

Complete solution of the Schrödinger equation for the time-dependent linear potential

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The complete solutions of the Schrödinger equation for a particle with time-dependent mass moving in a time-dependent linear potential are presented. One solution is based on the wave function of the plane wave, and the other is in the form of the Airy function. A comparison is made between the present solution and former ones to show the completeness of the present solution.

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The analytical solution of the Schrödinger equation with explicitly time-dependent potential has drawn much attention over past decades. Besides the intrinsic mathematical interest, this problem connects with various applications to many physical problems, for example, the degenerate parametric amplifier [1] and the quantum motion of trapped ions in the Paul trap [2]. To make clear the dynamical properties of the system with explicitly time-dependent potential, numerical simulation can be generally applied. However, some information about the system such as the Berry phase [3] and the squeezing property [4] will be probably neglected unless we can obtain the completely analytical solution of the system.

Not all systems with explicitly time-dependent potentials can be solved analytically. During the past several years, some efforts have been invested in finding the solution of the time-dependent harmonic-oscillator (TDHO) Hamiltonian. The most famous work in this respect is the invariant approach proposed by Lewis and Riesenfeld [5]. In terms of this idea and other elaborate methods, the TDHO Hamiltonian has been investigated from different angles and for different physical problems [6]. As far as we know, the general TDHO Hamiltonian with the potential of $g_2(t)x^2 + g_1(t)x + g_0(t)$, where $g_i(t)$ ($i=0,1,2$) are arbitrary time-dependent variables, has been solved, and the exact but very complicated form of the corresponding wave function has been presented [7]. Recently, a more general TDHO problem was studied in which the exact form of the propagator could be found [8]. Moreover, the investigation in this respect has been extended to the TDHO Hamiltonian with additional potentials [9,10], where the result in Ref. [9] can be used to approximately describe the dynamics of two trapped cold ions in the Paul trap [11].

Besides the TDHO problem, the linear potential model has also been frequently employed in some other studies [12,13]. Recently, this model was investigated quantum mechanically [14], in which an analytical-wave-function solution for such a system was presented by means of the invariant method. Although the author of [14] claimed that his result is the first presentation in this respect, such a problem has actually been studied before [15], in which the solution, with the form of the Airy function was presented. It was

shown [15,16] that the solution with the Airy function for describing the behavior of the free particle, corresponds to a wave packet moving acceleratively with no change of form. However, the acceleration of the Airy packet is not the behavior of any individual particle, but is caused by the caustic of the family of particle orbits. So there is no contradiction with Ehrenfest's theorem that no wave packet can accelerate in free space.

The purpose of the present paper is to undertake a completely analytical solution for the problem above along the idea in [15,16] by means of a simple algebra, named "time-space transformation method" [17]. With the time-space transformation method, in [17], we transformed the Schrödinger equation with TDHO into that with time-independent harmonic oscillator. But here we will try to transform a Schrödinger equation with a time-dependent linear potential into that of a free particle. According to [15,16], there are only two solutions with nonspreading properties for the quantum treatment of a free particle. One solution is based on the wave function of the plane wave and the other is with the form of the Airy function. However, as far as we know, no one has reported these two solutions simultaneously in treating the Hamiltonian with a time-dependent linear potential. Therefore, in what follows, we will consider a more general case than in Refs. [14,15], i.e., a particle with time-dependent mass moving in the time-dependent linear potential. It can be found that the solution in Ref. [14] is merely a particular case for a "standing" particle under the potential, in comparison with our result. Besides, we will present specifically the analytical solution of this problem with the form of the Airy function.

Consider the Schrödinger equation for a particle with time-dependent mass moving in a time-dependent linear potential, which can be described by the Schrödinger equation in units of $\hbar=1$,

$$i\frac{\partial}{\partial t}\Psi(x,t) = -\frac{1}{2M(t)}\frac{\partial^2}{\partial x^2}\Psi(x,t) + g_1(t)x\Psi(x,t), \quad (1)$$

where $M(t)$ and $g_1(t)$ are arbitrary time-dependent variables. Performing a unitary transformation $\Psi(x,t) = \Phi(x,t)e^{i\beta(t)x}$ with $\beta(t)$ being a time-dependent variable determined later, we have

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$$\begin{aligned}
i\frac{\partial}{\partial t}\Phi(x,t) &= -\frac{1}{2M(t)}\frac{\partial^2}{\partial x^2}\Phi(x,t) - i\frac{\beta(t)}{M(t)}\frac{\partial}{\partial x}\Phi(x,t) \\
&+ \frac{\beta(t)^2}{2M(t)}\Phi(x,t) + x\dot{\beta}(t)\Phi(x,t) \\
&+ g_1(t)x\Phi(x,t), \tag{2}
\end{aligned}$$

where the dot on the variable denotes the derivative with respect to time. If we perform the time and space transformation of $y=x+\nu(t)$ and $s=\int_0^t d\sigma/M(\sigma)$, where $\nu(t)$ will be determined later, Eq. (2) is changed to

$$\begin{aligned}
\frac{i}{M(t)}\frac{\partial}{\partial s}f(y,s) + i\dot{\nu}(t)\frac{\partial}{\partial y}f(y,s) &= -\frac{1}{2M(t)}\frac{\partial^2}{\partial y^2}f(y,s) \\
- i\frac{\beta(t)}{M(t)}\frac{\partial}{\partial y}f(y,s) + [g_1(t) + \dot{\beta}(t)][y - \nu(t)]f(y,s) \\
+ \frac{\beta(t)^2}{2M(t)}f(y,s) \tag{3}
\end{aligned}$$

in which $\Phi(x,t)=f(y,s)$ is used. To delete the term of $(\partial/\partial y)f(y,s)$, we set $\dot{\nu}(t)=-\beta(t)/M(t)$. Thus

$$\begin{aligned}
\frac{i}{M(t)}\frac{\partial}{\partial s}f(y,s) &= -\frac{1}{2M(t)}\frac{\partial^2}{\partial y^2}f(y,s) + [g_1(t) \\
+ \dot{\beta}(t)]yf(y,s) + G(t)f(y,s), \tag{4}
\end{aligned}$$

where $G(t)=\beta^2(t)/2M(t)-[g_1(t)+\dot{\beta}(t)]\nu(t)$. If we assume $g_1(t)+\dot{\beta}(t)=0$, and $f'(y,s)=f(y,s)\exp[-i\int_0^t G(t')dt']$, we will obtain the following equation for a free particle with mass equivalent to 1:

$$i\frac{\partial}{\partial s}f'(y,s) = -\frac{1}{2}\frac{\partial^2}{\partial y^2}f'(y,s). \tag{5}$$

From the usual textbook of quantum mechanics, we know that the simplest form of the solution is $f'(y,s)=(1/\sqrt{2\pi})\exp[i(Ay-A^2s/2)]$ with A being an arbitrary real number if we define the particle as propagating or counter-propagating along the direction of y . Reversing the procedure above, we can obtain

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\exp\{iA[x+\nu(t)]\}\exp\left\{-i\frac{A^2}{2}\int_0^t \frac{d\sigma}{M(\sigma)}\right\} \\
&\times \exp\left\{-i\int_0^t G(\sigma)d\sigma + ix\beta(t)\right\} \tag{6}
\end{aligned}$$

with $\beta(t)=-\int_0^t g_1(\sigma)d\sigma$ and $\nu(t)=-\int_0^t [\beta(\sigma)/M(\sigma)]d\sigma$. To compare with the solution in Ref. [14], we let $g_1(t)$ take the form of $q(\epsilon_0 + \epsilon \cos \omega t)$ and set $M(t)=m$, which yields

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\exp\left\{iA\left[x + \frac{q}{m}\left(\frac{\epsilon_0}{2}t^2 - \frac{\epsilon}{\omega^2}\cos \omega t + \frac{\epsilon}{\omega^2}\right)\right]\right. \\
&- i\frac{A^2t}{2m}\left.\right\}\exp\left\{-i\frac{q}{\omega}(\epsilon_0\omega t + \epsilon \sin \omega t)x\right\} \\
&\times \exp\left\{-i\frac{q^2}{2m\omega^3}\left[\frac{\epsilon_0^2(\omega t)^3}{3} + 2\epsilon_0\epsilon(\sin \omega t \right. \right. \\
&- \left. \left. \omega t \cos \omega t) + \epsilon^2\left(\frac{1}{2}\omega t - \frac{1}{4}\sin 2\omega t\right)\right]\right\}. \tag{7}
\end{aligned}$$

Obviously, when $A=0$, our solution is equivalent to Eq. (18) of Ref. [14] (Ref. [18]). As the physical meaning of A is the momentum component of the free particle along the propagating direction, the solution in Ref. [14] can be considered as a special case that the particle is ‘‘standing’’ in the potential of $g_1(t)x$.

In fact, for Eq. (5), besides the solution with the wave function of the plane wave, there is a remarkable but not widely known solution, called the ‘‘nonspreading wave packet’’ or the ‘‘Airy packet’’ solution in the form of $\text{Ai}(B[y-B^3s^2/4])\exp\{i(B^3s/2)(y-[B^3s^2/6])\}$, in which B is an arbitrary constant and Ai the Airy function [15,16]. So the wave function of Eq. (1) is

$$\begin{aligned}
\Psi(x,t) &= \text{Ai}\left(B\left[x + \int_0^t \frac{d\tau}{M(\tau)}\int_0^\tau g_1(\sigma)d\sigma - \left(\frac{B^3}{4}\int_0^t \frac{d\sigma}{M(\sigma)}\right)^2\right]\right) \\
&\times \exp\left\{i\frac{B^3}{2}\int_0^t \frac{d\sigma}{M(\sigma)}\left[x + \int_0^t \frac{d\tau}{M(\tau)}\int_0^\tau g_1(\sigma)d\sigma \right. \right. \\
&- \left. \left. \frac{B^3}{6}\left(\int_0^t \frac{d\sigma}{M(\sigma)}\right)^2\right]\right\}\exp\left\{-\frac{i}{2}\int_0^t \frac{d\tau}{M(\tau)}\right. \\
&\times \left.\left[\int_0^\tau g_1(\sigma)d\sigma\right]^2\right\}\exp\left\{-ix\int_0^t g_1(\sigma)d\sigma\right\}. \tag{8}
\end{aligned}$$

If we set $M(t)=m$, we will find that Eq. (8) here is formally different from Eq. (15) in Ref. [15] (Ref. [19]). However, the correctness of Eq. (8) can be tested simply by substituting Eq. (8) into Eq. (1) and using the Airy function’s properties [20]. It is easily seen that the probability density $|\Psi(x,t)|^2$ moves without change of form. The Airy packet propagates along the trajectory given by

$$x_0(t) = \frac{B^3}{4}\left[\int_0^t \frac{d\sigma}{M(\sigma)}\right]^2 - \int_0^t \frac{d\tau}{M(\tau)}\int_0^\tau g_1(\sigma)d\sigma. \tag{9}$$

If we choose $M(t)=m$ and $g_1(t)=B^3/2m$, then $x_0(t)=0$, which means the Airy packet is at rest. More specifically, setting $g_1(t)=q(\epsilon_0 + \epsilon \cos \omega t)$ will yield

$$\begin{aligned}
\Psi(x,t) = & \text{Ai} \left[B \left(x + \frac{q\epsilon_0}{2m} t^2 - \frac{q\epsilon}{m\omega^2} \cos \omega t + \frac{q\epsilon}{m\omega^2} - \frac{B^3 t^2}{4m^2} \right) \right] \\
& \times \exp \left\{ i \frac{B^3 t}{2m} \left[x + \frac{q\epsilon_0}{2m} t^2 - \frac{q\epsilon}{m\omega^2} \cos \omega t + \frac{q\epsilon}{m\omega^2} \right. \right. \\
& \left. \left. - \frac{B^3 t^2}{6m^2} \right] \right\} \times \exp \left\{ -ix \left(q\epsilon_0 t + \frac{q\epsilon}{\omega} \sin \omega t \right) \right\} \\
& \times \exp \left\{ -i \frac{q^2}{2m} \left[\frac{\epsilon_0^2}{3} t^3 + \frac{\epsilon^2}{2\omega^3} \left(\omega t - \frac{1}{2} \sin 2\omega t \right) \right. \right. \\
& \left. \left. + \frac{2\epsilon_0\epsilon}{\omega^3} (\sin \omega t - t\omega \cos \omega t) \right] \right\}. \quad (10)
\end{aligned}$$

In this case the Airy packet propagates sinusoidally with no change of form.

In summary, we have presented the completely analytical solution of a system with a time-dependent-mass particle moving in a time-dependent linear potential. The solution based on the wave function of the plane wave shows that the

result in Ref. [14] is merely the solution of the system in a particular case. The other solution with the form of the Airy function gives the specific form of a solution of the system, which has not been presented before. Each of the solutions has a free parameter representing the velocity of the wave packet of the solution. As it is exact and complete, the present result can be used to investigate quantum properties of the system with a time-dependent linear potential in a wider range of parameters, and serves as a comparison with other approximate works. We hope that the present work would be helpful for future exploration in this respect.

Note added. Recently the author was informed that a former work [21] for an electron moving in a general time-dependent electromagnetic field, whose result is very similar to that in the present work, had been carried out by generally constructing the evolution operators. The solution in [21] based on the wave function of the plane wave is the same as in the present work, whereas their solution with the Airy function is only for a special value of B defined in Eq. (10).

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