Pseudoforces in quantum mechanics

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Dynamical evolution is described as a parallel section on an infinite-dimensional Hilbert bundle over the base manifold of all frames of reference. The parallel section is defined by an operator-valued connection whose components are the generators of the relativity group acting on the base manifold. In the case of Galilean transformations we show that the property that the curvature for the fundamental connection must be zero is just the Heisenberg equations of motion and the canonical commutation relation in geometric language. We then consider linear and circular accelerating frames and show that pseudoforces must appear naturally in the Hamiltonian.

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I. INTRODUCTION

Evolution of a state vector in quantum mechanics can be viewed as a kind of parallel transport [1]. There have been suggestions to use the geometric language of vector bundles and parallel transport in various situations in quantum mechanics [2–6]. These ideas are natural in the discussion of the geometric or the Berry phase [7].

Despite these attempts to ''geometrize'' quantum mechanics there seems to be no common agreement in these approaches about the base space, or the structure group, let alone the connection or the curvature. Moreover it is not clear whether the extra mathematical machinery is justified by a new or clearer physical insight.

In this paper we give the *physical* reason why the bundle viewpoint is natural in quantum mechanics and illustrate it with application to accelerated frames.

If a physical system is observed in various frames of reference, the states described by them as vectors in their individual Hilbert spaces will form a section in a vector bundle with the Hilbert space as the standard fiber and the set of all frames of reference as the base manifold. There is no canonical identification of the fibers and we need a ''connection,'' a notion of covariant derivative or that of parallel transport.

We make use of the principle of relativity (all frames of reference are equally suitable for description) to provide the notion of parallelism and make the following assumption: States described by different frames of reference form a parallel section.

As each observer can apply an overall unitary operator on his Hilbert space and still obtain an equivalent description, we see that the structure group should be the group of all unitary operators $U(\infty)$ on the Hilbert space [8]. Thus there is an underlying ''gauge freedom'' that can be used to transform the natural parallel sections into constant sections and do away with the need to use all Hilbert spaces at once. This is the case in standard quantum mechanics where a single Hilbert space is used by all observers.

In this paper we develop our geometric picture and explicitly consider the case where Galilean group is the underlying relativity group. We find that Heisenberg equations of motion and the canonical commutation relation are contained in a single condition: that the fundamental connection is flat or that its curvature is zero.

Next we apply the geometric construction to accelerated frames and show that pseudoforce terms appear as expected. In the case of linearly accelerated frames we get a linear ''gravitational'' potential implying that equivalence principle must hold in quantum mechanics. In contrast, in the conventional formalism equivalence principle is obtained by an artificial time-dependent phase transformation of the wave function. In the case of rotating frames we show that both centrifugal and coriolis forces show up in the Hamiltonian. It is satisfactory to see that the coriolis force does not correspond to a potential because it does no work, being perpendicular to velocity, but naturally appears as a connection term added to the canonical momentum, much like the magnetic force. We are thus able to show that fibre bundles are the natural language in which to discuss quantum-mechanical effects of gravity.

II. GEOMETRIC SETTING

A. The bundle

Consider a quantum-mechanical system described by observers in different frames of reference. We assume that the set of all frames of reference forms a differentiable manifold. This is physically reasonable because frames of reference are related by symmetry transformations that form a group. This means that the frames can be labeled by coordinates x on the group manifold. A state of the system is described by a vector $\phi(x)$ in a Hilbert space $\mathcal{H}_x$ associated with the observer x. We, thus, have the ingredients of a vector bundle [9]. The base is a manifold M with coordinates x and a Hilbert space at each point. To every possible state of the system is associated a section or a mapping $x \rightarrow \phi(x)$ where $\phi(x)$ is the vector describing the state of the system by observer x. We assume there exist unitary operators $U(y,x)$ that connect the states $\phi(y)=U(y,x)\phi(x)$. These operators must satisfy consistency conditions

$$U(z,y)U(y,x)=U(z,x); \quad U(x,x)=1.$$

We must note that there is no canonical way of choosing states $\phi(x)$ to describe the system in the Hilbert space $\mathcal{H}_x$. If we were to apply a unitary operator to all vectors $\phi(x)$, $\psi(x)$, etc. in $\mathcal{H}_x$, the resulting states are equally well suited to describe the system provided the observables acting in $\mathcal{H}_x$ are similarly changed. In other words, we assume the *structure* or *gauge group* acting on the fiber to be the group of all unitary operators.

B. The connection

Let us choose a complete orthonormal set ϕ_α in the Hilbert space of some fixed observer, say at $x=0$, $\phi_\alpha \equiv \phi_\alpha(0)$. The sets $\phi_\alpha(x) = U(x,0)\phi_\alpha(0)$ then are complete orthonormal sets in all the other spaces $\mathcal{H}_x$. Any arbitrary section $\psi(x)$ can then be written as

$$\psi(x) = \sum_\alpha c_\alpha(x)\phi_\alpha(x), \tag{1}$$

where $c_\alpha(x)$ are the complex coefficients of expansion. Let Γ be the set of all sections. They can be added pointwise.

$$(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x) \tag{2}$$

and multiplied with smooth functions

$$(c\psi)(x) = c(x)\psi(x). \tag{3}$$

Let $\Lambda \otimes \Gamma$ be the tensor product of the space Λ of all one-forms on the base M and Γ. A connection on this bundle is a mapping D: $\Gamma \to \Lambda \otimes \Gamma$ such that

$$D(\psi_1 + \psi_2) = D\psi_1 + D\psi_2$$

and

$$D(c\psi) = cD\psi + dc\,\psi. \tag{4}$$

If $\phi_n(x)$ is a basis in Γ we can express $D(\phi_n)$ in terms of the basis $dx^\mu \otimes \phi_m$ in $\Lambda \otimes \Gamma$ as

$$(D\phi_n)(x) = \phi_m(x)\Gamma_{\mu mn}dx^\mu. \tag{5}$$

where coefficients $\Gamma_{\mu mn}(x)$ are the Christoffel symbols with respect to the basis $dx^\mu \otimes \phi_m$. We write this equation as

$$(D\phi_n)(x) = \phi_m \omega^\phi_{mn}(x), \tag{6}$$

where the complex matrix ω_{mn} can be obtained by taking inner product with ϕ_m in Eq. (5).

$$\omega^\phi_{mn} = (\phi_m, D\phi_n). \tag{7}$$

This matrix of one-forms is called the *connection matrix*. We require D to satisfy the Leibniz rule

$$D(\phi,\psi) = (D\phi,\psi) + (\phi,D\psi) = d(\phi,\psi), \tag{8}$$

which when applied to $\delta_{mn} = (\phi_m, \phi_n)$ shows that ω^ϕ is an anti-Hermitian matrix. Under a change of basis

$$\chi_n(x) = U(x)\phi_n(x), \tag{9}$$

we have

$$\chi_n(x) = \phi_m(x)(\phi_m(x), U(x)\phi_n(x)) = \phi_m(x)U_{mn}(x). \tag{10}$$

Thus, omitting the base point x for simplicity of notation

$$\begin{aligned}(D\chi_n)(x) &= D(\phi_s U_{sn}) = \phi_r \omega^\phi_{rs} U_{sn} + \phi_s dU_{sn} = \chi_m \omega^\chi_{mn} \\ &= \phi_r U_{rm}\omega^\chi_{mn}\end{aligned} \tag{11}$$

or

$$\omega^\chi_{mn} = U^{-1}_{mr}\omega^\phi_{rs}U_{sn} + U^{-1}_{mr}dU_{rn}. \tag{12}$$

Omitting matrix indices, we have

$$\omega^\chi = U^{-1}\omega^\phi U + U^{-1}dU. \tag{13}$$

The curvature two-form for the connection is given by

$$\Omega^\phi = d\omega^\phi + \omega^\phi \wedge \omega^\phi, \tag{14}$$

which transforms as

$$\Omega^\chi = U^{-1}\Omega^\phi U. \tag{15}$$

One may also note the Bianchi identity

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega. \tag{16}$$

III. PARALLEL SECTION AND THE FUNDAMENTAL CONNECTION

We now make the fundamental assumption that a system observed by different observers is represented by parallel sections. Let $\phi_m(x)$ be a family of parallel sections, that is

$$(D\phi)(x) = 0, \quad \text{for all } m. \tag{17}$$

This implies

$$\omega^\phi_{mn}(x) = 0 \tag{18}$$

everywhere.

We shall now see how does the connection matrix look like with respect to the basis of constant sections. The advantage of using constant sections is that one can give up the bundle picture altogether and identify all Hilbert spaces together to work in one common space. The constant section physically means that the state is represented by the same constant vector by all observers. This is the most general definition of the Heisenberg picture.

To get constant sections we use the fact that parallel sections are constructed by applying transformation $U(x,0)$ on $\phi(0)$ for all x:

$$\phi_m(x) = U(x,0)\phi_m(0) = U(x)\phi_m(0). \tag{19}$$

We can choose $\phi_m(0)$ as the new basis

$$\begin{aligned}\chi_m(x) &\equiv \phi_m(0) \\ &= U_x^{-1}\phi_m(x) \\ &= \phi_r(x)(\phi_r(x), U^{-1}(x)\phi_m(x)) \\ &= \phi_r(x)U^{-1}_{rm}.\end{aligned} \tag{20}$$

Then

$$\omega^{\chi} = U dU^{-1}, \tag{21}$$

which, as expected, is pure gauge.

IV. GALILEAN FRAMES

Let us consider a particle of mass m in one space dimension. We use units where $c=\hbar=1$. We consider the basis of sharp momentum states $|k\rangle$ such that

$$P|k\rangle = k|k\rangle \tag{22}$$

and

$$\langle k'|k\rangle = \delta(k-k'). \tag{23}$$

The time and space translations are given by the operators U_τ and U_ζ, respectively,

$$U_\tau|k\rangle = e^{-iH\tau}|k\rangle = e^{-ik^2/2m}|k\rangle \tag{24}$$

$$U_\zeta|k\rangle = e^{iP\zeta}|k\rangle = e^{ik\zeta}|k\rangle. \tag{25}$$

The boosts act as

$$U_\eta|k\rangle = |k-m\eta\rangle \tag{26}$$

given by

$$U_\eta = e^{-iK\eta}, \tag{27}$$

where K is the boost generator. The Lie algebra of the Galilean group is

$$[P,H]=0,\ \ [K,H]=iP,$$

$$[K,P]=imI. \tag{28}$$

The algebra is not closed. This is because unitary representation of the Galilean group in $\mathcal{H}$ is projective. The position operator X is related to K [10] as

$$K = mX \tag{29}$$

and it acts on the states $|k\rangle$ as

$$\langle k|X = i\frac{\partial}{\partial k}\langle k|. \tag{30}$$

Parallel sections can be constructed using U_τ, U_ζ, and U_η in a variety of ways. We choose the following convention that corresponds to the transformations

$$x' = x - \eta t - \zeta$$

$$t' = t - \tau \tag{31}$$

$$v' = v - \eta$$

between frame S and S'. If we take the standard frame at $x=0$, $t=0$, $v=0$ then

$$\phi(x',t',v') = \phi(-\zeta,-\tau,-\eta) = U_\tau U_\zeta U_\eta \phi(0,0,0). \tag{32}$$

We rename coordinates

$$\phi(x,t,v) = U_{-t}U_{-x}U_{-v}\phi(0,0,0) \tag{33}$$

and get, for the basis of constant sections,

$$\begin{aligned}\omega^{\chi} &= U_{-t}U_{-x}U_{-v}d(U_{-t}U_{-x}U_{-v})^{-1}\\ &= i[-Hdt + Pdx + X(t)mdv - mxdv].\end{aligned} \tag{34}$$

The curvature is zero, as it should be for a pure gauge connection. But it is worth seeing explicitly.

$$\Omega^{\chi} = d\omega + \omega \wedge \omega = 0. \tag{35}$$

This implies the following equations:

$$i\dot{P} = [P,H], \tag{36}$$

$$i\dot{X} = [X,H], \tag{37}$$

$$[X,P] = i. \tag{38}$$

Equations (34) and (35) are just the Heisenberg equations of motion for operators P and X while Eq. (38) is the canonical commutation relation for X and P.

One may argue that these equations are just reproductions of the algebra. Indeed the algebra is used in the calculation of the curvature. What is new is that in this differential geometric language all the information is contained in a single zero-curvature equation.

V. ACCELERATED FRAMES AND PSEUDOFORCES

Acceleration implies changing from one Galilean frame to another after every infinitesimal amount of time. This can be seen as a curve on the base manifold parametrized by time. We assume that an observer in the accelerating frame uses the same Hilbert space to describe a physical system as the observer at the base manifold point with which it coincides at each instant t. Moreover they assign the same state to the system [11].

A. Linearly accelerated frame and equivalence principle

The question of whether the principle of equivalence in classical mechanics also holds in quantum mechanics was discussed by Eliezer and Leach [12]. They studied the transformation of the Schrödinger equation under a change from an inertial frame of reference S to a uniformly accelerating one S'. Their argument goes as follows. Let

$$x' = x + \frac{1}{2}gt^2, \quad t' = t \tag{39}$$

be the change of coordinates to an accelerated frame. Then the equivalence principle holds provided the phase of the wave function of the system is redefined by a time-dependent expression. This means that the Schrödinger equation in the frame S

$$i\frac{d\psi}{dt}=-\frac{1}{2m}\nabla^2\psi \tag{40}$$

transforms to the equation for a particle moving in a uniform field

$$i\frac{d\psi'}{dt'}=-\frac{1}{2m}\nabla'^2\psi'-mgx'\psi' \tag{41}$$

with the redefinition of the phase of ψ given by

$$\psi'(x')=\exp\left(\frac{img}{\hbar}x't'-\frac{1}{6}gt'^3\right)\psi(x). \tag{42}$$

The phase factor has been chosen precisely to obtain equivalence principle. There is no explanation put forward for this factor.

In our formalism we find that the equivalence principle must hold in quantum mechanics in a straightforward manner. There is no need for any other condition such as the redefinition of the wave function by a time-dependent phase factor, like the one seen above.

Consider an observer in a linearly accelerated frame of reference. The linear acceleration corresponds to a curve on the base manifold parametrized by t and given by

$$x=\frac{1}{2}gt^2, \tag{43}$$

$$v=gt. \tag{44}$$

The parallel section is again specified by

$$|t,x,v;k\rangle\equiv U_vU_xU_t|k\rangle. \tag{45}$$

The rate of change of the vector along the curve should give the Hamiltonian for the accelerated observer. We get

$$\begin{aligned}i\frac{d}{dt}|t,x,v;k\rangle&=i\frac{d}{dt}(U_tU_xU_v|k\rangle)\\&=\left[\frac{(k-mv)^2}{2m}+(k-mv)t\frac{dv}{dt}-(k-mv)\right.\\&\quad\left.\times\frac{dx}{dt}-mx\frac{dv}{dt}-mgx-i\frac{dv}{dt}\frac{\partial}{\partial v}\right]|t,x,v;k\rangle\\&=\left[\frac{P(v)^2}{2m}-X(x)mg\right]|t,x,v;k\rangle,\end{aligned} \tag{46}$$

where $P(v)=k-mv$ and $X(x)=X+x$. Thus, the system "sees" an extra potential $X(x)mg$ that is the expected linear "gravitational" potential term. This is a manifestation of the equivalence principle in quantum mechanics.

The validity of the equivalence principle in the quantum regime has been experimentally tested in some beautiful experiments done with neutron interference [13].

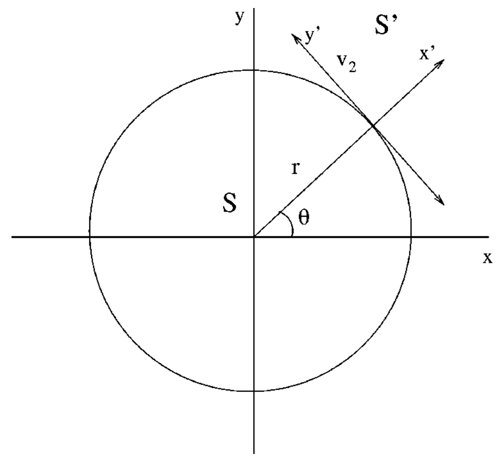

FIG. 1. S' is a frame that is rotating with angular velocity ω about origin of frame S with radius r.

B. Rotating frame, coriolis, and centrifugal forces

Consider a frame of reference S' that is rotating with constant angular velocity ω and radius r about the origin of coordinates on the xy plane of a frame S (see Fig. 1). The two frames are related as follows: we wait for time t, translate by $\mathbf{r}$ direction, rotate by angle $\theta=\omega t$, and finally give a boost in the y' direction by velocity v.

The parallel section is given by

$$U=U_vU_\theta U_{\mathbf{r}}U_t=e^{-iX_2mv}e^{iJ\theta}e^{iP\mathbf{r}}e^{-i(P^2/2m)t} \tag{47}$$

where $J=X_1P_2-X_2P_1$ is the angular momentum operator. The curve on the base manifold parametrized by t is

$$\mathbf{r}=(r\cos\theta,r\sin\theta),$$

$$\mathbf{v}=(-r\omega\sin\theta,r\omega\cos\theta). \tag{48}$$

The Hamiltonian H, as seen by an observer in the rotating frame, is given by the rate of change of the vectors specified along the curve on the base manifold.

$$\begin{aligned}H&=i\frac{dU}{dt}U^{-1}\\&=\frac{1}{2m}[P_1^2+(P_2+m\omega r)^2]-\omega r(P_2+m\omega r)-\omega(J+m\omega r)\end{aligned} \tag{49}$$

or

$$\begin{aligned}H=&\frac{1}{2m}[(P_1+m\omega X_2)^2+(P_2-m\omega X_1)^2]\\&-\frac{1}{2}m\omega^2[(X_1+r)^2+X_2^2].\end{aligned} \tag{50}$$

Thus the expected centrifugal and coriolis forces appear in the Hamiltonian. Since coriolis force does no work it cannot appear as an explicit potential term. Rather it appears as a connection in the canonical momentum.

VI. DISCUSSION

The bundle viewpoint is hinted in Dirac's work as early as 1932. In a most influential paper [14] Dirac puts forth the following argument: Let $q(t)$ be a complete set of commuting observables in the Heisenberg picture. The set of eigenvalues q' at each t forms a manifold M giving rise to "space-time" $B \equiv M \times T$ where T represents the time axis.

Evolution is determined by the moving basis $\langle q',t|$ at each (q',t). This can be interpreted as a section from the base B into a Hilbert space. Let $c:\tau \to (q'(\tau),t(\tau))$ be a curve in B. Then the change of basis vectors $\langle q',t|$ is given by

$$-i\frac{\partial}{\partial q'}\langle q',t| = \langle q',t|P(t),$$

$$i\frac{\partial}{\partial t}\langle q',t| = \langle q',t|H,$$

where $P(t) = e^{iHt}P(0)e^{-iHt}$.

Thus the change of a basis vector along the curve is

$$d\langle q',t| = i\langle q',t|dS,$$

$$dS = P(t)dq' - Hdt.$$

Thus in the bundle formalism Dirac's Lagrangian can be seen as an operator-valued one-form on the Hilbert vector bundle whose base manifold is spanned by the eigenvalues q' of a complete set of commuting operators $q(t)$ specified at all times and the standard fiber is Hilbert space. The components of this one-form are just the Hamiltonian and momentum operators. If the section $(q',t) \to \langle q',t|$ is assumed to be parallel then evolution can be seen as parallel transport. This is the theme on which our present work is based.

Asorey *et al.* [1] consider a Hilbert bundle with positive time axis $\mathcal{R}^+$ as the base manifold. Another viewpoint is that of Prugovecki [2] and Drechsler and Tuckey [3] whose bundle is the associated vector bundle for the principle bundle with Poincare group as structure group over space-time base manifold. The Hilbert space considered by them is the space of square integrable functions over phase space of space coordinates and the mass hyperboloid ($p^2 = m^2$, $p_0 > 0$). This approach allows them to consider parallel transport over curved spaces with possible applications to quantum gravity.

Graudenz [4] also has a Hilbert bundle with space-time base. Our approach agrees with that of Graudenz in that description of a physical system is always description by one observer. Yet another construction is given by Sardanashvily [5] who considers a C^* algebra at each point of the time axis $\mathcal{R}$.

Our geometric construction is different from others in the literature. For us the base manifold consists of all frames of reference. This means actually having a group of symmetry transformations as the base manifold with a frame of reference associated with each point on it. We have considered the case of Galilean group that makes the application specific to quantum mechanics.

Our objective here is to present a different geometric viewpoint which implies the validity of the equivalence principle in quantum mechanics. We have demonstrated this for both linearly accelerating and rotating frames.

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