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One-complex-plane representation approach to continuous variable quantum teleportation

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We formulate continuous variable quantum teleportation on a coherent-state basis. We present low-dimensional coherent state representation of the quadrature Bell states. This approach turns out to be suitable for investigating the teleportation process, yielding a simple direct description.

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The description of entangled states and the analysis of their applications has attracted a great deal of attention in quantum optics recently. One of the main motivations of this trend was the quantum teleportation phenomenon, which was originally introduced by Bennett [1]. The idea of continuous variable teleportation appeared quite soon after Bennett's original paper in a work by Vaidman [2], but this idea was put into the framework of quantum optics by Braunstein and Kimble quite a bit later than the discrete schemes [3]. However, first experimental realizations of discrete and continuous variable teleportation appeared quite simultaneously in both cases [4,5].

The formulation of Braunstein and Kimble in Ref. [3] utilizes the Wigner-function formalism. Their scheme may also be described in terms of either wave functions on a quadrature-state basis [6,7] or Fock states [8,7]. A general covariant description in terms of arbitrary canonically conjugate observables and their eigenstates is also possible [9].

Coherent states have proven to be extremely useful in quantum optics of single mode field. The overcompleteness of the coherent state basis allows us to introduce representations in lower dimensional, or even discrete, subspaces of the phase space [10–12]. A similar approach may be fruitful in the investigation of entangled multimode fields and their applications. In this paper we will show that teleportation can be treated in this manner: the process can be understood describing entangled states with coherent state integrals.

In what follows we consider the actual teleportation scheme of Braunstein and Kimble under ideal circumstances. The physical systems under consideration are single mode fields. As entangled states, we consider ideal Einstein-Podolsky-Rosen pairs obtained from squeezed vacuum in infinite squeezing limit, and perfect detection of quadrature amplitudes, which results in a projection onto quadrature eigenstates, according to the von Neumann principle. We also outline the effect of finite squeezing.

This paper is organized as follows. Using a one-dimensional representation of quadrature eigenstates, we obtain a one-complex-plane representation of the two mode entangled states playing an important role in teleportation. Then the description of continuous variable teleportation is provided.

Local measurements of a given field mode in the scheme under consideration are carried out by detectors measuring the value of either of the quadratures

$$\hat{x}=\frac{\hat{a}+\hat{a}^{\dagger}}{2},\quad \hat{p}=\frac{\hat{a}-\hat{a}^{\dagger}}{2i}. \tag{1}$$

According to the von Neumann projection principle, the measurement results in the projection to one of the eigenstates,

$$\hat{x}||X\rangle=X||X\rangle,\quad \hat{p}||P\rangle=P||P\rangle \tag{2}$$

depending on the measurement result, which is the value X or P, respectively. (The symbol $||\cdots\rangle$ denotes quadrature eigenstates.)

The Bell-state detector of the teleportation scheme in argument consists of an $\hat{x}$ detector and a $\hat{p}$ detector, combined with a beam splitter to convert two local quadrature measurements to a joint measurement on two modes. The whole apparatus then projects onto an entangled state of the two modes, the quadrature the Bell states, depending on the values X and P, measured.

With this picture in mind we construct the one-dimensional representation of quadrature eigenstates. (The word dimension stands for real, and not for complex dimension throughout this paper.) Let kets containing a single number denote coherent states. We start with the following states [10]:

$$|\text{Sq. vac. } p\rangle=\mathcal{N}(r)\int_{-\infty}^{\infty}dx\,G_r(x)|x\rangle,$$

$$|\text{Sq. vac. } x\rangle=\mathcal{N}(r)\int_{-\infty}^{\infty}dy\,G_r(y)|iy\rangle, \tag{3}$$

where

$$\mathcal{N}(r)=\frac{1}{\sqrt{\pi}}\frac{e^{r/2}}{\sqrt{e^{2r}-1}},\quad \text{and}\quad G_r(x)=e^{-(|x|^2/e^{2r}-1)}. \tag{4}$$

These are superpositions of coherent states placed on the real and imaginary axis of the phase space, respectively. It is straightforward to show that the mean values of the quadratures are 0, and for their variances

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$$\Delta\hat{p}^2_{|\text{Sq. vac.}p\rangle}=\Delta\hat{x}^2_{|\text{Sq. vac.}x\rangle}=\frac{e^{-2r}}{4},$$

$$\Delta\hat{x}^2_{|\text{Sq. vac.}p\rangle}=\Delta\hat{p}^2_{|\text{Sq. vac.}x\rangle}=\frac{e^{2r}}{4} \tag{5}$$

hold, and therefore these are squeezed vacuum states. If r tends towards infinity, the variance of the corresponding quadratures becomes zero, thus the states become quadrature eigenstates:

$$||P=0\rangle=\lim_{r\to\infty}|\text{Sq. vac. } p\rangle,$$

$$||X=0\rangle=\lim_{r\to\infty}|\text{Sq. vac. } x\rangle. \tag{6}$$

However, $\lim_{r\to\infty}\mathcal{N}(r)=0$, which expresses the fact that quadrature eigenstates need to be normalized in terms of probability densities instead of individual probabilities. For simplicity in what follows we omit this normalization factor. The states in Eq. (6) can then be written as

$$||P=0\rangle=\lim_{r\to\infty}\int_{-\infty}^{\infty}dxG_r(x)|x\rangle=\int_{-\infty}^{\infty}dx|x\rangle,$$

$$||X=0\rangle=\lim_{r\to\infty}\int_{-\infty}^{\infty}dyG_r(y)|iy\rangle=\int_{-\infty}^{\infty}dy|iy\rangle. \tag{7}$$

Finally, quadrature eigenstates can be obtained by shifting states in Eq. (7) using the Glauber displacement operator $\hat{D}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})$:

$$||P\rangle=\hat{D}(iP)||P=0\rangle=\int_{-\infty}^{\infty}dxe^{ixP}|x+iP\rangle,$$

$$||X\rangle=\hat{D}(X)||X=0\rangle=\int_{-\infty}^{\infty}dye^{-iXy}|X+iy\rangle. \tag{8}$$

Now we consider the Bell-state detector. Suppose that modes 1 and 2 interfere on the lossless beam splitter BS, and then the quadratures $\hat{x}$ of mode 1 and $\hat{p}$ of mode 2 are measured, and they were found to be X and P, respectively. The measurement projects the state of modes 1 and 2 at the output port of the beam splitter to

$$|\Psi_{\text{prod}_{X,P}}\rangle=||X\rangle_1||P\rangle_2. \tag{9}$$

This is a product state basis on the Hilbert space of these two modes. Our aim is to calculate the inverse beam-splitter transform of the states $|\Psi_{\text{prod}_{X,P}}\rangle$, which will yield an entangled state basis. We note that the connection of displacement and entangled states appears in other descriptions too [9,13].

Armed with the representations in Eq. (8), we may describe the action of a beam splitter quite simply. Two-mode coherent states interfere on beam splitters as classical fields, that is, their amplitudes transform as the annihilation operators. Particularly, we may consider a 50/50 beam splitter, with phase shifts chosen so that for the output state $|\alpha\rangle_1|\beta\rangle_2$ the corresponding input state is $|(\alpha+\beta)/\sqrt{2}\rangle_1|(\beta-\alpha)/\sqrt{2}\rangle_2$. Because of the linearity of the beam splitter, the inverse transform of arbitrary superpositions of coherent states may be written as

$$\int d^2\alpha\int d^2\beta\Phi(\alpha,\beta)|\alpha\rangle_1|\beta\rangle_2$$

$$\to\int d^2\alpha\int d^2\beta\Phi(\alpha,\beta)\left|\frac{\alpha+\beta}{\sqrt{2}}\right\rangle_1\left|\frac{\beta-\alpha}{\sqrt{2}}\right\rangle_2. \tag{10}$$

Here $\Phi(\alpha,\beta)$ is an arbitrary function, and the complex integrals may be replaced by any kinds of integrals or sums.

Equation (10) can be applied in our actual case: according to the von Neumann projection principle, the state of output modes of the beam splitter is projected onto

$$|\Psi_{\text{prod}_{X,P}}\rangle=||X\rangle_1||P\rangle_2$$

$$\to\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dxe^{i(xP-Xy)}|X+iy\rangle_1|x+iP\rangle_2, \tag{11}$$

thus its inverse transform, the corresponding Bell state reads

$$|B(X,P)\rangle=\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dxe^{i(xP-Xy)}\left|\frac{x+iy+X+iP}{\sqrt{2}}\right\rangle_1$$

$$\times\left|\frac{x-iy+iP-X}{\sqrt{2}}\right\rangle_2. \tag{12}$$

Introducing two complex variables

$$\gamma:=\frac{x+iy}{\sqrt{2}},\quad A:=\frac{X+iP}{\sqrt{2}}, \tag{13}$$

the state in Eq. (12) reads

$$|B(X,P)\rangle=\int d^2\gamma e^{A\gamma^*-A^*\gamma}|\gamma+A\rangle_1|\gamma^*-A^*\rangle_2. \tag{14}$$

These are the quadrature Bell states playing an important role in continuous variable quantum teleportation.

Starting from the one real dimensional representation in Eq. (8) of the quadrature eigenstates, we have obtained a representation of a two mode state which is a superposition of two mode coherent states with amplitudes on one single complex plane. Therefore we call it one-complex-plane representation.

If $A=0$, then we obtain the state

$$|\Psi_{\text{EPR id.}}\rangle=\int d^2\gamma|\gamma\rangle_1|\gamma^*\rangle_2. \tag{15}$$

It will be shown in a separate paper that this is the one-complex plane representation of a two-mode infinitely squeezed vacuum state. The effect of finite squeezing can be represented by the Gaussian factor $G_r(\sqrt{2}|\gamma|)$ in the integrand if Eq. (15) inherited from Eq. (7).

Let us now turn our attention to the teleportation process. Alice has an arbitrary quantum state $|\Psi_{\mathrm{in}}\rangle$ in mode 1, which she wants to teleport to Bob. A general pure state may be written in Glauber's analytic representation as

$$|\Psi_{\mathrm{in}}\rangle_1 = \int d^2\beta e^{-(|\beta|^2/2)} f(\beta^*)|\beta\rangle_1, \qquad (16)$$

where $f(\beta^*)$ is an analytic function of β^*. Alice and Bob share a two-mode squeezed vacuum state

$$|\Psi_{\mathrm{EPR}}\rangle_{23} = \int d^2\alpha G_r(\sqrt{2}|\alpha|)|\alpha^*\rangle_2|\alpha\rangle_3, \qquad (17)$$

as an EPR state for the teleportation. As discussed previously, in the ideal case $G_r(\sqrt{2}|\alpha|)\rightarrow 1$. The state of the whole system of all three modes is thus initially

$$\begin{aligned}|\Psi_i\rangle_{123} &= |\Psi_{\mathrm{in}}\rangle_1 \otimes |\Psi_{\mathrm{EPR}}\rangle_{23}\\ &= \int d^2\alpha \int d^2\beta G_r(\sqrt{2}|\alpha|) e^{-(|\beta|^2/2)}\\ &\quad \times f(\beta^*)|\beta\rangle_1|\alpha^*\rangle_2|\alpha\rangle_3. \qquad (18)\end{aligned}$$

In the next step, Alice then carries out a joint measurement, resulting a pair of values X,P, which is communicated to Bob via a classical channel. This measurement results in the projection of the state of modes 1 and 2 to one of the quadrature Bell states in Eq. (14). Therefore in what follows we shall omit all constant multiplying factors from the formulas. The (unnormalized) projected state of mode 3 reads

$$\begin{aligned}|\Psi_f\rangle_3 &= {}_{12}\langle B(X,P)|\Psi_i\rangle_{123}\\ &= \int d^2\alpha \int d^2\beta \int d^2\gamma G_r(\sqrt{2}|\alpha|) e^{-(|\beta|^2/2)}\\ &\quad \times f(\beta^*) e^{A^*\gamma - A\gamma^*}\langle \gamma + A|\beta\rangle\langle \gamma^* - A^*|\alpha^*\rangle|\alpha\rangle_3. \qquad (19)\end{aligned}$$

The integrals in β and γ can be evaluated via the successive application of the Glauber's integral identity

$$\frac{1}{\pi}\int d^2\alpha e^{-|\alpha|^2 + \alpha\beta^*} f(\alpha^*) = f(\beta^*), \qquad (20)$$

which is valid for any function f analytical in α^*. Applying Eq. (20) twice, integrating over γ and the over β, we obtain

$$\begin{aligned}|\Psi_f\rangle_3 &= \int d^2\alpha G_r(\sqrt{2}|\alpha|) e^{-(|\alpha|^2/2)} e^{-2A\alpha^*}\\ &\quad \times f(\alpha^* + 2A^*)|\alpha\rangle_3. \qquad (21)\end{aligned}$$

In the limit of ideal entanglement, $G_r(\sqrt{2}|\alpha|)\rightarrow 1$, this state becomes

$$|\Psi_f\rangle = D(-2A)|\Psi_{\mathrm{in}}\rangle. \qquad (22)$$

The state in mode 3 is a shifted version of the incoming state. Bob, in the knowledge of P and X, can carry out the inverse displacement to restore the original state.

Note that the displacement to be done by Bob is the identity operator if and only if $X=P=0$, and in this case the two-mode Bell state measured by Alice is the same as the shared entangled state. The same situation appears in the case of discrete variable teleportation.

If the entangled state is not ideal, the $G_r(\sqrt{2}|\alpha|)$ Gaussian smoothing factor appears in Eq. (21). After the inverse displacement, the result of the teleportation reads

$$|\Psi_f\rangle = \int d^2\alpha G_r(\sqrt{2}|\alpha - 2A|) e^{-|\alpha|^2/2} f(\alpha^*)|\alpha\rangle. \qquad (23)$$

The smoothing depends not only on r but also on A as a consequence of the finite number of photons contained in the entangled state. We remark that in order to calculate fidelity of teleportation (cf. Ref. [13]) in our formalism, one may average in A by forming a density matrix from the state in Eq. (23), and calculating the probability distribution of A from Eq. (21).

We have shown that by using one-complex-plane coherent state representation of quadrature states, quadrature Bell states can be represented by integrating on a single complex plane. Using this representation we have found that an alternative and rather plausible description of continuous variable quantum teleportation can be formulated. This approach is different from all previous considerations applying Wigner functions, photon number states, or quadrature wave functions. Regarding the role of coherent states, in the development of the theory of nonclassical states of light, our approach may prove to be useful in the further investigation of quantum teleportation and related phenomena.

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