \tag{44}$$

within the constraints of trace preserving and positivity,

$$a+2b=\frac{1}{2},$$

$$a,b\geqslant 0,\quad |c|\leqslant b,\quad |d|^2\leqslant\frac{a(b+c)}{2}. \tag{45}$$

Cloning of continuous variables

The parametrization of CP maps given in Sec. III and its specialization to the covariant case are useful tools for engineering measurements. The idea is to "spread" a quantum state on a larger system with a CP map $\mathcal{E}$, and then to perform a measurement on the spread state. The connection between the positive operator valued measure (POVM) $\{M_i\}$ on the larger space $\mathcal{K}$ and the resulting one $\{M_i^\vee\}$ on $\mathcal{H}$ is given by

$$M_i^\vee=\mathcal{E}^\vee(M_i)\doteq\mathrm{Tr}_{\mathcal{K}}[M_i\otimes\mathbb{1}R_{\mathcal{E}}^{T_{\mathcal{H}}}], \tag{46}$$

where $\mathcal{E}^\vee$ is the dual map of $\mathcal{E}$, and the symbol $T_{\mathcal{H}}$ stands for transposition with respect to $\mathcal{H}$ only (see Sec. III).

In Ref. [11], the cloning map for continuous variables of Ref. [21] is used to achieve the optimal POVM for the joint measurement of two conjugated quadratures X_0 and $X_{\pi/2}$ of an oscillator mode a (where $X_\phi=\frac{1}{2}[a^\dagger e^{i\phi}+ae^{-i\phi}]$) by measuring them separately on the two clones. Here, we will briefly show how our general method works on this problem.

Denote by $\mathcal{H}_3$ the input space and by $\mathcal{H}_1,\mathcal{H}_2$ the two output spaces of the oscillator modes a_3, a_1, and a_2, respectively. The cloning is described by

$$R_{\mathcal{C}}=\frac{1}{2}P_{12}\otimes \mathbb{1}_3\mathbb{1}_1\otimes(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)_{23}P_{12}\otimes\mathbb{1}_3, \tag{47}$$

where $P=V|0\rangle\langle 0|\otimes \mathbb{1}V^\dagger$, and V is the 50% beam splitter unitary transformation $V=\exp[(\pi/4)(a_1^\dagger a_2-a_1a_2^\dagger)]$.

A simple calculation shows that

$$P=\frac{2}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|^{\otimes 2}, \tag{48}$$

where $|\alpha\rangle=D(\alpha)|0\rangle$, and $D(\alpha)=e^{\alpha a^\dagger-\bar{\alpha}a}$ is the displacement operator generating the *Weyl-Heisenberg* (WH) group. By means of Eq. (48), the invariance of $R_{\mathcal{C}}$ defined in Eq. (47) with respect to permutations and displacements can easily be verified.

Using the dual cloning map as in Eq. (46), we should check that

$$\mathcal{C}^\vee(E_x^0\otimes E_y^{\pi/2})=\frac{1}{\pi}|\alpha\rangle\langle\alpha|,\quad \alpha=x+iy, \tag{49}$$

where $E_v^\phi=|v\rangle_{\phi\,\phi}\langle v|$ and $X_\phi|v\rangle_\phi=v|v\rangle_\phi$. In fact, the last term of Eq. (49) is the well-known optimal POVM for the joint measurement of conjugated quadratures, whereas E_v^ϕ is the POVM of the ϕ quadrature measurement. Hence identity (49) guarantees that the cloning achieves the optimal joint measurement of the two conjugated quadratures via commuting measurements on clones.

Noticing that

$$E_x^0\otimes E_y^{\pi/2}=D(\alpha)^{\otimes 2}E_0^0\otimes E_0^{\pi/2}D(\alpha)^{\otimes 2\dagger} \tag{50}$$

and exploiting the WH covariance, Eq. (49) reduces to

$$\mathcal{C}^\vee(E_0^0\otimes E_0^{\pi/2})=\frac{1}{\pi}|0\rangle\langle 0|. \tag{51}$$

Substituting Eq. (46) into this last equation, and taking matrix elements $\langle i|\cdots|j\rangle$, one finally must check that

$${}_0\langle 0|_{\pi/2}\langle 0|\langle i|R_{\mathcal{C}}|0\rangle_0|0\rangle_{\pi/2}|j\rangle=\frac{1}{\pi}\delta_{i0}\delta_{j0}. \tag{52}$$

Since $V|0\rangle_0|0\rangle_{\pi/2}=\sqrt{2/\pi}|\mathbb{1}\rangle\rangle$ and $V|0\rangle|0\rangle=|0\rangle|0\rangle$ (see Ref. [22]), one has that $P|0\rangle_{\pi/2}|0\rangle_0=\sqrt{2/\pi}|0\rangle|0\rangle$. Thus Eq. (52) holds, and the cloning really achieves the wanted POVM.

Universal cloning

Clearly, the universal covariant cloning of Werner [6] is a special case of covariant cloning for the covariance group $U(d)$, $d=\dim\mathcal{H}$, of all unitary operators on $\mathcal{H}$. Here, for the sake of comparison to Ref. [6], we consider more generally the cloning from M to $N>M$ copies. Hence the cloning is a CP map $\mathcal{C}$ from $\mathcal{L}(\mathcal{H}^{\otimes M})$ to $\mathcal{L}(\mathcal{H}^{\otimes N})$ such that for any $U\in U(d)$ and $\sigma\in\mathcal{L}(\mathcal{H})$

$$\mathcal{C}(U^{\otimes M}\sigma^{\otimes M}U^{\dagger\otimes M})=U^{\otimes N}\mathcal{C}(\sigma^{\otimes M})U^{\dagger\otimes N}. \tag{53}$$

The cost function for optimization is the (negative) fidelity between clones and input,

$$\Xi(R_{\mathcal{C}})=-F=-\mathrm{Tr}[\sigma^{\otimes N}\mathcal{C}(\sigma^{\otimes M})], \tag{54}$$

where σ is pure. Owing to covariance, the fidelity F does not depend on σ, since any pure state lies in the $U(d)$ orbit of any other pure state.

The optimal cloning map of Ref. [6] is given by

$$\mathcal{C}(\rho)=\frac{d(M)}{d(N)}S_N(\rho\otimes\mathbb{1}^{\otimes(N-M)})S_N, \tag{55}$$

where $\rho\in\mathcal{L}(\mathcal{H}^{\otimes M})$, S_N is the projector on the symmetric subspace $\mathcal{H}_+^{\otimes N}$, and $d(N)=\dim(\mathcal{H}_+^{\otimes N})$. In our framework, one has

$$R_{\mathcal{C}}=\frac{d(M)}{d(N)}\tilde{S}\mathbb{1}_{\mathcal{H}^{\otimes(N-M)}}\otimes(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)_{\mathcal{H}^{\otimes(M+M)}}\tilde{S}, \tag{56}$$

where $\tilde{S}=S_N\otimes\mathbb{1}^{\otimes M}$. It can be easily verified that $R_{\mathcal{C}}$ is both covariant and permutation invariant, as it must be.

ACKNOWLEDGMENTS

This work was supported by the European Union under the project EQUIP (Contract No. IST-1999-11053), and by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica (MURST) under the project ''Quantum Information Transmission and Processing: Quantum Teleportation and Error Correction.''

[1] D. Dieks, Phys. Lett. **92A**, 271 (1982); W. K. Wooters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[2] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[3] H. P. Yuen, Phys. Lett. **113A**, 405 (1986).
[4] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. **80**, 4999 (1998).
[5] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[6] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
[7] D. Bruß, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).
[8] C. Simon, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. **84**, 2993 (2000).
[9] G. M. D'Ariano, F. De Martini, and M. F. Sacchi, Phys. Rev.

Lett. **86**, 914 (2001).
[10] E. Galvao and L. Hardy, Phys. Rev. A **62**, 22301 (2000).
[11] G. M. D'Ariano, C. Macchiavello, and M. F. Sacchi, J. Opt. B: Quantum Semiclassical Opt. **3**, 44 (2001).
[12] D. Bruß, J. Calsamiglia, and N. Lütkenhaus, Phys. Rev. A **63**, 042308 (2001).
[13] A. Jamiolkowski, Rep. Math. Phys. **3**, 275 (1972).
[14] See, for example, G. M. D'Ariano, P. Lo Presti, and M. F. Sacchi, Phys. Lett. A **272**, 32 (2000).
[15] Remember that transposition is not a CP map; hence partial transposition is not a positive map. Thus the partial transposition of a positive operator can be nonpositive.
[16] K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin, 1983).
[17] M. Ozawa, J. Math. Phys. **25**, 79 (1984).
[18] H. F. Jones, *Groups, Representations, and Physics* (Institute of Physics Publishing, Bristol, 1990).
[19] R. Bhatia, *Matrix Analysis* (Springer-Verlag, New York, 1997).
[20] D. Bruß, M. Cinchetti, G. M. D'Ariano, and C. Macchiavello, Phys. Rev. A **62**, 012302 (2000).
[21] N. J. Cerf, A. Ipe, and X. Rottenberg, Phys. Rev. Lett. **85**, 1754 (2000).
[22] G. M. D'Ariano and M. F. Sacchi, Mod. Phys. Lett. B **11**, 1263 (1997).
[23] The optimal cloning fidelity in Ref. [20] is obtained under the same assumptions as in the present paper. All properties of the equatorial qubit covariant cloning here can be checked just by inspection of Eq. (37). Moreover, the present result agrees with the fidelity given in Ref. [24].
[24] H. F. Fan, K. Matsumoto, X. Wang, and M. Wadati, e-print quant-ph/0101101.