Enhancement of quantum tunneling oscillations due to nonlinear interactions

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We show that atomic-tunneling oscillations between two Bose-Einstein condensates can be increased as well as decreased by preparing the appropriate initial conditions and the nonlinear self-interaction. There is a trade-off relation between the oscillation amplitudes and the frequencies: smaller the oscillation amplitudes, higher the oscillation frequencies. The trajectory of the state vector on the Bloch sphere gives us an intuitive understanding of the increase of the tunneling oscillations. Similar effect predicted in this paper is expected for other nonlinear systems such as in the coupled quantum dots and waveguide coupling, in which the on-site energy and the self-phase-modulation, respectively, give rise to the nonlinear interactions.

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The problem of tunneling through the barrier of a double-well has received a considerable amount of attention as a fundamental problem in physics [1]. Considering a double-well potential with well-separated minima, if an atom is initially localized in one well, it can oscillate between the wells by quantum tunneling with the tunneling period $\tau_T = \pi/\kappa$, where κ is the tunneling coefficient. An attempt to enhance the tunneling oscillations by external fields has appeared in Ref. [2]. Guerin and Jauslin have theoretically shown that an enhancement of tunneling oscillations of the atom for a quartic double-well potential model of the NH_3 molecule could be realized by a pulse-shaped laser field. The mechanism of the enhancement is based on the photon-mediated tunneling through an excited state that has a larger tunneling coefficient than the ground states [2,3].

In this paper, we will show that the enhancement of the tunneling oscillations between two wells can be achieved in nonlinear systems without the external fields. Recently, the closely related topics have been reported by Wu and Nlu [4] and Zobay and Garraway [5]. The authors have investigated the influence of the nonlinearities (atomic self-interactions) on time-dependent tunneling (Landau-Zener tunneling) processes of Bose-Einstein condensates (BEC's). The authors discussed how the interactions modify the tunneling probability by a nonlinear Landau-Zener equation and claimed that the interactions can cause significant increase as well as decrease of tunneling probabilities.

Here, one interesting question arises: Can we enhance not the ultimate tunneling probabilities in nonlinear Landau-Zener tunneling, but the frequency of the tunneling oscillations itself? It seems apparently impossible in linear systems, but in the nonlinear systems it may be possible. As one of the nonlinear systems, we consider the tunneling of the neutral atoms between two BEC states in a double-well trap and show that the enhancement of the tunneling oscillations is really obtained for appropriate conditions of the initial populations in the double wells and of the magnitudes of the nonlinearities. In order to give the intuitive physical understanding of the enhancement, we will also show the visual trajectories of the state vector on the Bloch sphere.

Consider a condensate with N atoms in a trap potential $V(\vec{r})$. In the Hartree approximation, the state of the condensate is described by the Gross-Pitaevskii equation (GPE). It accurately describes the condensate wave function $\phi_N(\vec{r},t)$ in the presence of particle interactions in thermal equilibrium at temperatures well below the critical temperature,

$$i\hbar \frac{\partial \phi_N}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) + NU_0|\phi_N|^2 \right]\phi_N, \qquad (1)$$

where $U_0 = (4\pi\hbar^2 a/m)$ is the interatomic-scattering pseudopotential, a and m are atomic scattering length and mass, respectively.

We study the atomic tunneling at zero temperature between two weekly linked BEC's in a symmetric double-well trap shown in Fig. 1. The dynamics of this system can be governed by two Gross-Pitaevskii equations for the BEC amplitudes. That is, in the two-mode approximation we write $\phi_N(\vec{r},t) = b_1(t)u_1(\vec{r}) + b_2(t)u_2(\vec{r})$, satisfying the simple normalization $|b_1(t)|^2 + |b_2(t)|^2 = 1$. Substituting this into GPE (1), we can obtain nonlinear coupled-mode equations [6–9]

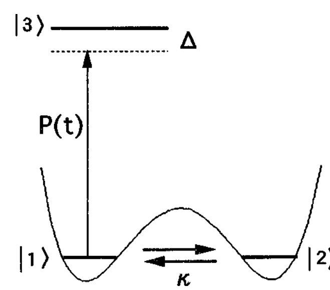

FIG. 1. Symmetric double-well trap for two Bose-Einstein-condensate states $|1\rangle$ and $|2\rangle$, which are coupled by the tunneling through the potential barrier, and states $|1\rangle$ and $|3\rangle$ are coupled by laser field, which prepares the initial phase ϕ_1 of state $|1\rangle$. The Rabi frequency of the laser field and the detuning are represented by Ω_L and Δ, respectively.

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$$\frac{db_1}{dt} = -i(\omega_1^0 + \Omega|b_1|^2)b_1 - i\kappa b_2, \tag{2a}$$

$$\frac{db_2}{dt} = -l(\omega_2^0 + \Omega|b_2|^2)b_2 - i\kappa b_1, \tag{2b}$$

where $\Omega = U_0 N/\hbar$ is proportional to the nonlinearity and κ is the coupling (tunneling) matrix element between the traps (see Fig. 1) given by $\kappa = \int[(\hbar^2/2m)(\nabla u_1 \nabla u_2) + u_1 V u_2]dr$ [10]. For a symmetric double-well trap, the zero-point energies in two traps are equal, i.e., $\omega_1^0 = \omega_2^0 = \omega^0$, and the tunneling coefficient κ can be determined from appropriate overlap integrals of the time-independent GPE eigenfunctions of the isolated traps. The total number of atoms $N = N_1 + N_2 = N(|b_1|^2 + |b_2|^2)$ is conserved. In Fig. 1, the excitation process by a laser pulse from level 1 to an excited level 3 is employed for the preparation of the initial phase (ϕ_1, see below), which is controllable by the detuning Δ and the area of the laser pulse [11].

The Gross-Pitaevskli equation describing the mean-field dynamics of a BEC is formally identical to the nonlinear Schrodinger equation that has appeared earlier in other fields, such as a model of polaron hopping in semiclassical approximation [12], nonlinear directional coupler of nonlinear optics [13], and Coulomb blockade of coherent electron oscillations in the coupled quantum dots [9], although they describe quite different physics. Therefore, many of the results, methods, and insights can be fruitfully applied from these other fields to the study of BEC dynamics and vice versa.

In what follows, we show some numerical results of Eq. (2) for the various initial conditions. The general form of the probability amplitudes of Eq. (2) is given by $b_1(0) = \sqrt{N_1(0)}e^{i\phi_1}$ and $b_2(0) = \sqrt{N_2(0)}e^{i\phi_2}$, where $N_{1,2}(0)$ and $\phi_{1,2}$ are the initial number of atoms and phases of the condensates in traps 1 and 2, respectively. We begin with numerical results for initial condition, all atoms being in one of two traps, i.e., $b_1(0) = \sqrt{N}$ $[N_1(0) = N]$ and $b_2(0) = 0$ $[N_2(0) = 0]$. In Fig. 2, we show the time evolution of the number of atoms in trap 1 and 2 with solid and dashed curves, respectively. The calculations were performed for a wide range of the nonlinear parameter, i.e., $\Omega/\kappa = 0$, 3, 3.9, 4, 4.1, and 5. As long as the nonlinearity does not exceed the threshold value ($\Omega/\kappa \leqq 4$), atoms initially localized on one trap can be completely transferred to the other well, and oscillate back and forth between the traps. If the nonlinearity exceeds the threshold, i.e., for $\Omega/\kappa > 4$, this oscillation becomes abruptly incomplete. We refer this to the quantum self-trapping transition in BEC [6,7,9], which is accompanied by the critical slowing down [7,9]. The oscillation frequency gradually decreases as the nonlinearity is increased and the oscillation period becomes very long at $\Omega/\kappa = 4$. Above this value, the oscillation mode changes abruptly and shows the quantum self-trapping oscillations with high frequency, larger than the bare tunneling frequency [Figs. 2(e) and 2(f)].

Here we are, however, interested in the symmetric population-exchange oscillations between two traps that have the time-averaged populations of 0.5N. We now consider the initial conditions of $N_1 = 0.7N$, $N_2 = 0.3N$, and $\phi_1 = \phi_2 = 0$. The time evolution of the populations is shown in Fig. 3 for $\Omega/\kappa = -50$, -45, -25, -10, 0, 2, 2.3, and 5. In the range $-4.5 \leqq \Omega/\kappa \leqq 2$, we can see symmetrical oscillations between two traps with a constant modulation amplitude 0.4. For $\Omega/\kappa = -50$ [Fig. 3(a)] and $\Omega/\kappa \geqq 2.3$ [Figs. 3(a) and 3(h)], the population dynamics exhibit two kinds of the macroscopic quantum self-trapping. In Fig. 3(a), the system is in the first type of trapped state (first type of π-phase

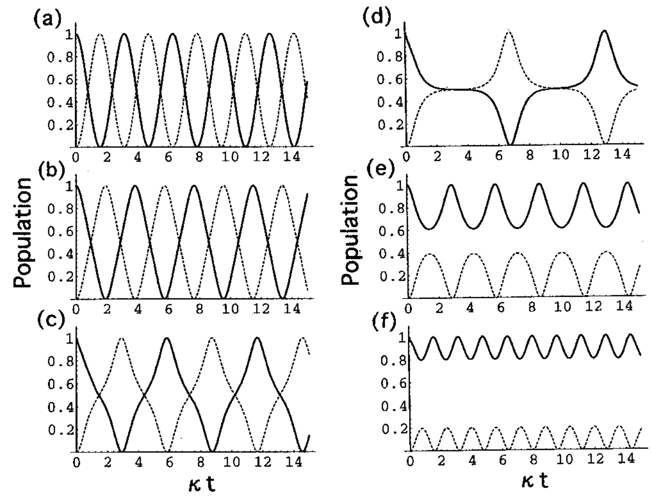

FIG. 2. Atomic-tunneling oscillations between two traps as a function of dimensionless time $\kappa\iota$ with the initial conditions $b_1(0) = \sqrt{N}$ and $b_2(0) = 0$. The value of the nonlinear self-interaction Ω/κ takes the values (a) 0, (b) 3, (c) 3.9, (d) 4, (e) 4.1, and (f) 5.

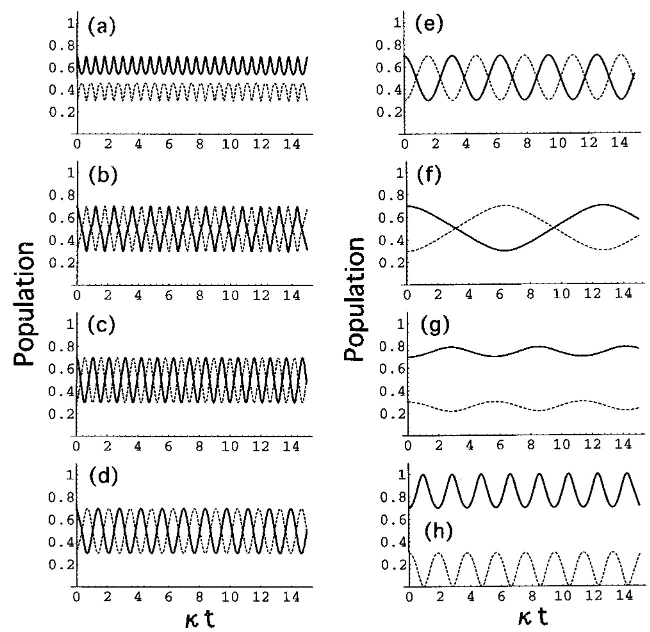

FIG. 3. Atomic-tunneling oscillations between two traps as a function of dimensionless time $\kappa\iota$ with the initial conditions $b_1(0) = \sqrt{0.7N}$ and $\phi = \phi_1 - \phi_2 = 0$. The value of the nonlinear self-interaction Ω/κ takes the values (a) -50, (b) -45, (c) -25, (d) -10, (e) 0, (f) 2, (g) 2.3, and (h) 5.

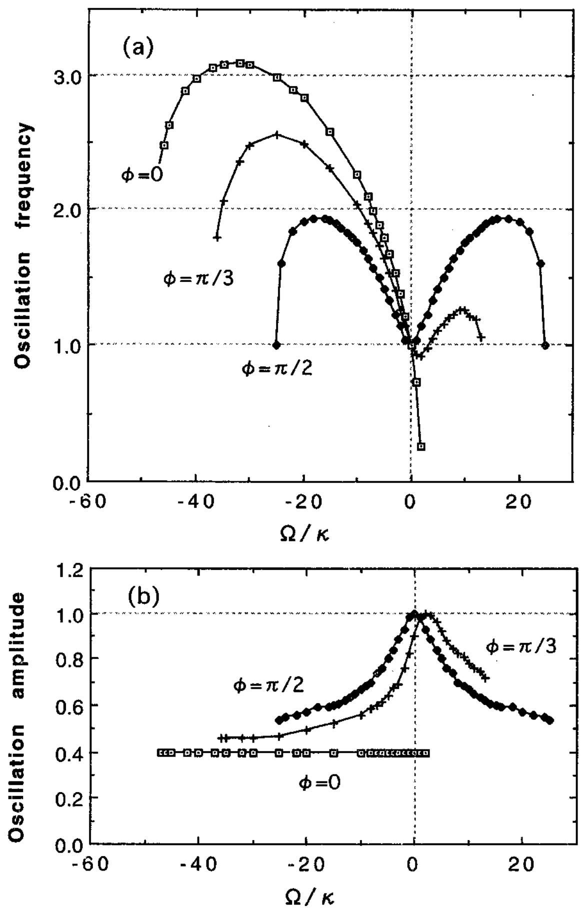

FIG. 4. The frequency (a) and the modulation amplitude (b) of the tunneling oscillations as a function of the nonlinear self-interaction Ω/κ for the initial phase $\phi=0$, $\pi/3$, and $\pi/2$ with $b_1(0)=\sqrt{0.7N}$.

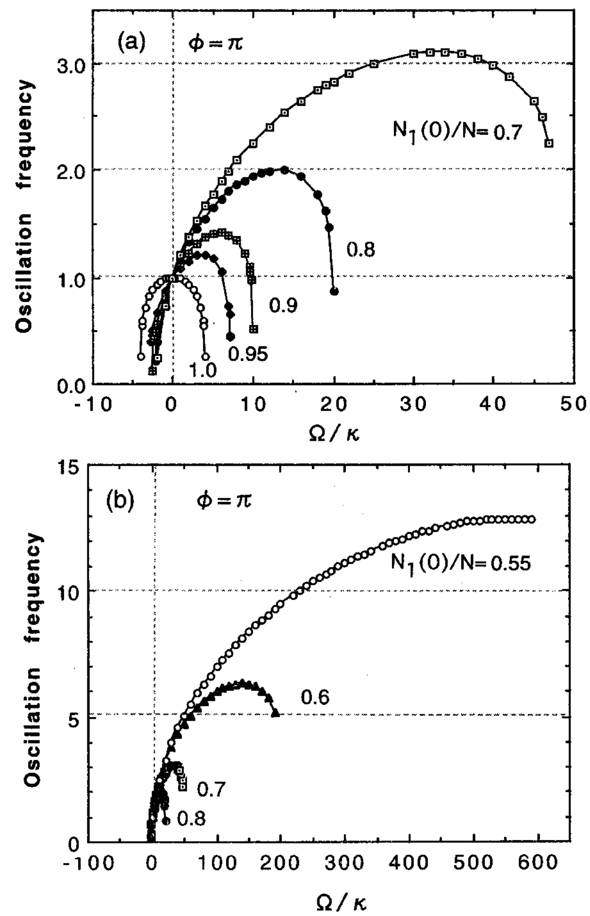

FIG. 5. The modulation amplitude of the tunneling oscillations as a function of the nonlinear self-interaction Ω/κ with $\phi=\pi$. (a) shows the results for the initial condition $b_1(0)=\sqrt{N}$, $\sqrt{0.95N}$, $\sqrt{0.9N}$, $\sqrt{0.8N}$, $\sqrt{0.7N}$ and (b) shows the results for $b_1(0)=\sqrt{0.6N}$ and $\sqrt{0.55N}$ together with $b_1(0)=\sqrt{0.8N}$ and $\sqrt{0.7N}$.

trapped state [7]) and in Figs. 3(g) and 3(h), the system shows the second type of π-phase trapped state [7]. Again, we are not interested in the self-tapping oscillations. In Figs. 3(b)–3(f), the atomic tunneling shows symmetric oscillations. The oscillation frequencies in Figs. 3(b)–3(d) become faster than those for the bare tunneling oscillations [Fig. 3(a)]. The oscillation frequency for $\Omega/\kappa=-25$ is about three times higher than the bare oscillation frequency ($f_0=\kappa/\pi$). For these initial conditions the enhancement of the tunneling is realized only for negative nonlinearity (or for attractive atom-atom interactions). As we can see below, however, the enhancement of the tunneling could also realized for positive nonlinearity by choosing the appropriate values of the phase $\phi=\phi_1-\phi_2$. This is the main results in this paper.

The oscillation frequency and the modulation amplitude against the nonlinear self-interactions (Ω/κ) are shown in Fig. 4 for the initial phase $\phi=\phi_1-\phi_2=0$, $\pi/3$, and $\pi/2$. Other parameters used are the same as in Fig. 2. As seen in Fig. 4(a), the enhancement of the tunneling oscillations can be observed only for negative values of the nonlinear self-interactions. For $\phi=0$, the maximum enhancement of the tunneling frequency is obtained at $\Omega/\kappa=-32$, where the tunneling oscillations are over three times faster than the bare tunneling oscillations for $\Omega/\kappa=0$, whereas the modulation amplitude maintains a constant value of 0.4. For $\phi=\pi/2$, the enhancement becomes symmetric for negative and positive nonlinearities. In contrast to the case of $\phi=0$, the modulation amplitude also varies as a function of Ω/κ. Comparing Figs. 4(a) and 4(b), we can see that the increasing of the oscillation frequency is accompanied by the decreasing of the modulation amplitude up to a point at which the oscillation frequency becomes maximum.

Figure 5 shows the oscillation frequencies against the negative and positive nonlinearity for various values of the initial populations $N_1/N=1.0$, 0.95, 0.9, 0.8, 0.7, 0.6, and 0.55 with $\phi=\pi$. Comparing both curves for $N_1/N=0.7$ in Figs. 4(a) and 5(a), we recognize that both the curves are symmetric about negative and positive nonlinearity. Accordingly, if we can prepare $\phi=\pi$ as the initial phase, the enhancement of the tunneling oscillations is also realized for the positive nonlinearity. This initial condition is easily realized by introducing an excitation process to the system. The initial phase ϕ of the condensate in trap 1 can be arbitrarily changed by applying a laser pulse that has appropriate pulse area and the detuning Δ [11]. In Fig. 5, we can see that as the initial population ($N_1(0)/N$) in trap 1 is reduced, the enhancement of the tunneling oscillations is increased. For example, if we choose $N_1(0)/N=0.55$ and $\phi=\pi$ as the initial conditions, we get extremely high tunneling oscillations that are 13 times higher than the bare tunneling oscillations. As far as the symmetric oscillations are observed, the modulation amplitude maintains a constant value, which is determined only by the value of $N_1(0)/N$; the oscillation amplitude is given by $2N_1(0)/N-1$.

In order to obtain an intuitive understanding for the enhancement of the atomic tunneling appearing in the nonlinear coupled systems, we rewrite $b_1(t)$ and $b_2(t)$ in Eqs. (2) by using the relations $u=b_1b_2^*+b_2b_1^*$, $v=-i(b_1b_2^*-b_2b_1^*)$, and $w=b_1b_1^*-b_2b_2^*$. The component w is directly related to the occupation probabilities of the traps 1 and 2, i.e., $|b_1|^2$ and $|b_2|^2$. After some simple procedures, we obtain a vector equation of the motion,

$$d\vec{\rho}/dt=\vec{\rho}\times\vec{T}, \tag{3}$$

in which $\vec{\rho}=(u,v,w)$ is a vector characterizing the state of the coupled system on the unit sphere, i.e., $u^2+v^2+w^2=1$. The state vector $\vec{\rho}$ rotates about the effective torque vector $\vec{T}=(2\kappa,0,2\Omega w)$. Equation (3) is similar to the Bloch-vector model, which has been widely used for quantum coherent optics in order to understand the interaction between atomic systems and resonant or near-resonant light, but contains an additional nonlinear term $(2\Omega w)$ arising from nonlinear self-interaction. Note that in our BEC system, the north pole ($w=1$) and south pole ($w=-1$) represent the states corresponding to all atoms localized in trap 1 and 2, respectively. The state vectors lying on the equator represent an equal division of atoms between the traps with different phases of the condensates.

In Fig. 6, we show the trajectories of the state vector $\vec{\rho}$ on the unit sphere corresponding to the evolution of the populations shown in Fig. 3. In this representation, the initial conditions used in Fig. 3 correspond to $w(0)=0.4$ and $v(0)=0$ together with $u(0)=\sqrt{1-w(0)^2}$. For these conditions, the state vector and the torque vector are, respectively, given by $\vec{\rho}=(u(0),0,w(0))$ and $\vec{T}=(2\kappa,0,2\Omega w(0))$ with $u(0)=[1-w(0)^2]^{-1/2}$. In the absence of the nonlinear self-interaction $\Omega/\kappa=0$, the torque vector $\vec{T}$ directs along the u axis and then the trajectory of $\vec{\rho}$ traces a circle with its center at the u axis [Figs. 3(e) and 6(a)]. For the small positive nonlinearity, i.e., $w(0)/u(0)>\Omega w(0)/\kappa$, the deviation angle of $\vec{T}$ from the w axis is smaller than that of the $\vec{\rho}$. Therefore, the vector $\vec{\rho}$ precessing about $\vec{T}$ rotates toward the equator so as to decrease the component $w(t)$ more and more. As a result, the state vector $\vec{\rho}$ rotates about the u axis making an elliptic trajectory, which corresponds to the symmetric-tunneling oscillations of the atoms between the two traps [Figs. 3(f) and 6(f)]. When the positive nonlinearity becomes large, i.e., $w(0)/u(0)<\Omega w(0)/\kappa$, the torque vector $\vec{T}$ makes a larger angle with the u axis than that of the vector $\vec{\rho}$. Therefore, the vector $\vec{\rho}$ rotates toward the north pole so as to increase $w(t)$ more and more. This positive feedback causes the self-trapping of the atomic distribution, resulting in the periodic localization of the condensate in one of the traps [see Figs. 3(g) and 6(g), Figs. 3(h) and 6(h)]. For the positive nonlinearity, there is the condition that both the vector $\vec{\rho}$ and the torque vector $\vec{T}$ become parallel and hence the vector $\vec{\rho}$ is locked in the torque vector $\vec{T}$, leading to the stationary state of the vector $\vec{\rho}$ [the medium state between Figs. 3(f) and 3(g)]. This phenomenon is similar to spin locking well known in nuclear magnetic resonance [14] and called the self-trapped stationary state or the z -symmetry-breaking state in Refs. [7,15]. This condition is obtained for $w(0)/u(0)=\Omega w(0)/\kappa$ or $\Omega/\kappa=[1-w(0)^2]^{-1/2}$. For very small $w(0)$, the condition is satisfied at $\Omega/\kappa=1$, which gives the exact threshold of the spontaneous symmetry breaking [9(b)].

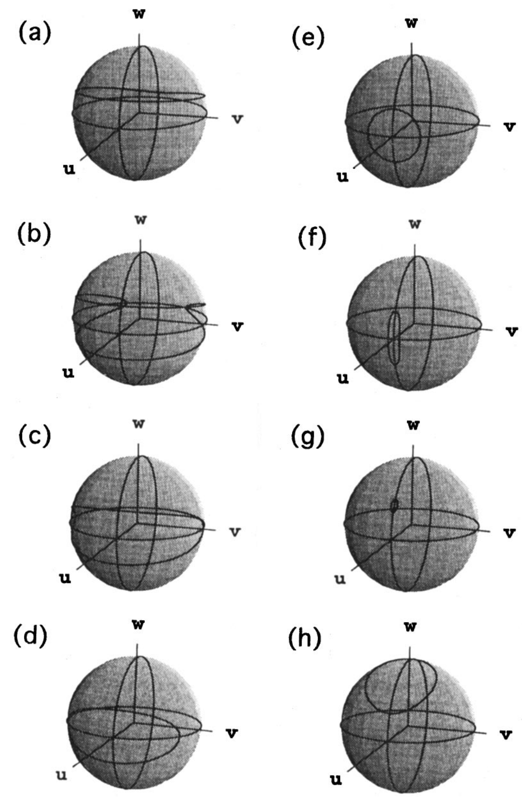

FIG. 6. Time evolution of the trajectories of the state vector $\vec{\rho}$ for the initial conditions $w(0)=0.4$ and $v(0)=0$ $[u(0)=\sqrt{1-w(0)^2}]$, which correspond to $b_1(0)=\sqrt{0.7N}$ and $\phi=0$ and the same initial conditions as in Fig. 3. The nonlinear self-interaction Ω/κ takes values (a) -50, (b) -45, (c) -25, (d) -10, (e) 0, (f) 2, (g) 2.3, and (h) 5.

For the negative self-interactions (say, Li atoms), the torque vector $\vec{T}$ is always in the lower hemisphere and the vector $\vec{\rho}$ is in the upper hemisphere. Therefore, $\vec{\rho}$ always rotates toward the equator and the self-trapping oscillations never occur [Figs. 3(b)–3(d) and Figs. 6(b)–6(d)]. As the nonlinearity increases, the trajectory shows oval orbits, which are squeezed along the longitude. The trajectories of $\vec{\rho}$ for the negative nonlinearity are easily envisaged in following way. The torque vector $\vec{T}$ deviates more and more from the u axis and approaches the $-w$ axis as the nonlinearity is increased. Therefore, the initial trajectory traces nearly along a latitude with a very large Rabi frequency $\sqrt{(2\kappa)^2+(2\Omega w)^2}-w$. As noted above, $\vec{\rho}$ always rotates toward the equator. Accordingly, the vector $\vec{\rho}$ reaches the equator sooner or later. At the equator, the population difference w becomes zero and the nonlinear part of the torque vector $\vec{T}$, i.e., a term of Ωw, vanishes and the two condensate states are in resonance, resulting in the torque vector Tv being along the u axis and the Rabi frequency 2κ. The oval trajectories have very large precession frequencies along the equator and have the bare tunneling frequency when the trajectories cross the equator. This is the reason why we could realize the enhancement of the atomic-tunneling oscillations in the nonlinear systems. The maximum enhancement of the oscillation frequency is obtained when the both edges of the trajectory on the equator align on the v axis.

Further increasing the nonlinear self-interaction to the negative side, the oscillation frequency decreases again [Figs. 3(b) and 6(b)] and then the both edges of the trajectory on the equator converge into one point. At this point, the second kind of self-trapping transition appears and the trajectory forms a circle in the upper hemisphere [Figs. 3(a) and 6(a)]. For the infinite nonlinearity, the torque vector $\vec{T}$ is directed along the $-w$ axis, resulting in the precession around the w axis. This means the vanishing of the tunneling oscillations, because the infinite nonlinearity causes infinite detuning between the two condensates.

In conclusion, the enhancement of tunneling oscillations due to nonlinear self-interactions in BEC's was predicted; sacrificing the oscillation amplitude, we can obtain a significant increase in the frequency of the tunneling oscillations. For instance, the frequency of the tunneling oscillations can be enhanced by factors 2, 3, and 13 compared with the bare tunneling oscillations for the modulation amplitudes of 0.6, 0.4, and 0.1, respectively. The trajectory of the state vector on the Bloch sphere gave us an intuitive understanding of the enhancement of the tunneling frequency. An effect similar to that predicted in this paper is expected for other nonlinear systems such as in coupled quantum dots and waveguide coupling in which the on-site energy and the self-phase modulation, respectively, give rise to the nonlinear interactions.

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