Implications of teleportation for nonlocality

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Adopting an approach similar to that of Zukowski [Phys. Rev. A **62**, 032101 (2000)], we investigate connections between teleportation and nonlocality. We derive a Bell-type inequality pertaining to the teleportation scenario and show that it is violated in the case of teleportation using a perfect singlet. We also investigate teleportation using "Werner states" of the form $\alpha P_s + (1-\alpha)I/4$, where P_s is the projector corresponding to a singlet state and I is the identity. We find that our inequality is violated, implying nonlocality, if $\alpha > 1/\sqrt{2}$. In addition, we extend Werner's local hidden variable model to simulation of teleportation with the $\alpha = \frac{1}{2}$ Werner state. Thus teleportation using this state does not involve nonlocality even though the fidelity achieved is $\frac{3}{4}$, which is greater than the "classical limit" of $\frac{2}{3}$. Finally, we comment on a result of Gisin's and offer some philosophical remarks on teleportation and nonlocality generally.

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I. INTRODUCTION

Quantum teleportation, first introduced in [1], is a quantum mechanical scheme that allows one participant (Alice) to transmit a quantum state in her possession to another participant (Bob). In the original version, it is a spin-$\frac{1}{2}$ state that is sent and the only resources required are a shared singlet and the capacity for Alice to send two classical bits to Bob. In this case, Bob ends up with a state identical with the one Alice begins with. The state in Alice's possession is randomized, so there is no contradiction with the no-cloning theorem [2]. A notable feature of the scheme is that it works even when Alice has no knowledge of the state she is sending.

Briefly, the procedure described in [1] (hereafter referred to as "the standard scheme") works as follows. Let the state Alice wants to teleport be $|\chi\rangle$ and the shared state have density matrix ρ. Alice performs a joint measurement on her half of ρ and $|\chi\rangle$. The measurement projects onto the Bell basis. Alice sends two classical bits to Bob informing him of which of the four possible outcomes she got. Bob then performs a corresponding unitary transformation. If ρ is maximally entangled then these four unitary transformations can be chosen such that the state Bob ends up with is identical with $|\chi\rangle$.

If ρ is not maximally entangled then typically, Bob will end up with something that is not identical with $|\chi\rangle$. Suppose that at the end Bob is in possession of a state whose density matrix is M. The "fidelity" of the teleportation is defined as $\langle\chi|M|\chi\rangle$. In general, the fidelity will depend on $|\chi\rangle$ but we can define an average fidelity by averaging over all possible values of $|\chi\rangle$. This average is often referred to as the fidelity for a particular teleportation scheme and shared state.

In Vaidman's view [3], quantum teleportation is so called because it involves the transfer of an "object" from one place to another without it ever being located in the intervening space. He troubles to argue that teleportation is indeed an appropriate name for this process. This might already lead us to entertain rather vague notions that teleportation must be intrinsically connected with another idea, viz., nonlocality. In particular, in the case in which the fidelity is 1 (corresponding to perfect teleportation), it seems intuitively obvious that some sort of nonlocality is at work.

The purpose of this study is to investigate this idea in more detail (other studies have been undertaken in a similar spirit by Gisin [4], Zukowski [5], Hardy [6], and Cerf *et al.* [7]). First, in Sec. II, we define more precisely what we mean by "nonlocal" — broadly speaking, we take it to mean "not simulable by local hidden variables." In the literature now, nonlocal is often used simply to mean entangled. We emphasize that it is the relation between teleportation and nonlocality in our sense that we shall investigate. In Sec. III, we discuss results of Hardy's [6] which show that teleportation and nonlocality are conceptually distinct and which give us an idea of how local hidden variables might simulate a teleportation procedure. This leaves open the question of whether teleportation and nonlocality are physically distinct — this is the question that we turn to in Sec. IV. Here, we adopt an approach similar to that of Zukowski [5] and derive an inequality, the violation of which shows that perfect teleportation using a singlet implies nonlocality. In Sec. VI we consider teleportation using "Werner states" instead of pure singlets. We extend Werner's original local hidden variable model in order to simulate a teleportation scenario with local hidden variables. We also consider how high the teleportation fidelity can be before teleportation using a Werner state violates our inequality and therefore implies nonlocality. Finally, in Sec. VII, we comment on a result of Gisin's that he claims has relevance to teleportation and nonlocality. We suggest that his interpretation of his result is slightly misleading but that the result is still interesting if interpreted differently.

II. WHAT WE MEAN BY "NONLOCALITY"

We would do well at the outset to specify precisely what we mean by nonlocality. Broadly, we take it to mean nonsimulablility by local hidden variables (but see the end of this section for a qualification). Consider a bipartite state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. The two subsystems are spatially separated, one being in the possession of an observer Alice and the other being in the possession of an observer Bob. Alice performs a measurement A on her subsystem while Bob performs a measurement B on his; these measurements occur at spacelike separation from one another. This procedure is repeated with a new system in the state ρ each time. We refer to this as a ''Bell-type experiment.'' If measurement *A* has an outcome a_i and measurement *B* has an outcome b_j, then a hidden variable model supposes that the probability of getting these two outcomes can be given in the form

$$\Pr(a_i,b_j|A,B,\rho)=\int d\lambda\ \omega^{\rho}(\lambda)\ \ \Pr(a_i,b_j|A,B,\lambda), \quad (1)$$

where $\omega^{\rho}(\lambda)$ is some distribution over a space Λ of hidden states λ. A local hidden variable model imposes the additional constraint

$$\Pr(a_i,b_j|A,B,\lambda)=\Pr(a_i|A,\lambda)\ \Pr(b_j|B,\lambda). \quad (2)$$

It was Bell who first discovered that some quantum states are nonlocal [8]. He derived an inequality involving the probabilities of measurement outcomes that must be satisfied by any local hidden variable model. He then showed that the quantum mechanical predictions for two spin-$\frac{1}{2}$ particles in a singlet state violate this inequality. Different versions of Bell's inequality were derived in [9,10]. We refer to these as the Clauser-Horne (CH) inequality and the Clauser-Horne-Shimony-Holt (CHSH) inequality, respectively.

These results have since been generalized. In [11], Peres considers experimental scenarios in which Alice and Bob each choose from an arbitrary finite number of possible measurements to perform, where each measurement has an arbitrary finite number of possible outcomes. Peres shows how to construct a list of inequalities, the idea being that if the outcomes can be simulated with a local hidden variable model then all the inequalities must be satisfied. He gives both a graphical method for the easy construction of these inequalities and an algorithm that produces a complete set — complete meaning that the satisfaction of all the inequalities is sufficient for the existence of a local hidden variable model for the particular experimental scenario considered. We refer to these inequalities as ''Bell-type inequalities.''

Both the formalism developed by Peres and that of Eqs. (1) and (2) above apply only to a scenario in which Alice and Bob each perform a single measurement on each run of the experiment. In a more complicated scenario, Alice and Bob might perform a sequence of measurements. In [12], Popescu gives examples of states for which nonlocality can be revealed in this manner even though local hidden variable models exist for single measurements that satisfy Eqs. (1) and (2) — he calls this ''hidden nonlocality'' (we discuss this briefly again in Sec. V). A new formalism would be needed to explain what is meant by a local hidden variable model here and what conditions would have to be violated for the nonexistence of such a model (see, e.g., [13]). We do not discuss this further. In what follows, we refrain from calling a state local unless it is completely local, i.e., has no hidden nonlocality (and indeed does not display what Teufel *et al.* call ''deeply hidden nonlocality'' [13]). We may, however, speak of a state having a local hidden variable model meaning only a model that satisfies Eqs. (1) and (2) — such a state may still have hidden nonlocality.

Note that in all of this we have been considering only measurements performed separately on each particle pair. In some cases, nonlocality may be revealed if Alice and Bob are allowed collective measurements on several particle pairs at once, even though the state would be local if this possibility were not allowed. We choose to ignore this possibility when classifying states as local or nonlocal — one could argue that, if collective measurements on *n* copies of a state ρ are needed to reveal nonlocality, then it is the state $\rho^{\otimes n}$ that is nonlocal rather than ρ.

III. CONCEPTUALLY VS PHYSICALLY DISTINCT

In [6], Hardy shows that teleportation and nonlocality are conceptually distinct. ''Conceptually distinct'' means that we ought to be able to imagine a scenario (not necessarily one complying with known physical laws but one that is at least logically consistent) in which perfect teleportation is realizable while locality is demonstrably preserved. Hardy constructs just such a scenario in the form of a toy theory which allows for notions of systems, states, measurements, a no-cloning theorem, and teleportation. It is also demonstrably local. Note that this is separate from the question of whether teleportation and nonlocality are physically distinct — in other words, it still may be the case that teleportation in quantum mechanics implies nonlocality.

In fact, Hardy gives two reasons why we might think that nonlocality is implied by quantum teleportation. The second reason he gives is that entanglement can be teleported in quantum mechanics. In other words if Alice and Bob share particles *C* and *D* in a singlet state, while Alice has particles *A* and *B* in her possession in some entangled state $|\psi\rangle_{AB}$, then Alice can teleport particle *B* to Bob so that at the end of the protocol *A* and *D* are in the state $|\psi\rangle_{AD}$. (This works essentially because teleportation is a linear operation applied to particle *B* [1].) It is then possible to obtain a violation of Bell's inequalities by performing measurements on *A* and *D*. However Hardy also responds to this point: ''We cannot necessarily assert on the basis of this fact that nonlocality plays a role in quantum teleportation. It is possible that the extra information which establishes the nonlocal correlations is only transmitted in the process of measuring the quantities in Bell's inequalities, and is not transmitted in the teleportation process.'' In recognition of this sense that the subsequent testing of entanglement that has been teleported is not part of the teleportation process itself, we will hereafter ignore the possibility of teleporting entanglement. We will consider teleportation of a single state, not entangled with any others, and the question of when this might imply nonlocality.

In giving his first reason why teleportation might imply nonlocality, Hardy notes, following Bennett, that ''the amount of information needed to specify a general qubit is much greater than the two bits of information which is classically communicated during quantum teleportation. One might speculate that when a qubit is teleported, the extra information is being carried by the nonlocal properties of the entangled state.'' As well as making this point, however, Hardy also has a response:

''On the other hand, it is not possible to extract more classical information from a qubit than the two classical bits communicated during teleportation and so there must remain questions about the reality of the quantum information apparently transmitted during teleportation.''

These two quotations, then, provide us with a motivation for a detailed investigation of the relationship between teleportation and nonlocality in quantum mechanics. This is undertaken in the next section. We adopt an approach similar to that of Zukowski [5]. We derive a Bell-type inequality and show that it is violated in the case of teleportation using a shared singlet and the standard scheme.

First, however, Hardy's toy theory is interesting because it provides some insight into how hidden variables might be able to describe a teleportation process, so it is worth examining a few of the details. A particle in Hardy's theory exists in one of four states, labeled 0, 1, 2, and 3. The states are hidden states because there is no measurement that will determine the state of a particle unambiguously and we cannot prepare a particle with an unambiguous state. We write the state of two particles as a vector (x_1,x_2), where $x_1,x_2 \in \{0,1,2,3\}$. Two particles may be correlated — for example, we can prepare them in such a way that their state is given by 25% chance of (0,0), 25% chance of (1,1), 25% chance of (2,2), and 25 % chance of (3,3).

This is the preparation used to perform teleportation and is clearly analogous to the maximally entangled quantum state used in the standard quantum mechanical scheme (except, of course, it has none of the nonlocal properties possessed by a maximally entangled quantum state). Suppose Alice and Bob share two particles, called particles 2 and 3, which have been prepared in this way. Suppose that Alice also has another particle, particle 1, in an unknown state, which she wishes to teleport to Bob. Alice can perform a joint measurement on particles 1 and 2 which can be characterized by four possible outcomes:

$$B_0=\begin{pmatrix}0&0\\1&1\\2&2\\3&3\end{pmatrix},\quad B_1=\begin{pmatrix}0&3\\1&0\\2&1\\3&2\end{pmatrix},$$

$$B_2=\begin{pmatrix}0&2\\1&3\\2&0\\3&1\end{pmatrix},\quad B_3=\begin{pmatrix}0&1\\1&2\\2&3\\3&0\end{pmatrix}.$$

The notation means that if outcome B_0 was obtained, for example, then the initial state of the two particles was either (0,0), (1,1), (2,2), or (3,3). Any possible initial state of the two particles leads to a unique measurement outcome. After the measurement, if outcome B_0 was obtained, then the state of the two particles is one of these four with equal probabilities, i.e., measurement disturbs the system (this leads to a proof of a no-cloning theorem). The measurement described here is clearly analogous to the Bell measurement used in the quantum mechanical standard scheme.

Alice now uses two classical bits to inform Bob of her measurement outcome. If x_3 is the initial state of particle 3 and x_1 the initial state of particle 1, Bob now knows the value of $i=(x_3-x_1) \mod 4$. Bob can perform an operation on particle 3 given by $U_i: x_3 \rightarrow (x_3-i) \mod 4$. These operations are analogous to the unitary operations of quantum mechanics. Teleportation has now been successfully completed — the final state of particle 3 is identical with the initial state of particle 1.

We can take a step back and ask: how exactly has this teleportation been accomplished? No individual particle has been transmitted from Alice to Bob (apart from in the preparatory stages and those used to transmit classical bits). Rather, correlations between hidden states have been used to ensure that the final state of the particle in Bob's possession is identical with the intial state of the particle that Alice teleported. Although Alice in this theory is unable to determine the state of a particle directly, it turns out that she is able to do a measurement that tells her the *difference* between the states of particles 1 and 2. The correlations between the hidden states ensure that particle 3 is initially in the same state as particle 2 and Bob is then able to perform an operation on particle 3 to make up the difference.

It is clear that this model works in the way it does only because there are only four different states that a particle can be in — this corresponds to the fact that Alice sends two classical bits. In this sense it is artificial and it is not miraculous that it works. It is, however, quite adequate to establish Hardy's claim that teleportation and nonlocality are conceptually distinct. In subsequent sections we turn to consider whether, or in what circumstances, teleportation in quantum mechanics could be effected by making use of correlations between hidden variables in a similar manner.

IV. TELEPORTATION AND NONLOCALITY

Consider the standard scheme for quantum mechanical teleportation. Alice's first action is to perform a joint measurement on her half of the shared system and the system she wishes to teleport. The measurement projects onto the Bell basis. There is another way of looking at this measurement (described in [14]). Recall that the most general type of quantum measurement corresponds to a positive operator valued decomposition of the identity [2]. We call such a measurement a POV measurement. In the special case that the measurement corresponds to a projective decomposition of the identity, we refer to it as a projective measurement. A POV measurement on some system can always be realized by attaching an ancilla and performing a projective measurement on the combined system [2]. Conversely, any joint projective measurement performed on the system and ancilla can be thought of as a POV measurement on just the system. This is exactly what is happening in the case of teleportation — we regard Alice's half of the shared state as the system and the state being teleported as the ancilla. If the state being teleported is $|\chi\rangle=\binom{a}{b}$, then Alice is performing a POV measurement on her half of ρ, the shared state, with elements as follows:

POV element	Bell outcome	Bob's state, if $\rho=$ singlet
$A_0=\frac{1}{2}\begin{pmatrix} \|b\|^2 & -ab^* \\ -a^*b & \|a\|^2 \end{pmatrix}$	$\|\psi_-\rangle$	$\begin{pmatrix} -a \\ -b \end{pmatrix}$
$A_1=\frac{1}{2}\begin{pmatrix} \|b\|^2 & ab^* \\ a^*b & \|a\|^2 \end{pmatrix}$	$\|\psi_+\rangle$	$\begin{pmatrix} -a \\ b \end{pmatrix}$
$A_2=\frac{1}{2}\begin{pmatrix} \|a\|^2 & -a^*b \\ -ab^* & \|b\|^2 \end{pmatrix}$	$\|\phi_-\rangle$	$\begin{pmatrix} b \\ a \end{pmatrix}$
$A_3=\frac{1}{2}\begin{pmatrix} \|a\|^2 & a^*b \\ ab^* & \|b\|^2 \end{pmatrix}$	$\|\phi_+\rangle$	$\begin{pmatrix} -b \\ a \end{pmatrix}$

The first column shows the POV element, the second the corresponding outcome if Alice's measurement is regarded as a projection onto the Bell basis, and the third the state that Bob ends up with (before he does any unitary transformation) assuming that ρ is a singlet.

Now suppose that the teleportation procedure is repeated many times but with a modification: Bob does not bother performing unitary transformations. Instead, he wishes to determine how close the states he receives are to those being teleported (he can always take into account the fact that he *would* have performed a unitary transformation had he waited for the two classical bits). Bob does this by performing ordinary projective measurements. He performs them at spacelike separation from Alice's POV (or Bell basis) measurements. Also, suppose that Alice does not know each time which state she is teleporting. *Someone* knows which states are being teleported by Alice and she can be called Clare. From the table above, it is clear that Alice is actually performing a four-element POV measurement each time and which measurement she is performing is determined by $|\chi\rangle$ (that is, by the values of a and b). We have now reduced the standard teleportation scenario to a typical experimental scenario for testing locality via Bell inequalities. On each run of the experiment, Alice performs one of a selection of incompatible measurements (although it is Clare who makes the choice for her) and so does Bob. After many runs, Alice and Bob can get together and, with Clare's help, see how their results are correlated. The point is that, if the teleportation is being carried out with high fidelity, then Bob's results will be strongly correlated with Alice's and we can expect that some Bell-type inequality will be violated. If no Bell-type inequality is violated then we can say that the whole procedure could have been simulated with a local hidden variable model and that no nonlocality is therefore being exhibited. The task now is to derive a Bell-type inequality pertaining to the teleportation scenario and investigate when it might be violated.

In order to look for a Bell-type inequality that may be violated, we impose some restrictions. Suppose that Alice only ever teleports one of two possible states, fed to her randomly by Clare. This means that in the terms of the analysis above Alice is always performing one from a choice of two possible POV measurements, each of which has four possible outcomes. We restrict Bob to a choice of two possible projective measurements, each of which has two possible outcomes. If a joint outcome includes both Alice's and Bob's outcomes, then there are $(4+4)\times(2+2)=32$ possible joint outcomes. As discussed in Sec. II, Peres shows how to construct Bell-type inequalities pertaining to this scenario in [11]. Unfortunately, there are a very large number of such inequalities (the number increases very rapidly with the number of joint outcomes) and it is hard to know where to start looking if we want to find one that is violated. For this reason, we impose another restriction. Suppose that, although Alice's measurements each have four distinct outcomes, we group them into pairs, so that, for example, A_0 or A_2 counts as outcome 1 and A_1 or A_3 counts as outcome 2. We then have that Alice has a choice of two measurements, each with two possible outcomes, and so does Bob. This means that we can apply the well known Clauser-Horne inequality [9] (or something equivalent to it) directly to this case.

First, some terminology. Alice performs one of two measurements, which we label T and U. Alice performing measurement T corresponds to Clare giving her a state $|\chi\rangle=\binom{a}{b}$ to teleport. Alice performing measurement U, on the other hand, corresponds to Clare giving her a state $|\chi'\rangle=\binom{a'}{b'}$ to teleport. Measurement T has two outcomes, labeled t and $\bar{t}$. Outcome t corresponds to Alice getting A_0 or A_2 in her measurement. Outcome $\bar{t}$ corresponds to her getting A_1 or A_3. Similarly, measurement U has two outcomes, u and $\bar{u}$. u corresponds to outcome A_0 or A_3 while $\bar{u}$ corresponds to A_1 or A_2. Note how the grouping of four possible outcomes into pairs is done differently according to which measurement Alice is performing. It turns out that unless we do it like this, we do not get an inequality that is violated.

Bob's two measurements are denoted R and S, with outcomes r, $\bar{r}$, s, and $\bar{s}$. R and S can correspond to spin measurements in different directions, r and s to spin up results and $\bar{r}$ and $\bar{s}$ to spin down results.

The inequality we use is

$$0\leqslant \Pr(t,\bar{s})+\Pr(\bar{u},r)+\Pr(u,s)-\Pr(t,r)\leqslant 1. \qquad (3)$$

Here, $\Pr(t,r)$ represents the probability of Alice obtaining outcome t and Bob obtaining outcome r in one run (given that Alice performs measurement T and Bob measurement R). This is seen to be equivalent to the Clauser-Horne inequality by adding and subtracting $\Pr(t,s)+\Pr(u,r)$, leaving

$$0\leqslant \Pr(t)+\Pr(r)+\Pr(u,s)-\Pr(t,s)-\Pr(u,r)-\Pr(t,r)\leqslant 1,$$

where $\Pr(t)$ is the probability of Alice getting outcome t and $\Pr(r)$ the probability of Bob getting outcome r.

Now we substitute some values to show that this inequality can be violated for teleportation using a singlet. Using the standard rule for obtaining probabilities in quantum mechanics, we get

$$\Pr(t,r)=\mathrm{Tr}[\rho(A_0+A_2)\otimes P],$$

where P is the projection operator corresponding to outcome r. We get similar expressions for the other outcomes. If ρ is a singlet, this gives

$$0 \leqslant \tfrac{1}{4}[2 - c(r_x + s_x) + d(r_y - s_y)] \leqslant 1, \qquad (4)$$

where $c = ab^* + a^*b$ and $d = -i(a'b'^* - a'^*b')$. We also have that $(r_x, r_y, r_z) = \vec{r}$ is the direction of spin measurement R and $(s_x, s_y, s_z) = \vec{s}$ is the direction of spin measurement S. If we set

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

$$\vec{r} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \quad \vec{s} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right),$$

then Eq. (4), and therefore Eq. (3), is violated (we get $\frac{1}{4}[2 - c(r_x + s_x) + d(r_y - s_y)] = (1 - \sqrt{2})/2 \approx -0.21 < 0$). This corresponds to Alice teleporting one of two states — either spin up along the x axis or spin up along the y axis. Bob is performing one of two possible spin measurements, oriented in the xy plane at $\pm 45°$ to the x axis.

Thus we see that perfect teleportation using the standard scheme does imply nonlocality. In the next section, we introduce Werner states and in Sec. VI we investigate when teleportation using a Werner state might imply nonlocality.

V. WERNER STATES

The Werner states were first introduced by Werner in [15]. They are states in $\mathcal{H}_d \otimes \mathcal{H}_d$, where $\mathcal{H}_d$ is a d-dimensional Hilbert space, which consist of a mixture of an entangled pure state in $d \times d$ dimensions and rotationally symmetric noise represented by the identity. The d-dimensional Werner state is given by

$$W_d = \frac{1}{d^3} I + \frac{2}{d^2} P^{anti},$$

where P^{anti} projects onto the totally antisymmetric subspace of the product space and I is the identity acting on the product space. In his original paper, Werner writes this as

$$W_d = \frac{1+d}{d^3} I - \frac{1}{d^3} V,$$

where V is the operator which, acting on any two-particle product state $|\phi\rangle \otimes |\psi\rangle$, exchanges it to give $|\psi\rangle \otimes |\phi\rangle$. [Note that $P^{anti} = (1 - V)/2$.]

In particular, the two-dimensional Werner state is given by

$$W_2 = \tfrac{1}{8} I + \tfrac{1}{2} P^s,$$

where P^s is the projector onto the singlet state.

In [15], Werner introduces a local hidden variable model for these states. We assume that Alice and Bob share a supply of d-dimensional Werner states and that both perform a single projective measurement on each run of a Bell-type experiment. Briefly, the model works as follows. Any given pair of particles is assigned a hidden state $|\lambda\rangle$. $|\lambda\rangle$ is a vector of unit magnitude in d-dimensional complex space. We define a measure $\omega(\lambda)\, d\lambda$ over the possible values of $|\lambda\rangle$ which is unchanged under $U(d)$ rotations. This gives us the distribution of $|\lambda\rangle$ values obtained over many runs of the experiment. Suppose Alice performs a measurement A with possible outcomes a_k and Bob a measurement B with possible outcomes b_l. (We assume that both Alice's and Bob's measurements are maximal projective measurements — see [16] for how to extend the model, fairly trivially, to nonmaximal projective measurements.) We have that the probability of obtaining the particular outcomes a_i and b_j is

$$\Pr(a_i, b_j | A, B) = \int d\lambda\, \omega(\lambda) \Pr(a_i | A, \lambda) \Pr(b_j | B, \lambda).$$

We also have

$$\Pr(a_i | A, \lambda) = \begin{cases} 1 & \text{if } \langle\lambda|P_{a_i}|\lambda\rangle < \langle\lambda|P_{a_{i'}}|\lambda\rangle \quad \forall i' \neq i \\ 0 & \text{otherwise,} \end{cases} \qquad (5)$$

and

$$\Pr(b_j | B, \lambda) = \langle\lambda|P_{b_j}|\lambda\rangle \qquad (6)$$

where P_{a_k} and P_{b_l} are the projection operators corresponding to measurement outcomes a_k and b_l. It is then reasonably easy to show that $\Pr(a_i, b_j | A, B) = \mathrm{Tr}[\, W_d P_{a_i} \otimes P_{b_j}]$, which is the required quantum probability (see both Werner's original paper [15] and Mermin [16]).

It is significant that the Werner states are mixed, entangled states. It might seem surprising that entangled states can have a local hidden variable model, especially in the light of a result that says that all entangled pure states are nonlocal [17,18]. The story became more complicated when Popescu showed that the d-dimensional Werner states exhibit the hidden nonlocality we spoke of earlier, provided $d \geqslant 5$ [12,16]. By this we mean that for these states nonlocality can be revealed if Alice and Bob each perform a sequence of two projective measurements on each run of the experiment — note that the local hidden variable model above only predicts the results for single projective measurements. (More precisely, the only way in which a local theory could account for the results of the sequential measurments would be for the outcome of Alice's first measurement to be causally related to the choice of which measurement to perform for her second measurement — presumably this is something that a sensible theory should disallow if Alice is a free observer.) Whether an alternative scheme might show that the Werner states with $d \leqslant 4$ have hidden nonlocality or whether the use of POV measurements might show these states to be nonlocal are open questions.

Rather than consider these questions here, we wish to restrict attention to the two-dimensional case but generalize slightly so that we consider states of the form

$$W_2^{\alpha} = \frac{1-\alpha}{4} I + \alpha P^s.$$

This corresponds to Werner's original state in two dimensions when $\alpha=\frac{1}{2}$. It is entangled if $\alpha>\frac{1}{3}$. Note that it follows trivially from Werner's local hidden variable model for the $\alpha=\frac{1}{2}$ case that there also exists a local hidden variable model (for single projective measurements) for any $\alpha<\frac{1}{2}$. Rather than consider in full generality which of these states may be local or nonlocal, in the next section we consider what teleportation using Werner states might tell us about nonlocality.

VI. WERNER STATES AND TELEPORTATION

A. Local hidden variable models

It has sometimes been argued that the state $W_2^{\alpha=1/2}$, to which Werner's original local hidden variable model applies, must be nonlocal in some sense because, if used for teleportation, the fidelity achieved is $\frac{3}{4}$, which is greater than the so-called classical limit of $\frac{2}{3}$ (see, e.g., [19]). This classical limit is obtained as follows. It is in fact the best fidelity that can be obtained when Alice and Bob do not share any entangled quantum state and Alice does not know what state she is teleporting. Fidelity $\frac{2}{3}$ is achieved by Alice performing a measurement of spin along the z axis on her particle and communicating the result to Bob classically. Bob then prepares a state that is correspondingly spin up or down along the z axis. From here on, we refer to teleportation that works when Alice is ignorant of the state she is teleporting as "unknown-state teleportation" (this includes, for example, the standard scheme) and teleportation that requires Alice to know the state she is teleporting as "known-state teleportation." We can say, then, that unknown-state teleportation with fidelity $>\frac{2}{3}$ requires Alice and Bob to share an entangled quantum state. Popescu [19] describes any instance of unknown-state teleportation with fidelity $>\frac{2}{3}$ as a type of nonlocality (presumably in view of the fact that shared entanglement is required). He is well aware, however, that this would not necessarily imply violation of a Bell-type inequality and notes therefore that there are two types of nonlocality which are "inequivalent." In the context of our investigation we prefer to reserve the term nonlocality strictly for nonsimulability by a local hidden variable model, as explained in Sec. II. We can say simply that unknown-state teleportation with fidelity $>\frac{2}{3}$ demonstrates that entanglement is present. The question of whether the $W_2^{\alpha=1/2}$ state might have hidden nonlocality or nonlocality that can be revealed by POV measurements remains open — what we wish to argue here is that its ability to teleport with fidelity $\frac{3}{4}$ does not bear on this question.

We do this via a simple modification of Werner's hidden variable model. Consider the teleportation scenario above in which Bob does not bother waiting for Alice's two classical bits or performing a unitary transformation but instead performs a measurement of some kind. If we can provide a local hidden variable model for this scenario, then we have effectively described teleportation of fidelity $\frac{3}{4}$ using local hidden variables (even though $\frac{3}{4}$ is greater than the classical limit).

Werner's original model does not immediately apply because it applies to projective measurements on the Werner state only. Here, Alice is performing a Bell-basis measurement on her half of the shared Werner state and the state to be teleported. As described above, however, for our purposes we can regard this as a POV measurement on Alice's half of the shared state alone. Our aim is to adapt Werner's model so that we recover the correct quantum mechanical predictions for POV measurements performed on Alice's half of the shared state at spacelike separation from projective measurements performed on Bob's half. We can do this as follows. Suppose that Alice performs an arbitrary POV measurement A, with elements A_m such that $\Sigma_m A_m = I$. Then, for a particular outcome A_n, we can define

$$\Pr(A_n|A,\lambda)=\langle\lambda|A_n|\lambda\rangle, \tag{7}$$

where on the right-hand side (RHS) A_n represents a positive operator and on the LHS the corresponding experimental outcome.

If we assume that Werner's original model works, it is easy to show that this modified model works for an arbitrary POV measurement performed by Alice. If a spectral decomposition of A_n is given by $A_n=\Sigma_i q_i^n Q_i^n$, where $0\leqslant q_i^n\leqslant 1$ and Q_i^n is a projector, then

$$\begin{aligned}\Pr(A_n,b_j|A,B)&=\int d\lambda\ \omega(\lambda)\Pr(A_n|A,\lambda)\Pr(b_j|B,\lambda)\\ &=\int d\lambda\ \omega(\lambda)\langle\lambda|A_n|\lambda\rangle\Pr(b_j|B,\lambda)\\ &=\sum_i q_i^n\int d\lambda\ \omega(\lambda)\langle\lambda|Q_i^n|\lambda\rangle\Pr(b_j|B,\lambda)\\ &=\sum_i q_i^n\,\mathrm{Tr}[\,W_2^{\alpha=1/2}Q_i^n\otimes P_{b_j}]\\ &=\mathrm{Tr}[\,W_2^{\alpha=1/2}A_n\otimes P_{b_j}],\end{aligned}$$

which is the required result. We got from the third line to the fourth line by assuming that Werner's original model for projective measurements works.

Incidently, the model can also easily be extended to include the case where Bob performs a POV measurement provided Bob's POV elements commute pairwise. In this case, there exists a single basis in terms of which we can write down the spectral decomposition of all of Bob's POV elements. Suppose $B_{n'}=\Sigma_j q_j^{n'}Q_j$ where the Q_j are the projectors onto the elements of this basis (and the point is that they are independent of n'). If a projective measurement corresponding to a decomposition of the identity $\Sigma_j Q_j=I$ is performed, Werner's original model gives us the outcome probabilities via rule (5). For now, we abuse notation slightly and write these outcome probabilities as $\Pr(Q_j|\lambda)$. We can define the outcome probabilities for Bob's POV measurement as

$$\Pr(B_{n'}|B,\lambda)=\sum_j q_j^{n'}\Pr(Q_j|\lambda). \tag{8}$$

Reasoning similar to that above for the extension on Alice's side shows that this will reproduce the correct quantum mechanical probabilities.

As mentioned above, to our knowledge, no one has yet constructed a local hidden variable model for $W_2^{\alpha=1/2}$ which correctly reproduces the quantum mechanical probabilities for arbitrary POV measurements on both Alice's and Bob's sides. What we have here is quite adequate for simulation of our modified teleportation scenario provided Bob is not allowed arbitrary POV measurements. It seems unlikely that allowing Bob this freedom would reveal nonlocality where there was none before, but we cannot claim to have ruled this out. Thus we conclude that with this slight qualification the ability of the $W_2^{\alpha=1/2}$ state to teleport with fidelity $\frac{3}{4}$ does not betoken nonlocality.

In the next section we consider Werner states with general α. We use the Bell-type inequality derived in Sec. IV to investigate when imperfect teleportation using Werner states might imply nonlocality.

B. When does teleportation imply nonlocality?

In this section we use all the same notation as in Sec. IV. Consider again our Bell-type inequality (3):

$$0 \leqslant \Pr(t,\overline{s}) + \Pr(\overline{u},r) + \Pr(u,s) - \Pr(t,r) \leqslant 1.$$

If Alice's and Bob's shared state is $\rho = \alpha P^s + [(1-\alpha)/4]I$, then we get

$$0 \leqslant \tfrac{1}{4}\{2 - \alpha[c(r_x + s_x) + d(r_y - s_y)]\} \leqslant 1,$$

again with $c = ab^* + a^*b$ and $d = -i(a'b'^* - a'^*b')$. If, as before, we set

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

$$\vec{r} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \vec{s} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right),$$

then we have a violation if

$$\alpha > \frac{1}{\sqrt{2}}.$$

Thus *teleportation using the standard scheme and a Werner state with* $\alpha > 1/\sqrt{2}$ *implies nonlocality*. This of course implies that these states are nonlocal.

In fact, it is precisely the Werner states with $\alpha > 1/\sqrt{2}$ that violate the ordinary CHSH inequality [10] for a suitable choice of projective measurements, so we did not need to consider teleportation or derive this inequality merely to find out that these states are nonlocal. That they violate the CHSH inequality can be shown using a result of Ref. [20] as follows. Any density matrix operating on $\mathcal{H}_2 \otimes \mathcal{H}_2$ can be written as

$$\rho = \tfrac{1}{4}(I \otimes I + V_i^1 \sigma_i \otimes I + V_i^2 I \otimes \sigma_i + T_{ij} \sigma_i \otimes \sigma_j),$$

where the σ_i are the Pauli σ matrices, V_i^1 and V_i^2 are real three-dimensional vectors, T_{ij} is a real 3×3 matrix, and repeated indices are summed over. (Note that there are other conditions that $\vec{V}^1$, $\vec{V}^2$, and T must satisfy for ρ to be a genuine density matrix.) Horodecki *et al.* show that ρ violates the CHSH inequality for some choice of measurements for Alice and Bob if and only if $t_1 + t_2 > 1$, where t_1 and t_2 are the largest two eigenvalues of $T^T T$.

It is easy to show that, in order to write a generalized two-dimensional Werner state in this form, we set

$$\vec{V}^1 = 0, \ \vec{V}^2 = 0, \quad \text{and} \quad T_{ij} = -\alpha\, \delta_{ij}$$

to get

$$W_2^\alpha = \tfrac{1}{4}(I \otimes I - \alpha\, \delta_{ij}\, \sigma_i \otimes \sigma_j).$$

This state violates the CHSH inequality if and only if $\alpha > 1/\sqrt{2}$, as claimed.

What we have shown in this section, then, is not just that the Werner states with $\alpha > 1/\sqrt{2}$ are nonlocal — we already knew this from [20]. We have shown, in addition, that consideration of teleportation using these states can also reveal them as nonlocal. This can be regarded as in keeping with a conjecture made by Zukowski [5] who suggests that "the quantum component of the teleportation process cannot be described in a local and realistic way as long as the initial [shared] state . . . neither admits such models."

To summarize, we have a local hidden variable model to describe teleportation using any Werner state with $\alpha \leqslant 1/2$ (with the slight caveat that we have not included the case in which Bob performs arbitrary POV measurements on the state that he receives). We have also shown that teleportation with the standard scheme and a Werner state with $\alpha > 1/\sqrt{2}$ does imply nonlocality. States with $1/2 < \alpha \leqslant 1/\sqrt{2}$ do not violate the CHSH inequality for any choice of projective measurements but apart from this, questions about their locality remain open. When used for teleportation, they do not violate our inequality but might violate some other inequality.

Note that a state with α just greater than $1/\sqrt{2}$ will teleport with a fidelity just greater than $\frac{1}{2}(1 + 1/\sqrt{2}) \approx 0.85$. The teleportation procedure will involve nonlocality even though this value for the fidelity is below a bound derived by Gisin, which is ~ 0.87 [4]. We discuss this further in Sec. VII.

Our investigation has been restricted to teleportation using Werner states and the standard scheme. It might be interesting to try to extend these results and find something more general such that unknown-state teleportation implies nonlocality if the fidelity is F or higher and can otherwise be described using a local hidden variable model. It may well be, however, that there is no such result to be found. There could exist two states ρ and ρ' and two teleportation schemes, which teleport with fidelities F and F', such that teleportation with ρ implies nonlocality while teleportation with ρ' does not even though $F < F'$.

VII. GISIN'S RESULT

Before concluding we would like to comment on a result of Gisin's which he claims has relevance to teleportation and nonlocality [4]. Gisin derives a value for teleportation fidelity that is given by

$$F=\frac{1}{2}+\sqrt{\frac{3}{2}}\frac{\arctan\sqrt{2}}{\pi}\approx 0.87.$$

He describes this as an "upper bound for the fidelity of quantum teleportation explainable by local hidden variables." The value of F is derived as follows.

First Gisin notes that the shared state D must be local, and "hence useless for teleportation." Then, "within the local hidden variable paradigm, Alice could measure the state ψ_{Alice} in the classical sense of 'measuring:' finding out what the state ψ_{Alice} is." Here, ψ_{Alice} is the state that Alice is teleporting (which we earlier called $|\chi\rangle$). In the light of this, to derive the value for F above, we assume that Alice and Bob share nothing, that Alice knows the quantum state she is trying to teleport, and that Alice sends two classical bits to Bob. The best they can do is to divide the surface of the Bloch sphere into four regions. Alice uses the two classical bits to let Bob know which region the state she is teleporting lies in. Bob then prepares a state in the center of this region. The optimal way of doing this is to inscribe a regular tetrahedron in the Bloch sphere. The areas of the surface of the sphere subtended by the faces of the tetrahedron are the four regions used. Calculation then gives the average fidelity obtained as F above.

We feel that the description of F as an "upper bound for the fidelity of quantum teleportation explainable by local hidden variables" is slightly misleading. On the assumption that "explainable" here can be replaced with "simulable," the fidelity explainable by local hidden variables rather depends on what is to count as a simulation of a quantum teleportation procedure. Under our and Zukowski's approach, we are happy if a local theory can predict the results of Alice's Bell measurement and of a spacelike-separated measurement made by Bob (for a completely different approach which is equally interesting, see [7]). Using our and Zukowski's approach, we found that the ability of the state $W_2^{\alpha=1/2}$ to teleport with fidelity $\frac{3}{4}$ does not betoken any form of nonlocality. On the other hand, the fact that under the standard scheme a state W_2^{α} with α just greater than $1/\sqrt{2}$ teleports with fidelity just greater than $\frac{1}{2}(1+1/\sqrt{2})\approx 0.85$ does betoken nonlocality. This is despite the value of ~ 0.85 being below Gisin's bound. It is true that, if Alice knows the state she is trying to teleport, then she can do better than this using only local means — this is what Gisin's result shows. If Alice does not know the state she wants to teleport, however, then the standard teleportation scheme is the best she can do and this will involve nonlocality. (Note that the standard scheme is indeed the optimal scheme for unknown-state teleportation using a Werner state; see, e.g., [21]).

In addition, there is no reason why teleportation with a fidelity greater than Gisin's bound should not be simulable by local hidden variables in some cases. The key here is that hidden variable theories are surely only required to reproduce probabilities for measurement results. In other words a hidden variable model need not specify an actual quantum state to be received by Bob on each run of the experiment — we must assume that Bob does some kind of measurement and it is only the outcome of this measurement that must be predicted by the model. So it is not quite correct to say that a shared state that is local must be "useless for teleportation." The local correlations may be useful in helping Alice and Bob achieve correlated measurement results.

At the least, we feel that Gisin's result is less genuinely to do with locality than our own or Zukowski's [5] or those of Cerf *et al.* [7]. This does not mean that Gisin's result is uninteresting, however. Interpreted as the best fidelity achievable when Alice and Bob share nothing at the start of the protocol, and Alice knows the state she is trying to teleport and is limited to the sending of two classical bits, it is correct. It can be contrasted with the value of $\frac{2}{3}$ for the fidelity which is the best Alice and Bob can do when they share nothing at the start of the protocol and Alice does not know the state she is trying to teleport.

Gisin's result is also useful. In the case that Alice and Bob share a nonmaximally entangled quantum state, they cannot achieve unit fidelity. Gisin's result shows that if Alice knows the state she is trying to teleport, is limited to the sending of two classical bits to Bob, and the best fidelity achievable with a quantum scheme is $< F\approx 0.87$, then they may as well not bother using the shared quantum state. They would do better to use the purely classical scheme above.

VIII. CONCLUSION

Perfect teleportation (i.e., teleportation with unit fidelity) initially seems paradoxical because only two classical bits are sent, yet Bob ends up with a quantum system in a state identical with the state of Alice's input system — and it would take an infinite amount of classical information to specify precisely a quantum state. It is concluded (rather vaguely) that some sort of nonlocality must be involved — the extra "information" must be transmitted nonlocally. Vaidman argues that, correctly interpreted, quantum teleportation involves the transfer of an object from one place to another without it ever being located in the intervening space [3]. This also sounds vaguely paradoxical and might suggest nonlocality (although Vaidman himself is more concerned to reconcile this view of teleportation with his own belief in a many-worlds type interpretation of quantum mechanics). Rather than adopt either of these two viewpoints, we are more inclined to dissolve these paradoxes (at least partially) by sharing Hardy's doubts concerning the reality of the information apparently transmitted during teleportation (see the quotations in Sec. III). We suggest that the paradox is resolved if we consider a quantum state as being a description of an ensemble of systems, rather than a single system — Bob can identify the state and any information contained therein by performing measurements on the whole ensemble. But to teleport the whole ensemble, Alice does indeed send an infinite number of classical bits.

Having said this, teleportation might still involve nonlocality. If we define nonlocality to mean nonsimulability by local hidden variables, then to speak meaningfully of teleportation being local or nonlocal, we must have Bob performing a measurement of some sort on the state that he receives (because hidden variable models are required only to reproduce the results of measurements). Bob's measurement is at spacelike separation from Alice's. We can speak of the teleportation as being nonlocal if Bob's results are correlated with Alice's in a way that cannot be simulated with a local model.

In investigating this, we have considered perfect teleportation using a singlet, derived an appropriate Bell-type inequality, and shown that it is violated. So perfect teleportation is nonlocal. We have also considered teleportation using Werner states, of the form $\rho=\alpha P^s+[(1-\alpha)/4]I$. Using the same inequality, we found that the teleportation is nonlocal precisely for those Werner states that violate the CHSH inequality, i.e., those with $\alpha>1/\sqrt{2}$. These teleport with fidelity $F>\frac{1}{2}(1+1/\sqrt{2})\approx 0.85$. We also extended Werner's local hidden variable model for the $\alpha=\frac{1}{2}$ states to give a local model describing teleportation using these states (the fidelity of which is $\frac{3}{4}$). We concluded that the ability to teleport with fidelity $\frac{3}{4}$ does not confer nonlocality in this case, with the qualification that we have not allowed Bob arbitrary POV measurements.

Broadly speaking, the status of the (two-dimensional) Werner states with respect to locality remains unknown. Teleportation shows that those with $\alpha>1/\sqrt{2}$ are nonlocal — but we already knew this. Teleportation using the $\alpha=\frac{1}{2}$ state can be simulated locally — but the state may still have a hidden nonlocality to be revealed by other means. We do not know whether (unknown-state) teleportation using the $\frac{1}{2}<\alpha\leqslant 1/\sqrt{2}$ states can be simulated locally or not, or whether they might have nonlocality to be revealed by other means.

Related independent results have been very recently circulated by Clifton and Pope [22].

Note added in proof. A local hidden variable model allowing for POV measurements on Bob's side has recently been constructed for the $\alpha=\frac{5}{12}$ state, which teleports with fidelity $\frac{17}{24}$ [23]. Thus we can simulate teleportation with fidelity $\frac{17}{24}$ without the qualification that Bob is restricted to projective measurements.

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