Coherent control of atom dynamics in an optical lattice

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(Received 5 March 2001; published 20 August 2001)

On the basis of a simple exactly solvable model we discuss the possibilities for state preparation and state control of atoms in a periodic optical potential. In addition to the periodic potential a uniform force with an arbitrary time dependence is applied. The method is based on a formal expression for the full evolution operator in the tight-binding limit. This allows us to describe the dynamics in terms of operator algebra, rather than in analytical expansions.

DOI: 10.1103/PhysRevA.64.033424 PACS number(s): 32.80.Qk, 05.60.−k, 42.50.Vk

I. INTRODUCTION

The energy eigenvalues of a quantum particle moving in a periodic potential form energy bands (the Bloch bands) that are separated by band gaps. The eigenstate within a band is characterized by the quasimomentum, which determines the phase difference between two points separated by a period. An initially localized wave packet typically propagates through space, leading to unbounded motion. When an additional uniform force is applied, the Bloch bands break up into a ladder of equally spaced energy levels, called the Wannier-Stark ladder. In this case, a wave packet of the particle extending over several periods can exhibit bounded oscillatory motion, termed Bloch oscillation, at a frequency determined by the level separation in the ladder. These early results of the quantum theory of electrons in solid crystals [1–4] have regained interest recently due to the advent of optical lattices for atoms. These lattices are formed when cold atoms are trapped in the periodic potential created by the superposition of a number of traveling light waves [5–8]. In contrast to the case of electrons in crystal lattices, these optical lattice fields have virtually no defects, they can be switched on and off at will, and dissipative effects can be largely controlled. The phenomenon of Bloch oscillations was first observed for cesium atoms in optical lattices [9]. The uniform external force is mimicked by a linear variation of the frequency of one of the counterpropagating traveling waves, thereby creating an accelerated standing wave. By applying a modulation on the standing-wave position, Rabi oscillations between Bloch bands as well as the level structure of the Wannier-Stark ladder have been observed for sodium atoms in an optical lattice [10]. Theoretical studies of transitions between ladders have also been presented [11]. Bloch oscillations have also been demonstrated for a light beam propagating in an array of waveguides, with a linear variation of the refractive index imposed by a temperature gradient [12].

When the applied uniform force is oscillating in time, the motion of a particle in a periodic potential is usually unbounded. However, it has been predicted that the motion remains bounded for specific values of the ratio of the modulation frequency and the strength of the force [13]. Similar effects of dynamical localization, including routes to chaos, have been studied experimentally for optical lattices, including both amplitude and phase modulation of the uniform force [14]. Phase transitions have been predicted for atoms in two incompatible periodic optical potentials imposed by bichromatic standing light waves [15].

In the present paper we discuss the Wannier-Stark system with a time-dependent force, as a means of preparing the state of particles in a periodic potential. We derive an exact expression for the evolution operator of the particle, with an arbitrary time-dependent force. This allows one to apply the combination of delocalizing dynamics in the absence of the uniform force with the periodic dynamics induced by a uniform force for coherent control of the state of the particles. Exact solutions in the case of a constant uniform force have been obtained before by analytical techniques [17,18]. The operator method allows phenomena induced by an oscillating force to be described exactly in a unified scheme. Examples are dynamical localization and fractional Wannier-Stark ladders.

The model is described in one dimension (1D). However, this is no real restriction. Under the assumption of nearest-neighbor interaction, the corresponding 2D or 3D problem exactly factorizes into a product of 1D solutions.

II. MODEL SYSTEM

A. Periodic potential

The quantum-mechanical motion of atoms in a periodic optical potential $V(x)$ with period a is described by the Hamiltonian

$$H_0 = \frac{P^2}{2M} + V(x). \tag{1}$$

We assume that the atoms are sufficiently cooled so that only the lowest energy band is populated. The ground state in well n located at $x = na$ is indicated as $|n\rangle$. These states play the role of the basis of localized Wannier states. For simplicity we take the tight-binding limit, where only the ground levels in neighboring wells are coupled. When we choose the zero of energy at the ground level in a well, the Hamiltonian (1) projected on these ground levels is defined by

$$H_0 = \frac{1}{2}\hbar\Omega(B_+ + B_-), \quad B_\pm|n\rangle = |n \pm 1\rangle. \tag{2}$$

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The raising and lowering operators B_+ and B_- are each other's Hermitian conjugates, and each one of them is unitary. The frequency Ω measures the coupling between neighboring wells, due to tunneling through the barriers. We shall allow the coupling to depend on time. The eigenstates of H_0 are directly found by diagonalizing the corresponding matrix. These states are the Bloch states $|k\rangle$, with energy $E(k)=\hbar\Omega\cos(ka)$. Their expansion in the Wannier states and the inverse relations can be expressed as

$$|k\rangle=\sqrt{\frac{a}{2\pi}}\sum_n e^{inka}|n\rangle,\quad |n\rangle=\sqrt{\frac{a}{2\pi}}\int dk\, e^{-inka}|k\rangle. \tag{3}$$

Obviously, the states $|k\rangle$ are periodic with period $2\pi/a$, and the quasimomentum k can be chosen from the Brillouin zone $[-\pi/a,\pi/a]$. The integration in Eq. (3) extends over this Brillouin zone. From the translation property $\langle x|n\rangle=\langle x+a|n+1\rangle$ of the Wannier wave functions it follows that the states (3) do indeed obey the Bloch condition $\langle x+a|k\rangle=\exp(ika)\langle x|a\rangle$. When the states $|n\rangle$ are normalized as $\langle n|m\rangle=\delta_{nm}$, the Bloch states obey the continuous normalization relation $\langle k|k'\rangle=\delta(k-k')$.

B. Uniform force

An additional uniform force is described by adding to the Hamiltonian the term

$$H_1=\frac{\hbar x\Delta}{a}, \tag{4}$$

where the (possibly time-dependent) force of size $\hbar\Delta(t)/a$ is in the negative direction. On the basis of the Wannier states, this term is diagonal, and it is represented as

$$H_1=\hbar\Delta B_0,\quad B_0|n\rangle=n|n\rangle. \tag{5}$$

Hence the evolution of a particle occurs under the influence of the total Hamiltonian

$$H=H_0+H_1, \tag{6}$$

with H_0 and H_1 defined by Eqs. (2) and (5), in terms of the operators $B_\pm$ and B_0. We shall also need expressions for the operators $B_\pm$ and B_0 acting on a Bloch state. These can be found from the definition of the operators and the expansions (3). One easily finds that

$$B_\pm|k\rangle=e^{\mp ika}|k\rangle,\quad e^{-i\beta B_0}|k\rangle=\left|k-\frac{\beta}{a}\right\rangle. \tag{7}$$

In Bloch representation the operators have the significance $B_\pm=\exp(\mp ika)$, $B_0=(i/a)(d/dk)$, which is confirmed by the commutation rules (8). The Wannier states may be viewed as discrete position eigenstates, with B_0 the corresponding position operator. The Bloch states play the role of momentum eigenstates, and the finite range of their eigenvalues within the Brillouin zone reflects the discreteness of the position eigenvalues.

C. Operator algebra

The basic operators $B_\pm$ and B_0 obey the commutation rules

$$[B_0,B_\pm]=\pm B_\pm,\quad [B_+,B_-]=0. \tag{8}$$

In order to derive exact expressions for the evolution operator corresponding to the Hamiltonian (6), we need several operator identities involving these operators B_0 and $B_\pm$. The identities

$$e^{i\beta B_0}B_\pm e^{-i\beta B_0}=e^{\pm i\beta}B_\pm \tag{9}$$

directly follow from the commutation rules (8), and they lead to the transformation rules

$$e^{i\beta B_0}\exp\left(-i\frac{1}{2}\alpha(B_++B_-)\right)e^{-i\beta B_0}=\exp\left(-i\frac{1}{2}\alpha(e^{i\beta}B_++e^{-i\beta}B_-)\right) \tag{10}$$

for arbitrary values of α and β. We shall also need the equalities

$$\exp\left(\frac{i}{2}\alpha B_\pm\right)B_0\exp\left(-\frac{i}{2}\alpha B_\pm\right)=B_0\mp\frac{i}{2}\alpha B_\pm, \tag{11}$$

which are verified after differentiation with respect to α, while using the commutation rules (8).

III. OPERATOR DESCRIPTION OF EVOLUTION

A. Evolution operator

In this section we derive expressions for the evolution operator $U(t,0)$, which transforms an arbitrary initial state $|\Psi(0)\rangle$ as $|\Psi(t)\rangle=U(t,0)|\Psi(0)\rangle$. The results are valid for any time dependence of the uniform force and the coupling between neighboring wells as specified by $\Delta(t)$ and $\Omega(t)$. A time-dependent coupling represents the case that the intensity of the lattice beams is varied. We express the evolution operator in the factorized form

$$U(t,0)=U_1(t,0)U_0(t,0), \tag{12}$$

where $U_1(t,0)=\exp[-i\phi(t)B_0]$ gives the evolution corresponding to the Hamiltonian H_1 alone, in terms of the phase shift

$$\phi(t)=\int_0^t dt'\,\Delta(t'). \tag{13}$$

From the evolution equation for U with the Hamiltonian (6) while using the transformation (9) we find the evolution equation

$$\frac{dU_0}{dt}=-\frac{i\Omega(t)}{2}(e^{i\phi(t)}B_++e^{-i\phi(t)}B_-)U_0(t). \tag{14}$$

Since this equation contains only the commuting operators B_+ and B_-, it can easily be integrated. In fact, the solution is given by Eq. (10) with the time-dependent values of the real parameters α and β defined by the relations

$$\alpha(t)e^{i\beta(t)}=\int_0^t dt'\,\Omega(t')e^{i\phi(t')}. \tag{15}$$

Combining this solution with the definition of U_1 leads to a closed expression for the evolution operator $U(t,0)$ for an arbitrary time dependence of the uniform force, in terms of the parameters α, β, and ϕ defined in Eqs. (13) and (15). The result is $U(t,0)\equiv R(\alpha,\beta,\phi)$, with R defined by

$$R(\alpha,\beta,\phi)=e^{i(\beta-\phi)B_0}\exp[-i\tfrac{1}{2}\alpha(B_++B_-)]e^{-i\beta B_0}. \tag{16}$$

This defines the unitary operator R as a function of the three parameters α, β, and ϕ. The result is valid for an arbitrary time dependence of the force and the coupling, described by $\Delta(t)$ and $\Omega(t)$. The characteristics of the evolution of an arbitrary initial state are determined by the properties of the operators R as a function of α, β, and ϕ. Mathematically, these operators form a three-parameter group, which is generated by the three operators $B_\pm$ and B_0.

On the basis of the Wannier states, the contribution of the operator B_0 in Eq. (16) is trivial, whereas the effect of the exponent containing $B_\pm$ can be evaluated by first expanding a Wannier state in Bloch states, for which the action of this exponent is simple. Then, re-expressing the Bloch states in Wannier states, we find

$$\exp[-i\tfrac{1}{2}\alpha(B_++B_-)]|m\rangle=\sum_n i^{-n+m}J_{n-m}(\alpha)|n\rangle, \tag{17}$$

where we used the defining expansion $\exp(i\xi\sin\phi)=\Sigma_n\exp(in\phi)J_n(\xi)$ of the ordinary Bessel functions. Hence the matrix elements of the operator (16) between Wannier states are

$$\langle n|R(\alpha,\beta,\phi)|m\rangle=(ie^{-i\beta})^{-n+m}e^{-in\phi}J_{n-m}(\alpha). \tag{18}$$

For the evolution operator (16) in Bloch representation we can just use the form of the operators $B_\pm$ and B_0, as given in Sec. II B. This leads to the result

$$R(\alpha,\beta,\phi)|k\rangle=e^{-i\alpha\cos(ka-\beta)}|k-\phi/a\rangle. \tag{19}$$

This shows that the quasimomentum as a function of time varies as $k(t)=k(0)-\phi(t)/a$, with $\phi(t)$ given in Eq. (13). The parameter ϕ determines the shift of the quasimomentum during the evolution. The expressions (18) and (19) clarify the significance of the three parameters α, β, and ϕ that specify the evolution operator.

B. Heisenberg picture

The transport properties of any initial state are conveniently described by the evolution of the operators in the Heisenberg picture. Since any evolution operator can be written in the form of $R(\alpha,\beta,\phi)$ for the appropriate values of the parameters, we can view $R^\dagger BR$ as the Heisenberg operator corresponding to any operator B. The Heisenberg operators corresponding to $B_\pm$ can be expressed as

$$R^\dagger(\alpha,\beta,\phi)B_\pm R(\alpha,\beta,\phi)=e^{\pm i\phi}B_\pm, \tag{20}$$

which is directly shown by using Eq. (9). Since $B_\pm=\exp(\mp ika)$ in Bloch representation, this confirms the significance of ϕ as the shift of the value of the quasimomentum.

After using the transformation property (11), one finds the Heisenberg operator corresponding to the position operator B_0 as

$$R^\dagger(\alpha,\beta,\phi)B_0R(\alpha,\beta,\phi)=B_0+\frac{i\alpha}{2}(e^{-i\beta}B_--e^{i\beta}B_+). \tag{21}$$

This implies that the expectation value of the position after evolution is determined by

$$\langle n\rangle=\langle B_0\rangle+\frac{i\alpha}{2}(e^{-i\beta}\langle B_-\rangle-e^{i\beta}\langle B_+\rangle), \tag{22}$$

where the averages in the right-hand side should be taken with respect to the inital state. Hence no displacement of a wave packet can occur whenever $\langle B_+\rangle=\langle B_-\rangle^*=0$. This is true whenever the initial state is diagonal in the Wannier states $|n\rangle$. Conversely, average motion of a wave packet can occur only in the presence of initial phase coherence between neighboring Wannier states. The width of a wave packet is determined by the expectation value of the square of the Heisenberg position operator (21). This gives the expression

$$\begin{aligned}\langle n^2\rangle={}&\langle B_0^2\rangle+\frac{\alpha^2}{4}(2-e^{-2i\beta}\langle B_-^2\rangle-e^{2i\beta}\langle B_+^2\rangle)\\&+\frac{i\alpha}{2}(e^{-i\beta}\langle B_0B_-+B_-B_0\rangle-e^{i\beta}\langle B_0B_++B_+B_0\rangle).\end{aligned} \tag{23}$$

IV. LOCALIZED INITIAL STATES

A. Arbitrary wave packets

A fairly localized initial state $|\Psi(0)\rangle=\Sigma_nc_n|n\rangle$ with a reasonably well-defined quasimomentum can be modeled by assuming that neighboring states have a fixed phase difference θ, so that

$$c_n^*c_{n+1}=|c_nc_{n+1}|e^{i\theta}. \tag{24}$$

Thus the quasimomentum is initially centered around the value $k_0=\theta/a$. For simplicity, we assume moreover that the distribution over Wannier states is even in n, so that $|c_n|=|c_{-n}|$. The initial average position of the particle is located at $n=0$. In order to evaluate the time-dependent average position and spreading of the packet, we can apply Eqs. (22) and (23). The symmetry of the distribution implies that

$\langle B_0\rangle=0$, while $\langle B_0^2\rangle=\sigma_0^2$ is the initial variance of the position. When we introduce the quantities

$$\sum_n |c_{n+1}c_n|\equiv b_1, \quad \sum_n |c_{n+2}c_n|\equiv b_2, \tag{25}$$

we obtain the simple identities

$$\langle B_+\rangle=b_1 e^{-i\theta}, \quad \langle B_+^2\rangle=b_2 e^{-2i\theta},$$

$$\langle B_0 B_+\rangle=-\langle B_+ B_0\rangle=\frac{1}{2}b_1 e^{-i\theta}. \tag{26}$$

The last identity is proved by using the fact that the quantity $f_{2n+1}\equiv|c_{n+1}c_n|$ is even in its index (which takes only odd values). Therefore, $\Sigma_l l f_l=0$, which is equivalent to the statement that $2\langle B_+B_0\rangle+\langle B_+\rangle=0$. The other expectation values occurring in Eqs. (22) and (23) are found by taking the complex conjugates of the identities (26). This leads to the simple exact results

$$\langle n\rangle=\alpha b_1\sin(\beta-\theta),$$

$$\langle n^2\rangle=\sigma_0^2+\frac{\alpha^2}{2}[1-b_2\cos 2(\beta-\theta)], \tag{27}$$

so that the variance of the position is found as

$$\sigma^2\equiv\langle n^2\rangle-\langle n\rangle^2=\sigma_0^2+\frac{\alpha^2}{2}[1-b_1^2-(b_2-b_1^2)\cos 2(\beta-\theta)]. \tag{28}$$

Notice that the parameters b_1 and b_2 are real numbers between 0 and 1. In the limit of a wide initial wave packet, determined by coefficients c_n whose absolute values vary slowly with n, the parameters b_1 and b_2 will both approach 1, and the width σ will not vary during the evolution. In the opposite special case that the initial state is the single Wannier state $|0\rangle$, one finds that $b_1=b_2=0$, so that the width $\sigma=\alpha/\sqrt{2}$.

In the special case that the particle is initially localized in the single Wannier state at $x=0$, so that $|\Psi(0)\rangle=|0\rangle$, the parameters b_1, b_2, and σ_0 vanish, so that

$$\langle n\rangle=0, \quad \sigma^2=\langle n^2\rangle=\alpha^2/2. \tag{29}$$

This shows that the average position of the wave packet does not change, and that its width is determined by the parameter α alone. This is in line with the fact that the population distribution over the Wannier states after the evolution is $p_n=|\langle n|R|0\rangle|^2=J_n^2(\alpha)$, as follows from Eq. (18). Hence the (time-dependent) value of α determines the spreading of an initially localized particle.

B. Gaussian wave packet

When the initial distribution over the sites is Gaussian with a large width, we can evaluate the full wave packet after evolution. Suppose that the initial state is specified by the coefficients

$$c_n=\frac{1}{\sqrt{\sigma_0\sqrt{2\pi}}}e^{in\theta}\exp\left(-\frac{n^2}{4\sigma_0^2}\right), \tag{30}$$

which obey the condition (24). This state is properly normalized provided that $\sigma_0\gg 1$. When the evolution operator is expressed as in Eq. (16), the time-dependent state is expanded as $|\Psi(t)\rangle=R|\Psi(0)\rangle=\Sigma_n f_n\exp[in(\theta-\phi)]|n\rangle$. Summation expressions for the coefficients f_n are directly obtained by using the expression (18) of R in Wannier representation. We use similar techniques to those applied in Ref. [16] in the context of the diffraction of a Gaussian momentum distribution of atoms by a standing light wave. The technique is based on differentiation of the expression for f_n with respect to n, while using the property $\alpha[J_{n+1}(\alpha)+J_{n-1}(\alpha)]=2nJ_n(\alpha)$ of Bessel functions. When the width is sufficiently large, so that the difference $f_{n+1}-f_n$ can be approximated by the derivative, this leads to the differential equation

$$2\sigma_0^2\frac{df_n}{dn}\approx[\alpha\sin(\beta-\theta)-n]f_n+i\alpha\cos(\beta-\theta)\frac{df_n}{dn}. \tag{31}$$

By solving this equation, we arrive at the closed expression

$$f_n=\frac{1}{\mathcal{N}}\exp\left(\frac{-n^2/2+\alpha n\sin(\beta-\theta)}{2\sigma_0^2-i\alpha\cos(\beta-\theta)}\right), \tag{32}$$

with the normalization constant determined by

$$\mathcal{N}^4=\pi\left(2\sigma_0^2+\frac{\alpha^2\cos^2(\beta-\theta)}{2\sigma_0^2}\right). \tag{33}$$

We find that the distribution is Gaussian at all times, with a time-varying average position and variance. These are given by the expressions

$$\langle n\rangle=\alpha\sin(\beta-\theta), \quad \sigma^2=\sigma_0^2+\frac{\alpha^2}{8\sigma_0^2}[1+\cos 2(\beta-\theta)]. \tag{34}$$

These results are in accordance with Eqs. (27) and (28), as one checks by using the approximate expressions $b_l=\exp(-l^2/8\sigma_0^2)\approx 1-l^2/8\sigma_0^2$, while neglecting terms of order $(1/\sigma_0)^4$ and higher. The width of the packet never gets smaller than its initial value. The phase difference between neighboring sites is mainly determined by $\theta-\phi$. This shows that a phase difference can be created or modified in a controlled way, simply by imposing a time-dependent force that gives rise to the right value of ϕ. Notice that in these expressions (34) θ and β enter in an equivalent fashion. The position and the width of the Gaussian distribution can be controlled at will by adapting the force to the desired value of β.

We recall that the results of this section are valid for an arbitrary time-dependent force $\Delta(t)$, which determines the time-dependent values of the parameters α, β, and ϕ as specified in Eqs. (13) and (15). In the subsequent sections, we specialize these expressions for constant or oscillating values of the uniform force.

V. CONSTANT UNIFORM FORCE AND BLOCH OSCILLATIONS

A. Wannier-Stark ladder of states

The case of a constant force is the standard situation where Bloch oscillations occur. When Δ and Ω are constant, the Hamiltonian is time independent, and then it is convenient to introduce the normalized eigenstates $|\psi_m\rangle$ of H. When we expand these eigenstates in the Wannier states as $|\psi_m\rangle=\Sigma_n|n\rangle c_n^{(m)}$, the eigenvalue relation $H|\psi_m\rangle=E_m|\psi_m\rangle$ with $E_m=\hbar\omega_m$ leads to the recurrence relations for the coefficients

$$\frac{1}{2}\Omega(c_{n-1}^{(m)}+c_{n+1}^{(m)})+\Delta n c_n^{(m)}=\omega_m c_n^m. \tag{35}$$

We introduce the generating function

$$Z_m(k)=\sqrt{\frac{a}{2\pi}}\sum_n c_n^{(m)}e^{-inka}, \tag{36}$$

which is normalized for integration over the first Brillouin zone. In fact, from the expression (3) of the Bloch state, one notices that the generating function $Z_m(k)=\langle k|\psi_m\rangle$ is equal to the Bloch representation of the eigenstate $|\psi_m\rangle$. The relations (35) are found to be equivalent to the differential equation

$$\Omega\cos(ka)Z_m(k)-\frac{\Delta}{ia}\frac{d}{dk}Z_m(k)=\omega_m Z_m(k), \tag{37}$$

with the obvious normalized solution

$$Z_m(k)=\sqrt{\frac{a}{2\pi}}\exp\left(\frac{i}{\Delta}[\Omega\sin(ka)-ak\omega_m]\right). \tag{38}$$

Since the functions $Z_m(k)$ as defined by Eq. (36) are periodic in k with period $2\pi/a$, the same must be true for the expressions (38). Hence, the frequency eigenvalues must be an integer multiple of Δ, so that we can choose $\omega_m=m\Delta$, with integer m. For these values of the eigenfrequencies, the coefficients $c_n^{(m)}$ follow from the Fourier expansion of Z_m, with the result

$$c_n^{(m)}\equiv\langle n|\psi_m\rangle=J_{m-n}(\Omega/\Delta). \tag{39}$$

We find that the total Hamiltonian H has the same eigenvalues as H_1. Apparently, the energy shifts due to the coupling between the Wannier states as expressed by H_0 cancel each other. Since the energy eigenvalues are integer multiples of Δ, each solution of the Schrödinger equation is periodic in time with period $2\pi/\Delta$, and the same is true for the evolution operator $U(t)$ given in Eq. (16). This also implies that an initial localized state remains localized at all times, due to the addition of the uniform external force. The eigenstates $|\psi_m\rangle$ are the Wannier-Stark ladder of states [10]. They form a discrete orthonormal basis of the first energy band, and they are intermediate between the Wannier and the Bloch bases of states.

B. Oscillations of localized states

The definitions (13) and (15) show that

$$\alpha=(2\Omega/\Delta)\sin(\Delta t/2),\quad \beta=\Delta t/2,\quad \phi=\Delta t. \tag{40}$$

In the Wannier representation, the matrix elements of U are found from Eq. (16) as

$$\langle n|U(t,0)|m\rangle=i^{-n+m}e^{-i\Delta t(n+m)/2}J_{n-m}\left(\frac{2\Omega}{\Delta}\sin\frac{\Delta t}{2}\right), \tag{41}$$

which represents the transition amplitude from an initial state $|m\rangle$ to the final state $|n\rangle$. For the initial Wannier state $|\Psi(0)\rangle=|0\rangle$, the time-dependent state is $|\Psi(t)\rangle=\Sigma_n f_n(t)|n\rangle$ with

$$f_n(t)=i^{-n}e^{-i\Delta tn/2}J_n\left(\frac{2\Omega}{\Delta}\sin\frac{\Delta t}{2}\right). \tag{42}$$

This is in accordance with Eq. (50) of Ref. [17], which was obtained by a rather elaborate analytical method, rather than an algebraic one. Equation (29) shows that the time-dependent average position $\langle n\rangle$ of the wave packet remains zero at all times, whereas the mean-square displacement $\sigma=|\alpha|/\sqrt{2}$ displays a breathing behavior, and returns to zero after the Bloch period $2\pi/\Delta$. Moreover, according to Eq. (42), the phase difference between neighboring sites varies continuously with time.

This is already quite different when only two Wannier states are populated initially. Consider the initial state

$$|\Psi(0)\rangle=\frac{1}{\sqrt{2}}(|0\rangle+e^{i\theta}|1\rangle). \tag{43}$$

Then the average position can be evaluated from Eq. (22), for the values of α and β given in Eq. (40). The result is

$$\langle n\rangle=\frac{1}{2}+\frac{\Omega}{2\Delta}[\cos\theta-\cos(\Delta t-\theta)], \tag{44}$$

which shows that the packet displays a harmonically oscillating behavior. The amplitude of the oscillation is governed by the ratio Ω/Δ, which is one-half the maximum amplitude for Bloch oscillations of a wave packet with a large width (see Sec. V C). This amplitude must be appreciable in order that interband coupling induced by the uniform force remains negligible, as we have assumed throughout this paper. The distribution $p_n=|f_n|^2$ after one-half a Bloch period, both for the initial single Wannier state and for the inital state (43), is illustrated in Fig. 1. This demonstrates that a strong displacement can already be induced by evolution of a superposition state of just two neighboring Wannier states, with a specific phase difference. This displacement arises from the

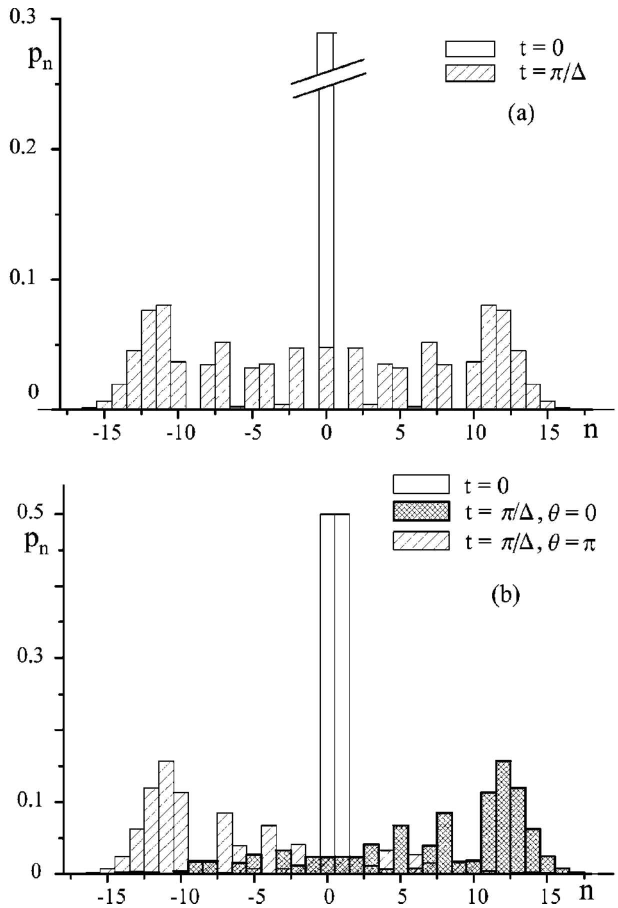

FIG. 1. (a) Plot of the breathing population distribution for an initial Wannier state $|0\rangle$. (b) Plot of the oscillating population distribution, for two initial superpositions of Wannier states $|0\rangle$ and $|1\rangle$, and two different values of the relative phase θ. Both plots are evaluated for $\Omega/\Delta=6$. Shaded distributions hold after one-half a Bloch period $t=\pi/\Delta$.

interference between the transition amplitudes from the two initial states to the same final state $|n\rangle$.

C. Bloch oscillations and breathing of a Gaussian wave packet

The evolution of a Gaussian wave packet as discussed in Sec. IV B is specialized to the present case of a constant force after substituting the expressions (40) in Eqs. (32)–(34). We find for the average position $\langle n\rangle$ the identity

$$\langle n(t)\rangle=\frac{\Omega}{\Delta}[\cos\theta-\cos(\theta-\Delta t)]. \tag{45}$$

This demonstrates that the wave packet oscillates harmonically in position with frequency Δ and with amplitude Ω/Δ in units of the lattice distance a. The velocity of the wave packet is found from the time derivative of Eq. (45), with the result

$$v(t)=-a\Omega\sin(\theta-\Delta t). \tag{46}$$

It is noteworthy that this expression (46) coincides exactly with the expression for the group velocity $dE/\hbar dk$, with the derivative evaluated at the time-dependent value of the quasimomentum $(\theta-\Delta t)/a$, with $E=\hbar\Omega\cos(ka)$ the dispersion relation between energy and quasimomentum in the absence of the uniform force, as given in Sec. II A. Apparently, the expression for the group velocity retains its validity in the presence of the uniform force also. Of course, the concept of Bloch oscillations of the wave packet as a whole has significance only when the amplitude Ω/Δ of the oscillation is large compared with the width σ of the packet, which in turn must extend over many lattice sites.

The time-dependent width σ of the Gaussian packet is found from Eq. (34) in the form

$$\sigma^2=\sigma_0^2+\frac{\Omega^2}{4\sigma_0^2\Delta^2}(1-\cos\Delta t)[1+\cos(\Delta t-2\theta)]. \tag{47}$$

Hence the variance of the position deviates from its initial value by an oscillating term. The amplitude of this oscillation is governed by the ratio $(\Omega/2\Delta\sigma_0)^2$. The initial width is restored whenever one of the terms in brackets vanishes. This happens twice during every Bloch period, except when $\theta=\pi/2$, when these two instants coincide. This combined breathing and oscillating behavior is illustrated in Figs. 2 and 3, for various values of the relative phase θ. Notice that the oscillation is always harmonic with the Bloch frequency Δ. This is due to the simple form of the dispersion relation for the case of nearest-neighbor interaction. The time dependence of the variance is a superposition of terms with frequencies Δ and 2Δ.

D. Zero external force

In the absence of an external force, we can take the limit $\Delta\to 0$ in the results of the previous subsections. In particular, this gives $\phi=\beta=0$, $\alpha(t)=\Omega t$. Then the evolution of an initial Wannier state $|\Psi(0)\rangle=|0\rangle$ is given by

$$|\Psi(t)\rangle=R|\psi(0)\rangle=\sum_n i^{-n}J_n(\Omega t)|n\rangle, \tag{48}$$

which shows that the free spreading of an initial Wannier state after a time t gives Wannier populations equal to $p_n=|J_n(\Omega t)|^2$ [19]. The mean-square displacement increases linearly in time, as $\sigma=\Omega t/\sqrt{2}$. This shows that the spreading is unbounded in the absence of an external force. The self-propagator $p_0(t)$ decays to zero for large times. The phase difference between neighboring sites is $\pm\pi/2$ at all times. For only two coupled wells, the coupling would give rise to Rabi oscillations with frequency Ω. Equation (48) can be viewed as the generalization to the case of an infinite chain of wells.

For a Gaussian wave packet with initial width σ_0 and initial quasimomentum determined by θ, expressions (45) and (47) take the form

$$\langle n(t)\rangle=-\Omega t\sin\theta,\quad \sigma^2=\sigma_0^2+\frac{\Omega^2t^2}{8\sigma_0^2}(1+\cos 2\theta). \tag{49}$$

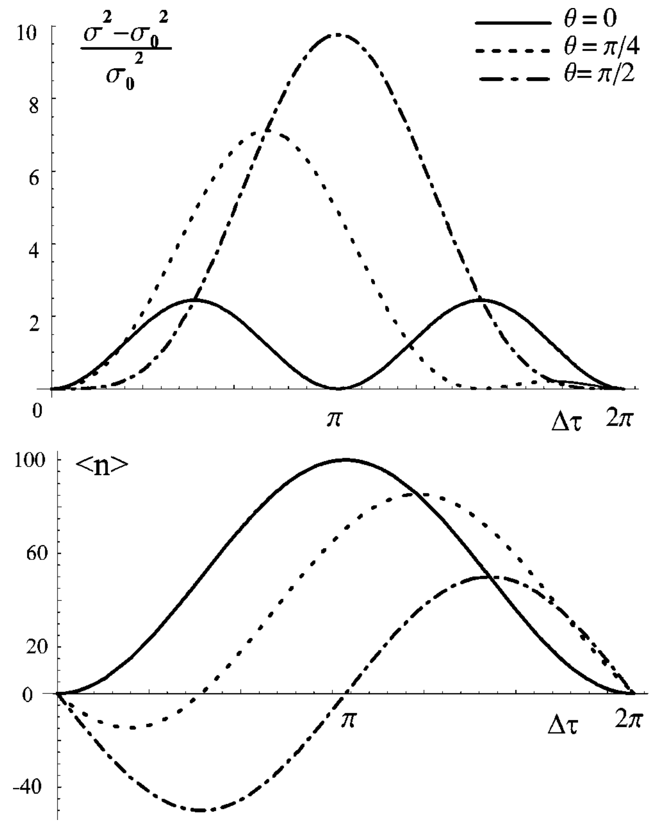

FIG. 2. Periodic behavior of the width and the average position of a Gaussian wave packet for various initial values of the phase difference θ between neighboring states. Initial value of the width is $\sigma_0=4$ and $\Omega/\Delta=50$.

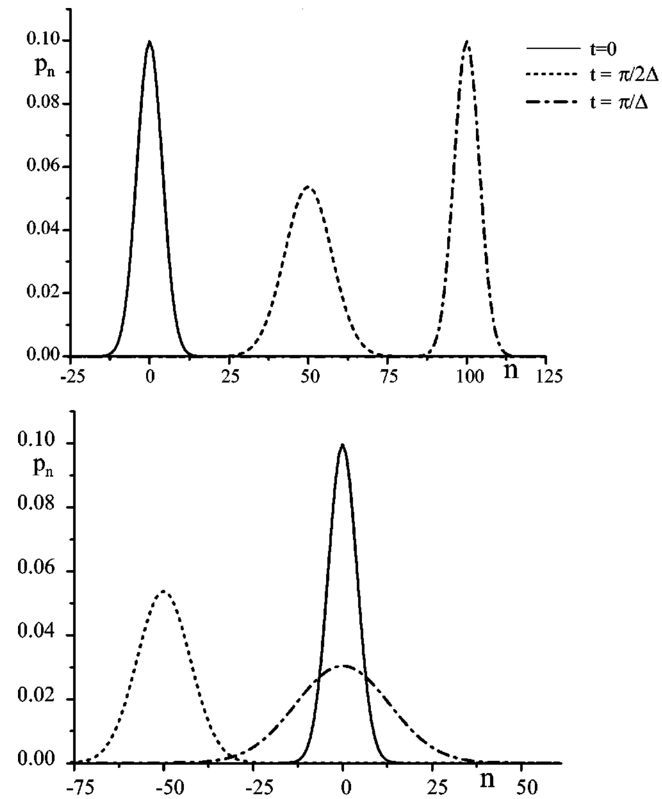

FIG. 3. Bloch oscillation and corresponding breathing behavior of a Gaussian wave packet in a constant uniform force. Values of σ_0, Ω, and Δ as in Fig. 2. Upper part: $\theta=0$. Lower part: $\theta=\pi/2$.

As one would expect in the absence of a uniform force, the group velocity takes the constant value $v=-a\Omega\sin\theta$, which leads to unbounded motion of the packet (except for $\theta=0$ or $\pm\pi$). Usually, the width increases indefinitely during the propagation. However, for the special values $\theta=\pm\pi/2$ the width is constant, and the packet propagates as a solitary wave. Notice that such a phase difference between neighboring Wannier states arises spontaneously when a single Wannier state spreads in the absence of a uniform force.

VI. OSCILLATING FORCE

Other situations of practical interest arise when the uniform force has an oscillating component. Examples are the coupling between the states in the Wannier-Stark ladder [10], and dynamical localization for special values of the amplitude-frequency ratio of the oscillation [13,14]. The situation of an oscillating force is also decribed by the operator description of Sec. III A. We give some results below.

A. ac force only

The situation of a harmonically oscillating uniform force can be expressed as

$$\Delta(t)=\delta\cos(\omega t), \tag{50}$$

so that $\phi=(\delta/\omega)\sin(\omega t)$. Then according to Eq. (15) the parameters α and β are specified by the equalities

$$\alpha e^{i\beta}=\Omega t J_0\left(\frac{\delta}{\omega}\right)+\Omega\sum_{n\neq 0} J_n\left(\frac{\delta}{\omega}\right)\frac{1}{in\omega}(e^{in\omega t}-1), \tag{51}$$

where we used the expansion defining the ordinary Bessel functions, given in Sec. III A.

The first term in Eq. (51) increases linearly with time, whereas the summation is bounded and periodic in time with period $T=2\pi/\omega$. The behavior of α and β as defined by Eq. (51) is quite complicated in general. However, for large times the value of α, and thereby the spreading of an initial Wannier state, is the same as in the absence of the uniform force, with Ω replaced by the reduced effective coupling $\Omega J_0(\delta/\omega)$. After one period T, the values of the parameters become simple, and we find $\beta=\phi=0$, $\alpha=\Omega T J_0(\delta/\omega)$. The evolution operator $U(T)$ during one period T is simply given by the operator R defined in Eq. (16), at these values of the parameters. The eigenstates of the evolution operator $R=U(T)$ are simply the Bloch states $|k\rangle$. The eigenvalues can be expressed as $\exp[-i\mathcal{E}(k)T/\hbar]$, with

$$\mathcal{E}(k)=\hbar\Omega J_0\left(\frac{\delta}{\omega}\right) \tag{52}$$

the corresponding values of the quasienergy, which are strictly speaking defined only modulo $\hbar\omega$. The quasienergy bandwidth is reduced by the factor $J_0(\delta/\omega)$, compared with the energy bandwidth in the absence of the uniform force.

When the ratio δ/ω of the amplitude and the frequency of the oscillating force coincide with a zero of the Bessel function J_0, no unbounded spreading occurs, and an initially localized state remains localized at all times, with a periodically varying mean-square displacement. The quasienergy bandwidth is reduced to zero in this case. This effect of dynamical localization has been discussed before for electrons in crystals [13]. The related effect of an effective switch-off of atom-field coupling occurs for a two-level atom in a frequency-modulated field when the ratio of the amplitude-frequency ratio of the modulation equals a zero of the Bessel function J_0. This effect, which leads to population trapping in a two-level atom, has recently been discussed by Agarwal and Harshawardhan [20].

B. ac and dc force

A constant uniform force creates Wannier-Stark states with equidistant energy values. An additional oscillating force can induce transitions between these states. Therefore, we consider the force specified by

$$\Delta(t) = \Delta_0 + \delta\cos(\omega t). \tag{53}$$

Then the values of the parameters ϕ, α, and β are

$$\phi(t) = \Delta_0 t + (\delta/\omega)\sin(\omega t),$$

$$\alpha e^{i\beta} = \Omega\sum_n J_n\left(\frac{\delta}{\omega}\right)\frac{1}{i(\Delta_0+n\omega)}\left(e^{i(\Delta_0+n\omega)t}-1\right). \tag{54}$$

In general, each term in the summation is bounded and periodic, but the different periods can be incompatible. Moreover, whenever $\Delta_0 + n\omega = 0$, the corresponding summand attains the unbounded form $\Omega t J_n(\delta/\omega)$. At such a resonant value of Δ_0, the spreading of an initially localized state becomes unbounded, and the particle becomes delocalized. This delocalization is suppressed again when the ratio δ/ω is equal to a zero of the corresponding Bessel function J_n. This is a simplified version of the phenomenon of fractional Wannier-Stark ladders, which has recently been observed and discussed [21,22].

The quasienergy values are again determined by the eigenstates of the evolution operator $U(T)$ for one period of the oscillating force. This operator is equal to the general operator R defined in Eq. (16), with the parameters

$$\alpha = 2\Omega\sin(\Delta_0 T/2)\sum_n J_n\left(\frac{\delta}{\omega}\right)\frac{1}{\Delta_0+n\omega},$$

$$\beta(T) = \Delta_0 T/2, \quad \phi(T) = \Delta_0 T. \tag{55}$$

These expressions are correct whenever $\Delta_0 + n\omega$ is nonzero for all values of n. Since these values of the parameters can be directly mapped onto the values (40) specifying the evolution with a constant uniform force, the eigenvectors and corresponding quasienergies are also immediately found. The eigenvectors of R can be expressed as $|\psi_m\rangle = \Sigma_n |n\rangle c_n^{(m)}$, with the expansion coefficients $c_n^{(m)} = J_{m-n}(\zeta)$. Here the argument ζ of the Bessel functions must be chosen as the sum

$$\zeta = \Omega\sum_n J_n\left(\frac{\delta}{\omega}\right)\frac{1}{\Delta_0+n\omega}, \tag{56}$$

which replaces the simple argument Ω/Δ in Eq. (39). The eigenvalues of $R = U(T)$ are $\exp(-i\mathcal{E}_m T/\hbar)$, with the discrete quasienergy values $\mathcal{E}_m = \hbar m \Delta_0$ (modulo $\hbar\omega$).

In the resonant case that $\Delta_0 + n_0\omega = 0$ for some integer n_0, one summand in the expression for α and β is modified, as indicated above. When $T = t$, only this modified summand is nonzero, and the evolution operator $U(T) = R$ for one time period is characterized by the values

$$\alpha = \Omega T J_{n_0}, \quad \beta = 0, \quad \phi = -2\pi n_0. \tag{57}$$

The eigenvectors of R are the Bloch states $|k\rangle$, and the corresponding quasienergy values are

$$\mathcal{E}(k) = \hbar\Omega J_{n_0}\left(\frac{\delta}{\omega}\right)\cos(ka). \tag{58}$$

VII. DISCUSSION AND CONCLUSIONS

We have analyzed the Wannier-Stark system, which is characterized by the Hamiltonian (6), in terms of the operators $B_\pm$ and B_0. The present interest in this model arises from the dynamics of atoms in a periodic optical potential, with an additionally applied uniform external force. We adopted the tight-binding limit, which implied nearest-neighbor interaction only. This gives rise to an explicit simple dispersion relation between energy and quasimomentum, which makes the model exactly solvable. From the commutation properties of the basic operators we obtain Eq. (16) for the evolution operator for an arbitrary time dependence of the uniform force, where the three parameters are defined in Eqs. (13) and (15). As shown in Secs. III B and IV, the parameter ϕ determines the shift in the value of the quasimomentum, whereas α and β determine the evolution of the average position and the width of a wave packet. A particle starting in a single Wannier state has a uniform distribution over the quasimomentum, and cannot change its average position, whereas the width of its wave packet is simply measured by α. On the other hand, even when only two neighboring states are populated initially, the wave packet can display an appreciable motion. In Sec. IV B it is demonstrated that an initially Gaussian packet remains Gaussian at all times. This remains true when the initial state has a nonzero expectation value of the quasimomentum, which is described as an initial phase difference between neighboring Wannier states.

These results, which are valid for a uniform force with an arbitrary time dependence, unify and extend earlier results obtained for a constant or an oscillating uniform force. A constant force induces Bloch oscillations of a wave packet, and we obtain a simple expression for the amplitude of the oscillation and for the time dependence of the width of the wave packet. For an oscillating force, the operator method shows that the quasienergy bands can be evaluated directly in terms of the value of the parameter α after one oscillation period. This produces an exactly solvable model for dynamical localization and fractional Wannier-Stark ladders. In general, by selecting a proper time dependence of the force or of the coupling between wells, thereby realizing the desired values of the parameters α, β, and ϕ, we can coherently control the width and the position of a wave packet, as well as the phase difference between neighboring sites.

ACKNOWLEDGMENT

This work is part of the research program of the ''Stichting voor Fundamenteel Onderzoek der Materie'' (FOM).

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