Relativistic corrections to the electromagnetic polarizabilities of compound systems

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The low-energy amplitude of Compton scattering on the bound state of two charged particles of arbitrary masses, charges, and spins is obtained. A case in which the bound state exists due to electromagnetic interaction is considered. The term, proportional to ω^2, is obtained taking into account the first relativistic correction. It is shown that the complete result for this correction differs essentially from the commonly used term $\Delta\alpha$, proportional to the rms charge radius of the system. We propose that the same situation can take place in the more complicated case of hadrons.

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I. INTRODUCTION

The electromagnetic polarizabilities $\bar{\alpha}$ and $\bar{\beta}$ are fundamental characteristics of the bound system. Their magnitudes depend not only on the quantum numbers of the constituents, but also on the properties of the interaction between these constituents. Therefore, the experimental and theoretical investigation of the electromagnetic polarizabilities are of a great importance. In particular, their prediction and the comparison with experimental data may serve as a sensitive tool for tests of hadron models. Correspondingly, a large number of researchers have been attracted by this fascinating possibility. The electromagnetic polarizabilities can be obtained from the low-energy Compton scattering amplitude. In the lab frame, the amplitude of Compton scattering on the compound system of total angular momentum $S=0$, 1/2 up to $O(\omega^2)$ terms reads [1,2]

$$T=T_{Born}+\bar{\alpha}\omega_1\omega_2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*+\bar{\beta}(\mathbf{k}_1\times\boldsymbol{\epsilon}_1)\cdot(\mathbf{k}_2\times\boldsymbol{\epsilon}_2^*), \tag{1.1}$$

where ω_i, $\mathbf{k}_i$, and $\boldsymbol{\epsilon}_i$ are the energy, momentum, and polarization vector of incoming $(i=1)$ and outgoing $(i=2)$ photons $(\hbar=c=1)$. The contribution T_{Born} corresponds to the amplitude of Compton scattering off a pointlike particle with spin, mass, charge, and magnetic moment equal to those of the compound system. For spin $S\geqslant 1$, the $O(\omega^2)$ part of the Compton scattering amplitude contains additional terms, proportional to quadrupole and higher multipoles of the bound system [3]. In particular, for $S=1$ there is a contribution proportional to the quadrupole moment operator.

The investigation of electromagnetic polarizabilities is interesting not only for systems, bound by the electromagnetic interaction, like atoms, but also for those bound by strong interaction, such as atomic nuclei [4] or hadrons [5]. At present there are many different approaches used for the description of the electromagnetic polarizabilities of hadrons: the MIT bag model [6,7], the nonrelativistic quark model [8–12], the chiral quark model [13,14], the chiral soliton model [15,16], and the Skyrme model [17,18]. Here, we mentioned only a small part of the publications on these topics (see also review [19]). Though much effort has been devoted to these calculations, all of them cannot be considered as completely satisfactory. In particular, there is a problem in the explanation of the magnitudes of proton and neutron electric polarizabilities within a nonrelativistic quark model. It was derived many years ago [2,20,21] that $\bar{\alpha}$ can be represented as a sum

$$\bar{\alpha}=\frac{2}{3}\sum_{n\neq 0}\frac{|\langle n|\mathbf{D}|0\rangle|^2}{E_n-E_0}+\Delta\alpha=\alpha_{\bigcirc}+\Delta\alpha, \tag{1.2}$$

where $\mathbf{D}$ is the internal electric dipole operator, $|0\rangle$ and $|n\rangle$ are the ground and excited states in terms of internal coordinates, and E_n and E_0, the corresponding energies. The term $\Delta\alpha$ in $\bar{\alpha}$ has a relativistic nature and its leading term is equal to

$$\Delta\alpha=\frac{er_E^2}{3M}, \tag{1.3}$$

where e and M are the particle charge and mass, r_E is the electric radius defined through the Sachs form factor G_E. Our definition of r_E^2 absorbs a total charge e of the system. The calculation of the quantity $\alpha_{\bigcirc}$ in the nonrelativistic quark model without relativistic corrections taken into account leads to the same magnitude of $\alpha_{\bigcirc}$ for proton and neutron. Since $\Delta\alpha$ is equal to zero for the neutron but gives a significant contribution to $\bar{\alpha}$ for the proton, one has a contradiction between the theoretical prediction of $\bar{\alpha}$ for nucleons and their experimental values, since the latter are close to each other. In fact, this approach is not consistent, because there are relativistic corrections to $\alpha_{\bigcirc}$ that are of the same order as $\Delta\alpha$.

Starting from second-order perturbation theory, one gets the following expression for $\alpha_{\bigcirc}$:

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$$\alpha_{\bigcirc}=\frac{2}{3}\sum_{n\neq 0}\frac{|\langle n|\mathbf{J}|0\rangle|^2}{(E_n-E_0)^3}, \tag{1.4}$$

where $\mathbf{J}$ is the internal electromagnetic current. Using the identity $\mathbf{J}=i[H,\mathbf{D}]$, where H is the Hamiltonian, one comes to the form Eq. (1.2) for $\alpha_{\bigcirc}$. The relativistic corrections to $\alpha_{\bigcirc}$ in Eq. (1.4) come from the corrections to wave functions and energies of the ground and excited states, and correction to current $\mathbf{J}$. In papers [22,23], the relativistic corrections to the current operator and electric dipole operator were obtained. In [24–26], some general expressions for the relativistic corrections to electric polarizabilities were obtained, but no explicit calculations for a realistic system were made and the importance of these corrections was not realized.

Due to the relation between $\mathbf{J}$ and $\mathbf{D}$, it is clear that there also is a relativistic correction to the electric dipole moment operator (see below) which is connected with the appropriate relativistic definition of the center-of-mass coordinate. The neglect of this relativistic correction leads to an incomplete expression for $\overline{\alpha}$, and the missing piece that is calculated in the following turns out to be very essential. We expect that the inclusion of all relativistic corrections allows one to remove the big difference between the predictions of the nonrelativistic quark model for proton and neutron electric polarizabilities due to the difference in $\Delta\alpha$.

The expression for the magnetic polarizability in the nonrelativistic quark model with no exchange and momentum-dependent forces has the form [21,27,28]

$$\overline{\beta}=\beta_{para}+\beta_{dia}=\frac{2}{3}\sum_{n\neq 0}\frac{|\langle n|\mathbf{M}|0\rangle|^2}{E_n-E_0}-\left(\sum_i\frac{e_i^2\langle r_i^2\rangle}{6m_i}+\frac{\langle\mathbf{D}^2\rangle}{2M}\right), \tag{1.5}$$

where $\mathbf{M}$ is the internal magnetic dipole operator and the summation in the second term on the right-hand side (rhs) is performed over the constituent quarks, $\mathbf{r}_i$ being the corresponding internal radius vector.

In order to understand the importance of the different relativistic corrections for the polarizabilities, it is useful to consider the example of a system where the relativistic corrections can be obtained *ab initio*. In this paper, we calculate the low-energy Compton scattering amplitude for a system of two particles with masses $m_{1,2}$ and charges $e_{1,2}$, bound by electromagnetic forces. We consider the case $e_{1,2}^2\ll 1$, which provides the validity of the nonrelativistic expansion. We consider, in detail, the cases of spin 0 and 1/2 of the particles and give the result for general case of arbitrary spins.

II. SCATTERING AMPLITUDES

For the electromagnetic interaction between particles, a simple estimate shows that the part t of Compton scattering amplitude, proportional to ω^2 has the form

$$t=\omega^2a^3\left(c_1+c_2\frac{\varepsilon_0}{\mu}+c_3\frac{\varepsilon_0^2}{\mu^2}+\cdots\right), \tag{2.1}$$

where $a=1/(\mu g)$ is the Bohr radius, $g=-e_1e_2>0$, $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass, $\varepsilon_0=-\mu g^2/2$ is the ground-state binding energy in the nonrelativistic approximation, and c_i are some quantities, bilinear with respect to $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2^*$ and depending on the ratio of charges and masses. Here, for definiteness, we assume $\omega=\omega_1$. The two first terms of this expansion contain the parameter $g\ll 1$ in the denominators and, therefore, come from the contribution of big distances $r\sim a$ (or small momenta $p\sim 1/a=\mu g$) to the matrix element. These two terms that have no contributions from the Born amplitude are the ones we are going to calculate in this article. Since they are determined by a contribution from big distances (small momenta), it is possible to use the nonrelativistic expansion in the calculations. In fact, the first term is known and contains the contributions of $\alpha_{\bigcirc}$, Eq. (1.2) and β_{para}, Eq. (1.5), calculated in the leading nonrelativistic approximation (see below). Some contributions to the second term are also known, namely, those containing the magnetic polarizability β_{dia}, Eq. (1.5) and the correction $\Delta\alpha$, Eq. (1.3). These contributions come from the expansion of the photon wave functions over $\mathbf{kr}\sim\omega/\mu g$ and from the expansion of the propagator of the system with respect to the photon energy and the center-of-mass kinetic energy of intermediate states. The corresponding results for β and $\Delta\alpha$ can be obtained using the nonrelativistic Hamiltonian of the system. As was mentioned above, the other source of the contributions to c_2, which has not been investigated so far, is the relativistic correction to the Hamiltonian of the system and the corresponding corrections to the wave functions, energy levels, and currents. We will obtain the complete result for the second term in the expansion Eq. (2.1). In the expression for the electromagnetic current, we neglect for a while the dependence of the form factors on the momentum transfer. We will take this dependence into account at the consideration of the general case of arbitrary spins.

A. The system of two spin-0 particles

Let us consider first the bound state of two spin-0 particles. In order to calculate the Compton scattering amplitude, it is convenient to put the system into the external electromagnetic field $\mathbf{A}(\mathbf{x},t)$. In this case, the nonrelativistic Hamiltonian has the form

$$\tilde{H}_{nr}[\mathbf{A}]=\frac{\boldsymbol{\pi}_1^2}{2m_1}+\frac{\boldsymbol{\pi}_2^2}{2m_2}-\frac{g}{|\mathbf{r}_1-\mathbf{r}_2|}, \tag{2.2}$$

where $\boldsymbol{\pi}_i=\mathbf{p}_i-e_i\mathbf{A}(\mathbf{r}_i,t)$. Let us pass to the variables $\mathbf{r}$ and $\mathbf{R}$, corresponding to the relative and center-of-mass coordinate:

$$\mathbf{r}_1=\mathbf{R}+\frac{m_2}{M}\mathbf{r},\quad \mathbf{r}_2=\mathbf{R}-\frac{m_1}{M}\mathbf{r},\quad M=m_1+m_2. \tag{2.3}$$

Then, the momenta $\mathbf{p}_i$ are

$$\mathbf{p}_1=\frac{m_1}{M}\mathbf{P}+\mathbf{p},\quad \mathbf{p}_2=\frac{m_2}{M}\mathbf{P}-\mathbf{p}, \tag{2.4}$$

where $\mathbf{P}=-i\nabla_R$ and $\mathbf{p}=-i\nabla_r$. For $\mathbf{A}=\mathbf{0}$ we have

$$\tilde{H}_{nr}[\mathbf{A}=0]=\frac{\mathbf{P}^2}{2M}+H_{nr}=\frac{\mathbf{P}^2}{2M}+\frac{\mathbf{p}^2}{2\mu}-\frac{g}{r}. \quad (2.5)$$

The first relativistic correction $\tilde{H}_B$ (Breit Hamiltonian, see, e.g., [29]) to Eq. (2.2) reads

$$\tilde{H}_B[\mathbf{A}]=-\frac{(\boldsymbol{\pi}_1^2)^2}{8m_1^3}-\frac{(\boldsymbol{\pi}_2^2)^2}{8m_2^3}+\frac{g}{2m_1m_2}\left(\frac{\delta^{ij}}{r}+\frac{r^ir^j}{r^3}\right)\pi_1^i\pi_2^j. \quad (2.6)$$

The first term in Eq. (2.6) is the correction to the kinetic energy and the second one is the correction due to the magnetic quanta exchange, corresponding to the space component of the photon propagator in the Coulomb gauge. If $\mathbf{A}=0$, then in the center-of-mass frame where the eigenvalue of the operator $\mathbf{P}$ is equal to zero we have

$$\tilde{H}_B[\mathbf{A}=0]|_{\mathbf{P}=0}\equiv H_B$$

$$=-\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\frac{(\mathbf{p}^2)^2}{8}-\frac{g}{2m_1m_2}\times\left(\frac{\delta^{ij}}{r}+\frac{r^ir^j}{r^3}\right)p^ip^j. \quad (2.7)$$

The terms, containing the operator $\mathbf{P}$ in the Hamiltonian, determine the contribution of recoil effect to the Compton scattering amplitude. Within the precision of the present calculations, these terms should be taken into account only in the Hamiltonian $\tilde{H}_{nr}$ and can be omitted in $\tilde{H}_B$ (see below). The correction $\delta\varepsilon_0$ to the ground-state energy, related to the Hamiltonian H_B reads

$$\delta\varepsilon_0=\langle 0|H_B|0\rangle=-g^4\left[\frac{5}{8}\mu^4\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)+\frac{\mu^3}{m_1m_2}\right]. \quad (2.8)$$

Let us start the calculation of the Compton scattering amplitude with the amplitude T_{nr} obtained with the use of the nonrelativistic Hamiltonian Eq. (2.5). This amplitude can be represented as a sum $T_{nr}=T_{res}+T_s$ of resonance and seagull parts. The part T_{res} is determined by the second order of perturbation theory with respect to the terms in $\tilde{H}_{nr}[\mathbf{A}]$, linear in the vector potential $\mathbf{A}$. In the laboratory frame it has the form

$$T_{res}=-\langle\psi_0|\exp[-i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{R}]\boldsymbol{\epsilon}_2^*\cdot\left[\frac{e_1}{m_1}\mathbf{p}_1\exp(-i\mathbf{k}_2\cdot\mathbf{r}_1)+\frac{e_2}{m_2}\mathbf{p}_2\exp(-i\mathbf{k}_2\cdot\mathbf{r}_2)\right]\left[\varepsilon_0+\omega_1-\frac{\mathbf{P}^2}{2M}-H_{nr}\right]^{-1}\times\boldsymbol{\epsilon}_1\cdot\left[\frac{e_1}{m_1}\mathbf{p}_1\exp(i\mathbf{k}_1\cdot\mathbf{r}_1)+\frac{e_2}{m_2}\mathbf{p}_2\exp(i\mathbf{k}_1\cdot\mathbf{r}_2)\right]|\psi_0\rangle+(\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2^*,\ \omega_1\leftrightarrow-\omega_2,\ \mathbf{k}_1\leftrightarrow-\mathbf{k}_2). \quad (2.9)$$

Here $\psi_0(\mathbf{r})=\pi^{-1/2}(\mu g)^{3/2}\exp(-\mu gr)$ is the wave function of the ground state, depending on the relative coordinate $\mathbf{r}$. The final momentum of the bound system is equal to $\mathbf{k}_1-\mathbf{k}_2$. Using the relations Eq. (2.3) and Eq. (2.4) and making a simple transformation in order to cancel the exponents containing $\mathbf{R}$, we obtain

$$T_{res}=-\langle\psi_0|\boldsymbol{\epsilon}_2^*\cdot\left[\frac{e_1}{m_1}\left(\mathbf{p}+\frac{m_1}{M}\mathbf{k}_1\right)\exp\left(-i\frac{m_2}{M}\mathbf{k}_2\cdot\mathbf{r}\right)-\frac{e_2}{m_2}\left(\mathbf{p}-\frac{m_2}{M}\mathbf{k}_1\right)\exp\left(i\frac{m_1}{M}\mathbf{k}_2\cdot\mathbf{r}\right)\right]G(\omega_1)\times\boldsymbol{\epsilon}_1\cdot\mathbf{p}\left[\frac{e_1}{m_1}\exp\left(i\frac{m_2}{M}\mathbf{k}_1\cdot\mathbf{r}\right)-\frac{e_2}{m_2}\exp\left(-i\frac{m_1}{M}\mathbf{k}_1\cdot\mathbf{r}\right)\right]\times|\psi_0\rangle+(\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2^*,\ \omega_1\leftrightarrow-\omega_2,\ \mathbf{k}_1\leftrightarrow-\mathbf{k}_2). \quad (2.10)$$

Here, $G(\omega)=[\varepsilon_0+\omega-\omega^2/2M-H_{nr}]^{-1}$ is the nonrelativistic propagator of the system in the operator form. The seagull amplitude T_s is determined by first order of perturbation theory with respect to the terms in $\tilde{H}_{nr}[\mathbf{A}]$ which are quadratic in $\mathbf{A}$. Similar to Eq. (2.10), we obtain

$$T_s=-\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\langle\psi_0|\left[\frac{e_1^2}{m_1}\exp\left(i\frac{m_2}{M}(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}\right)+\frac{e_2^2}{m_2}\exp\left(-i\frac{m_1}{M}(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}\right)\right]|\psi_0\rangle. \quad (2.11)$$

Performing the expansion of Eqs. (2.9) and (2.11) with respect to $\mathbf{k}_{1,2}$ and $\omega_{1,2}$ up to quadratic terms and using the relation $\omega_1-\omega_2=(\mathbf{k}_1-\mathbf{k}_2)^2/2M$, we obtain

$$T_{nr}=-\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\frac{(e_1+e_2)^2}{M}+\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\omega^2\left[\frac{9}{2\mu g^4}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2+\frac{e_1+e_2}{Mg^2}\left(\frac{e_1}{m_1^2}+\frac{e_2}{m_2^2}\right)\right]-[\boldsymbol{\epsilon}_1\times\mathbf{k}_1]\cdot[\boldsymbol{\epsilon}_2^*\times\mathbf{k}_2]\times\left[\frac{1}{2g^2}\left(\frac{e_1^2}{m_1^3}+\frac{e_2^2}{m_2^3}\right)+\frac{3}{2Mg^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\right]. \quad (2.12)$$

There is no need here to distinguish between ω_1 and ω_2 in the $O(\omega^2)$ term. Therefore, we set $\omega_1=\omega_2=\omega$ in Eq. (2.12).

The result Eq. (2.12) is in agreement with Eq. (1.2) and Eq. (1.5), with $\alpha_{\bigcirc}$ calculated in the nonrelativistic approximation, since in our model

$$\mathbf{D}_{nr}=\mu\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)\mathbf{r},$$

$$\alpha_{\bigcirc nr}=\frac{2}{3}\langle\psi_0|\mathbf{D}_{nr}G_0\mathbf{D}_{nr}|\psi_0\rangle=\frac{9}{2\mu g^4}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2,$$

$$\Delta\alpha=\frac{e_1+e_2}{3M}\langle\psi_0|e_1r_1^2+e_2r_2^2|\psi_0\rangle=\frac{e_1+e_2}{Mg^2}\left(\frac{e_1}{m_1^2}+\frac{e_2}{m_2^2}\right),\tag{2.13}$$

$$\langle\psi_0|\mathbf{M}|\psi_{n\neq0}\rangle=0,\quad\langle\psi_0|\mathbf{D}^2|\psi_0\rangle=\frac{3}{g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2,$$

$$\langle\psi_0|\left[\frac{e_1^2r_1^2}{m_1}+\frac{e_2^2r_2^2}{m_2}\right]|\psi_0\rangle=\frac{3}{g^2}\left(\frac{e_1^2}{m_1^3}+\frac{e_2^2}{m_2^3}\right).$$

Here, G_0 is the reduced Green function in the operator form:

$$G_0=[\varepsilon_0-H_{nr}+i0]^{-1}(1-|\psi_0\rangle\langle\psi_0|).\tag{2.14}$$

The details of calculations of different matrix elements, containing the operator G_0 are presented in Appendix A.

We pass now to the calculation of the relativistic corrections to the electromagnetic polarizabilities, connected with the Breit Hamiltonian $\tilde{H}_B$ Eq. (2.6). We perform the calculations in the same way as at the derivation of Eq. (2.9) and Eq. (2.11), but for the Hamiltonian $\tilde{H}=\tilde{H}_{nr}+\tilde{H}_B$. At the calculation of these corrections within our accuracy the terms of the Hamiltonian $\tilde{H}_B[\mathbf{A}]$, quadratic in $\mathbf{A}$ do not contribute to the electromagnetic polarizabilities, i.e., the seagull contribution from $\tilde{H}_B[\mathbf{A}]$ is absent. In the corrections to the electromagnetic polarizabilities from the resonance part of the amplitude, we can take the second order of expansion with respect to ω of the operator Green function and put $\mathbf{k}_{1,2}=0$ elsewhere. This means that within our accuracy, we can neglect in $\tilde{H}_B[\mathbf{A}]$ the terms containing the total momentum $\mathbf{P}$ and replace the exponents in the photon wave function by unity. Therefore, the terms in the Hamiltonian $\tilde{H}_B[\mathbf{A}]$, linear in $\mathbf{A}$, can be represented in the form $-\mathbf{A}(0)\mathbf{J}_B$, where

$$\mathbf{J}_B=-\left(\frac{e_1}{m_1^3}-\frac{e_2}{m_2^3}\right)\frac{\mathbf{p}^2}{2}\mathbf{p}-\frac{g(e_1-e_2)}{2m_1m_2r}\left(\mathbf{p}+\frac{\mathbf{r}}{r^2}(\mathbf{r}\cdot\mathbf{p})\right)\tag{2.15}$$

is the correction to the operator of the total internal current $\mathbf{J}$:

$$\mathbf{J}=\mathbf{J}_{nr}+\mathbf{J}_B=(e_1/m_1-e_2/m_2)\mathbf{p}+\mathbf{J}_B.\tag{2.16}$$

Let us now discuss the relativistic correction to the electric dipole moment operator. In the laboratory frame, it is equal to $\mathbf{D}_{tot}=e_1\mathbf{r}_1+e_2\mathbf{r}_2$. For the total Hamiltonian of the system the following relation holds

$$i[\tilde{H}_{nr}[\mathbf{A}=0]+\tilde{H}_B[\mathbf{A}=0],\ \mathbf{D}_{tot}]\tag{2.17}$$

$$=e_1\frac{\mathbf{p}_1}{m_1}\left(1-\frac{\mathbf{p}_1^2}{2m_1^2}\right)-\frac{ge_1}{2m_1m_2r}\left(\mathbf{p}_1+\frac{\mathbf{r}}{r^2}(\mathbf{r}\cdot\mathbf{p}_1)\right)+(1\leftrightarrow2).$$

The rhs of Eq. (2.17) is nothing but the operator of total current in the lab frame, containing the total momentum $\mathbf{P}$. This can be verified by differentiating the Hamiltonian $\tilde{H}_{nr}[\mathbf{A}]+\tilde{H}_B[\mathbf{A}]$ over $\mathbf{A}$ at $\mathbf{A}=\mathbf{0}$. Therefore, there are no relativistic corrections to the total electric dipole operator $\mathbf{D}_{tot}$. The internal electric dipole moment operator is defined as

$$\mathbf{D}=\mathbf{D}_{tot}-(e_1+e_2)\mathbf{R}_{cm},\tag{2.18}$$

where $\mathbf{R}_{cm}$ is the center-of-mass vector. This vector is defined in such a way that it satisfies the following relations:

$$[\mathbf{R}_{cm},\ \mathbf{P}]=i,\quad i[H_{tot},\ \mathbf{R}_{cm}]=\frac{\mathbf{P}}{H_{tot}},\tag{2.19}$$

where H_{tot} is the total relativistic Hamiltonian of the system, and $\mathbf{P}$ is the total momentum. Within our accuracy, the second relation in Eq. (2.19) reads:

$$i[\tilde{H}_{nr}[\mathbf{A}=0]+\tilde{H}_B[\mathbf{A}=0],\ \mathbf{R}_{cm}]=\frac{\mathbf{P}}{M}\left(1-\frac{\tilde{H}_{nr}[\mathbf{A}=0]}{M}\right).\tag{2.20}$$

It is known (see, e.g., [30]) that there is a relativistic correction to $\mathbf{R}_{cm}$ in classical electrodynamics. For the case of two particles interacting due to electromagnetic field, the corresponding operator that satisfies the relations Eq. (2.19) has the form [22,23]

$$\mathbf{R}_{cm}=\mathbf{R}+\frac{1}{2M}\left(\left\{\mathbf{r}_1,\ \frac{\mathbf{p}_1^2}{2m_1}-\frac{g}{2r}\right\}+\left\{\mathbf{r}_2,\ \frac{\mathbf{p}_2^2}{2m_2}-\frac{g}{2r}\right\}-\left\{\mathbf{R},\ \frac{\mathbf{p}_1^2}{2m_1}+\frac{\mathbf{p}_2^2}{2m_2}-\frac{g}{r}\right\}\right).\tag{2.21}$$

Here, we took into account the first relativistic correction and use the notation $\{a,b\}=ab+ba$. In terms of the variables $\mathbf{r}$ and $\mathbf{p}$ [see Eq. (2.3) and Eq. (2.4)] we obtain

$$\mathbf{R}_{cm}=\mathbf{R}+\frac{(m_2-m_1)}{2M^2}\left(\{\mathbf{r},H_{nr}\}+g\frac{\mathbf{r}}{r}\right),\tag{2.22}$$

where the term proportional to the total momentum $\mathbf{P}$ is omitted. Substituting this expression into Eq. (2.18), we obtain the relativistic correction to the internal electric dipole moment:

$$\mathbf{D}_B=\frac{(e_1+e_2)(m_1-m_2)}{2M^2}\left(\{\mathbf{r},H_{nr}\}+g\frac{\mathbf{r}}{r}\right).\tag{2.23}$$

Note that, as should be the case, the operator of total internal current $\mathbf{J}=\mathbf{J}_{nr}+\mathbf{J}_B$ satisfies within our accuracy the relation

$$\mathbf{J}=i[H_{nr}+H_B,\mathbf{D}_{nr}+\mathbf{D}_B], \tag{2.24}$$

where $\mathbf{D}_{nr}$ is defined in Eq. (2.13), and $H_{nr}+H_B$ is the internal part of Hamiltonian [see Eq. (2.5) and Eq. (2.7)].

Let us write down now the corrections to the $O(\omega^2)$ term of Compton scattering amplitude, related to the Breit Hamiltonian and the corresponding current. The correction due to $\mathbf{J}_B$ reads

$$t_c=-\omega^2\langle\psi_0|[\boldsymbol{\epsilon}_2^*\cdot\mathbf{J}_B G_0^3\boldsymbol{\epsilon}_1\cdot\mathbf{J}_{nr}+\boldsymbol{\epsilon}_2^*\cdot\mathbf{J}_{nr}G_0^3\boldsymbol{\epsilon}_1\cdot\mathbf{J}_B]|\psi_0\rangle+(\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2^*). \tag{2.25}$$

The $O(\omega^2)$ correction to the amplitude, connected with the expansion of the propagator with respect to H_B, has the form

$$t_p=-\omega^2\langle\psi_0|\boldsymbol{\epsilon}_2^*\cdot\mathbf{J}_{nr}[G_0^2H_BG_0^2+G_0^3H_BG_0+G_0H_BG_0^3]\boldsymbol{\epsilon}_1\cdot\mathbf{J}_{nr}|\psi_0\rangle+(\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2^*). \tag{2.26}$$

The contribution due to the correction to wave function is

$$t_w=-\omega^2\langle\psi_0|[\boldsymbol{\epsilon}_2^*\cdot\mathbf{J}_{nr}G_0^3\boldsymbol{\epsilon}_1\cdot\mathbf{J}_{nr}G_0H_B+H_BG_0\boldsymbol{\epsilon}_2^*\cdot\mathbf{J}_{nr}G_0^3\boldsymbol{\epsilon}_1\cdot\mathbf{J}_{nr}]|\psi_0\rangle+(\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2^*). \tag{2.27}$$

At last, the contribution corresponding to the correction to the ground-state energy reads:

$$t_e=3\omega^2\delta\varepsilon_0\langle\psi_0|\boldsymbol{\epsilon}_2^*\cdot\mathbf{J}_{nr}G_0^4\boldsymbol{\epsilon}_1\cdot\mathbf{J}_{nr}|\psi_0\rangle+(\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2^*). \tag{2.28}$$

In order to calculate the matrix elements in Eq. (2.25)–Eq. (2.28) it is convenient to use the following relations (see Appendix A):

$$\begin{aligned}
G_0\mathbf{r}\psi_0&=-\frac{\mathbf{r}}{g}\left(\frac{r}{2}+a\right)\psi_0,\\
G_0r\mathbf{r}\psi_0&=-\frac{\mathbf{r}}{g}\left(\frac{r^2}{3}+\frac{5ar}{6}+\frac{5a^2}{3}\right)\psi_0,\\
G_0r^2\psi_0&=-\frac{1}{g}\left(\frac{r^3}{3}+ar^2-\frac{11a^3}{2}\right)\psi_0,\\
G_0r^3\psi_0&=-\frac{1}{g}\left(\frac{r^4}{4}+\frac{5ar^3}{6}+\frac{5a^2r^2}{2}-\frac{155a^4}{8}\right)\psi_0,
\end{aligned} \tag{2.29}$$

where $a=1/\mu g$. Using (2.29) and also the relation $\mathbf{p}=i\mu[H_{nr},\mathbf{r}]$, we get the following results for the contributions Eq. (2.25)–Eq. (2.28)

$$\begin{aligned}
t_c&=-\frac{2\mu\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*}{9g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)\times\left[20\left(\frac{e_1}{m_1^3}-\frac{e_2}{m_2^3}\right)+\frac{89}{4}\frac{(e_1-e_2)}{\mu m_1m_2}\right],\\
t_p&=\frac{\mu^2\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*}{g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\times\left[\frac{1061}{288}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)+\frac{25}{3\mu m_1m_2}\right],\\
t_w&=\frac{\mu^2\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*}{g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left[\frac{3}{4}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)+\frac{14}{9\mu m_1m_2}\right],\\
t_e&=-\frac{\mu^2\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*}{g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\frac{129}{4}\times\left[\frac{5}{8}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)+\frac{1}{\mu m_1m_2}\right].
\end{aligned} \tag{2.30}$$

Representing the sum of all contributions in Eq. (2.30) as $\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\alpha_{\bigcirc B}$ we obtain the following result for $\alpha_{\bigcirc B}$:

$$\alpha_{\bigcirc B}=-\frac{1}{g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left(\frac{121}{6\mu}-\frac{113}{4M}\right)-\frac{(e_1+e_2)}{Mg^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)\frac{(m_1-m_2)}{2m_1m_2}. \tag{2.31}$$

If one starts the calculation of $\alpha_\bigcirc$ from Eq. (1.2), then it is easy to check that the first term in Eq. (2.31) corresponds to the sum of corrections due to modification of wave function, propagator, and ground-state energy. The second term in Eq. (2.31) corresponds to the contribution due to the relativistic correction to the electric dipole moment Eq. (2.23). Therefore, this correction appears due to the correct description of the center-of-mass motion. It is seen that the first term in Eq. (2.31) has the same dependence on charges as $\alpha_{\bigcirc nr}$ while the second term is similar to $\Delta\alpha$, Eq. (2.13). Taking a sum of $\alpha_{\bigcirc B}$, $\alpha_{\bigcirc\, nr}$, and $\Delta\alpha$, we come to the following result for $\overline{\alpha}$ for the system under consideration:

$$\overline{\alpha}=\frac{1}{\mu g^4}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left[\frac{9}{2}-g^2\left(\frac{121}{6}-\frac{113\mu}{4M}\right)\right]+\frac{(e_1+e_2)}{Mg^2}\left[\frac{3}{2}\left(\frac{e_1}{m_1^2}+\frac{e_2}{m_2^2}\right)-\frac{(e_1+e_2)}{2m_1m_2}\right]. \tag{2.32}$$

Thus, the relativistic corrections to $\alpha_\bigcirc$ has reduced to a renormalization of $\alpha_{\bigcirc nr}$ and essentially to a modification of $\Delta\alpha$, Eq. (1.3). One can expect that the last statement is valid not only for the system under consideration. Indeed, due to the definition of **D**, the correction to $\overline{\alpha}$ related to the modification of $\mathbf{R}_{cm}$ is proportional to Q/M, where Q is the total charge of the system, and, therefore, has the same structure as $\Delta\alpha$.

As a nontrivial test of our method of calculation we checked the fulfillment of the low-energy theorem for the Compton scattering amplitude. At $\omega=0$, this amplitude should have the form

$$T(\omega=0)=-\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\frac{(e_1+e_2)^2}{E_0}\approx-\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\frac{(e_1+e_2)^2}{M+\varepsilon_0}$$
$$\approx-\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\frac{(e_1+e_2)^2}{M}\left(1-\frac{\varepsilon_0}{M}\right), \tag{2.33}$$

where $E_0=M+\varepsilon_0$ is the mass of the system. It is interesting, that the term in rhs of Eq. (2.33), proportional to the nonrelativistic energy ε_0, appears as a contribution of terms from the Breit Hamiltonian $\widetilde{H}_B[\mathbf{A}]$, which we checked by explicit calculations (see Appendix B).

B. The system of a spin-0 particle and a spin-1/2 particle

Let the first particle have the spin 1/2 and the second particle have the spin 0. Then we should add the term

$$\delta\widetilde{H}_{nr}[\mathbf{A}]=-\frac{e_1(1+\kappa_1)}{m_1}\mathbf{s}_1\cdot\mathbf{H} \tag{2.34}$$

to the nonrelativistic Hamiltonian $\widetilde{H}_{nr}[\mathbf{A}]$, Eq. (2.2). Here, $\mathbf{H}$ is the external magnetic field, $\mathbf{s}_1=\boldsymbol{\sigma}_1/2$ is the spin operator of the first particle, and κ_1 is its anomalous magnetic moment in units $e_1/2m_1$. There is also some additional contribution $\delta\widetilde{H}_B[\mathbf{A}]$ to $\widetilde{H}_B[\mathbf{A}]$, Eq. (2.7) (see, e.g., [29]). The terms of $\delta\widetilde{H}_B[\mathbf{A}]$ linear in $\mathbf{s}_1$ as well as Eq. (2.34) determine the $O(\omega)$ terms of the Compton amplitude. These terms are well known and follow, together with the ω-independent term, from the low-energy theorem [31]. As it was explained in the previous subsection, it is sufficient within our accuracy to account for the Breit Hamiltonian only in the long-wave limit, i.e., at $\omega_{1,2}=0$ in order to obtain the $O(\omega^2)$ terms of Compton amplitude. In this limit, the Hamiltonian $\delta\widetilde{H}_B[\mathbf{A}]$ reads

$$\delta\widetilde{H}_B[\mathbf{A}]=-\frac{e_1e_2(1+2\kappa_1)}{2m_1^2}\left(\pi\delta(\mathbf{r})+\frac{1}{r^3}\mathbf{s}_1\cdot(\mathbf{r}\times\boldsymbol{\pi}_1)\right)$$
$$+\frac{e_1e_2(1+\kappa_1)}{m_1m_2r^3}\mathbf{s}_1\cdot(\mathbf{r}\times\boldsymbol{\pi}_2). \tag{2.35}$$

The explicit calculation shows that the contribution of $\delta\widetilde{H}_{nr}[\mathbf{A}]$ given by Eq. (2.34) and the terms in Eq. (2.35) linear in σ_1 do not lead to any contributions to $c_{1,2}$ in Eq. (2.1), i.e., they can be neglected in the calculation of polarizabilities within our accuracy. In particular, there are no terms linear in the spin in the quantities $c_{1,2}$, which is in agreement with the general conclusion on the absence of terms $O(\omega^2)$ linear in spin in the non-Born part of the Compton amplitude [32]. The only term that should be taken into account in addition to those considered in the previous subsection, is the spin-independent term in Eq. (2.35) (Darwin term):

$$\delta_D H_B=\frac{\pi g(1+2\kappa_1)}{2m_1^2}\delta(\mathbf{r}). \tag{2.36}$$

It follows from Eq. (2.35) that there is no correction to the current associated with the Hamiltonian $\delta_D H_B$. Using the expressions (2.26–2.28) with the replacement $H_B\to\delta_D H_B$ and the relations Eq. (2.29) we obtain

$$\delta_D t_c=0,\quad \delta_D t_w=-\frac{5\mu\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*}{8m_1^2g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2(1+2\kappa_1), \tag{2.37}$$
$$\delta_D t_p=0,\quad \delta_D t_e=\frac{129\mu\omega^2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*}{8m_1^2g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2(1+2\kappa_1).$$

As a result, the correction to the electric polarizability associated with the Breit Hamiltonian in the system of spin 0 and spin 1/2 will be the sum of $\alpha_{\bigcirc B}$ Eq. (2.31) and

$$\delta\alpha_{\bigcirc B}=\frac{31\mu}{2m_1^2g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2(1+2\kappa_1). \tag{2.38}$$

C. The system of two spin-1/2 particles

In the case of two spin-1/2 particles, it is necessary to account for two Darvin terms in addition to the Breit Hamiltonian Eq. (2.7), corresponding to both particles

$$\delta_D H_B=\frac{\pi g(1+2\kappa_1)}{2m_1^2}\delta(\mathbf{r})+\frac{\pi g(1+2\kappa_2)}{2m_2^2}\delta(\mathbf{r}) \tag{2.39}$$

and the Hamiltonian, corresponding to spin-spin interaction [29]:

$$\delta_s H_B=\frac{g(1+\kappa_1)(1+\kappa_2)}{m_1m_2}\times\left[\frac{3(\mathbf{n}\cdot\mathbf{s}_1)(\mathbf{n}\cdot\mathbf{s}_2)-\mathbf{s}_1\cdot\mathbf{s}_2}{r^3}+\frac{8\pi}{3}\delta(\mathbf{r})\mathbf{s}_1\cdot\mathbf{s}_2\right], \tag{2.40}$$

with $\mathbf{n}=\mathbf{r}/r$. It is more convenient to rewrite $\delta_s H_B$ in terms of the total spin operator $\mathbf{S}=\mathbf{s}_1+\mathbf{s}_2$:

$$\delta_s H_B=\frac{g(1+\kappa_1)(1+\kappa_2)}{2m_1m_2}\left[\frac{3n_in_jQ_{ij}}{2r^3}+4\pi\delta(\mathbf{r})\left(\frac{2}{3}\mathbf{S}^2-1\right)\right], \tag{2.41}$$

where the operator Q_{ij}, quadratic in $\mathbf{S}$, is equal to

$$Q_{ij}=S_iS_j+S_jS_i-\tfrac{2}{3}\mathbf{S}^2\delta_{ij}. \tag{2.42}$$

Note that in such a system as positronium, it is necessary to add the contribution of the annihilation diagram, which results in the replacement $(2\mathbf{S}^2/3-1)\to(7\mathbf{S}^2/6-1)$ in the coefficient of the δ-function in Eq. (2.41) (of course, in this case $m_1=m_2$, $e_1=-e_2$). As in the previous subsection, the terms proportional to the δ-function in Eq. (2.39) and Eq. (2.41) give the contribution $\delta\alpha_{\bigcirc B}$, which should be added to $\alpha_{\bigcirc_B}$, Eq. (2.31):

$$\delta\alpha_{\bigcirc B}=\frac{31\mu}{2g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left[\frac{1+2\kappa_1}{m_1^2}+\frac{1+2\kappa_2}{m_2^2}+\frac{4(1+\kappa_1)(1+\kappa_2)}{m_1m_2}\left(\frac{2}{3}S(S+1)-1\right)\right]. \tag{2.43}$$

Here, we replaced $\mathbf{S}^2$ by its eigenvalue $S(S+1)$, where $S=0,1$ is the total spin of the system. The term in Eq. (2.43) containing the tensor operator $3(\mathbf{n}\cdot\mathbf{S})^2-\mathbf{S}^2$ determines the contribution to the $O(\omega^2)$ part of the Compton amplitude, which has the form

$$t_{(tensor)}=\omega^2\alpha_T\epsilon_1^i\epsilon_2^{j*}\langle Q_{ij}\rangle, \tag{2.44}$$

where $\langle\cdots\rangle$ denotes the averaging over the spin part of the wave function. Of course, $t_{(tensor)}$ vanishes if $S=0$. Since there is no correction to the current or to the energy of the ground state due to the tensor part of $\delta_s H_B$, the contributions to α_T come only from the corrections to the propagator and to the wave function. Using Eqs. (2.26) and (2.27), and the relations (see Appendix A)

$$G_0(3r_ir_j-r^2\delta_{ij})|\psi_0\rangle=-\frac{3r_ir_j-r^2\delta_{ij}}{g}\left(\frac{r}{3}+\frac{a}{2}\right)|\psi_0\rangle,$$

$$G_0(3r_ir_j-r^2\delta_{ij})r|\psi_0\rangle=-\frac{3r_ir_j-r^2\delta_{ij}}{g}\left(\frac{7a^2}{8}+\frac{7ar}{12}+\frac{r^2}{4}\right)|\psi_0\rangle, \tag{2.45}$$

we obtain

$$\alpha_T=-\frac{47(1+\kappa_1)(1+\kappa_2)}{40Mg^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2. \tag{2.46}$$

In the system of two spin-1/2 particles there is a big paramagnetic contribution to the magnetic polarizability from the first term in Eq. (1.5). The main contribution corresponds to the transition from the ground state with the total spin $S=0$ to the state with $S=1$, with both states having the same angular momenta $l=0$, and radial quantum numbers $n_r=0$ (hyperfine splitting). Representing the spin part of the magnetic moment operator in the form

$$\mathbf{M}_s=f_1\mathbf{s}_1+f_2\mathbf{s}_2,\quad f_i=\frac{e_i(1+\kappa_i)}{m_i},$$

and using Eq. (2.40), we obtain

$$\beta_1=-\frac{3(f_1-f_2)^2a^3}{16f_1f_2},\quad a=\frac{1}{\mu g}. \tag{2.47}$$

As was pointed out in the previous section, for positronium, it is necessary to change the coefficient of δ-function in Eq. (2.41). As a result, the contribution of the first term in Eq. (1.5) to the magnetic polarizability of positronium is

$$\beta_1=\pm\tfrac{3}{7}a^3, \tag{2.48}$$

where upper sign corresponds to parapositronium $(S=0)$, and the lower sign to orthopositronium $(S=1)$.

D. The system of two particles with arbitrary spins

Let the particles have the spins $s_{1,2}$ and magnetic moments $\mu_{1,2}$, which we represent in the form

$$\mu_a=\frac{e_as_a}{m_a}(1+\kappa_a),\quad a=1,2. \tag{2.49}$$

The electromagnetic current for each particle has the form (see, e.g., [33,34])

$$j_\mu=\overline{\psi}(p')\left[F_e\frac{p_\nu+p'_\nu}{2m}+\frac{G_m}{2m}\Sigma_{\mu\nu}q^\nu\right]\psi(p), \tag{2.50}$$

where $q=p'-p$. The operator $\Sigma_{\mu\nu}$ is a generalization of the corresponding matrix for spin 1/2. The indices numerating the particles have been omitted. The quantities F_e and G_m depend on q^2 and $(s_\mu q^\mu)^2$, where s_μ is the four-vector of the spin operator. These quantities are normalized as follows:

$$F_e(q=0)=1,\quad G_m(q=0)=1+\kappa. \tag{2.51}$$

If we neglect the q dependence of the form factors, then, in addition to the Breit Hamiltonian for two spin-0 particles, Eq. (2.6), it is necessary to take into account the Hamiltonian Eq. (2.40) (with the corresponding spin operators) and two other contributions [33]. Namely, the Darwin Hamiltonian

$$\delta_DH_B=\sum_{a=1,2}\frac{2\pi g}{3m_a^2}(1+2\kappa_a)(s_a+\zeta_a)\delta(\mathbf{r}), \tag{2.52}$$

$\zeta=0$ for integer spin and $\zeta=1/4$ otherwise, and the term containing the quadrupole moments of the particles

$$\delta_QH_B=\sum_{a=1,2}\frac{g(1+2\kappa_a)\xi_a}{2m_a^2r^3}[3(\mathbf{n}\cdot\mathbf{s}_a)^2-\mathbf{s}_a^2], \tag{2.53}$$

$\xi=1/(2s-1)$ for integer spin and $\xi=1/(2s)$ otherwise. It is clear that all matrix elements can be calculated in the same way as in the previous subsection. The averaging over the spin variables can be done using the following relations

$$\langle SS_z|\left[s_{1i}s_{2j}+s_{2i}s_{1j}-\frac{2}{3}\delta_{ij}\mathbf{s}_1\cdot\mathbf{s}_2\right]|S,S'_z\rangle=A(S,s_1,s_2)\langle SS_z|Q_{ij}|S,S'_z\rangle,$$

$$\langle SS_z|\left[s_{1i}s_{1j}+s_{1i}s_{1j}-\frac{2}{3}\delta_{ij}\mathbf{s}_1^2\right]|S,S_z'\rangle$$
$$=B(S,s_1,s_2)\langle SS_z|Q_{ij}|S,S_z'\rangle, \tag{2.54}$$

where $\mathbf{S}=\mathbf{s}_1+\mathbf{s}_2$ is the total spin operator, Q_{ij} is defined in Eq. (2.42), and for $S\geqslant 1$

$$A(S,s_1,s_2)=\frac{\Lambda^2+2\Lambda(\lambda_1+\lambda_2)-3(\lambda_1-\lambda_2)^2}{2\Lambda(4\Lambda-3)},$$
$$B(S,s_1,s_2)=\frac{3\Lambda^2+\Lambda(2\lambda_1-6\lambda_2-3)+3(\lambda_1-\lambda_2)(\lambda_1-\lambda_2-1)}{2\Lambda(4\Lambda-3)}. \tag{2.55}$$

Here, $\Lambda=S(S+1)$, $\lambda_{1,2}=s_{1,2}(s_{1,2}+1)$ are the eigenvalues of the operators $\mathbf{S}^2$ and $\mathbf{s}_{1,2}^2$, respectively. For $S=0,1/2$ we put $A=B=0$. As a result, we obtain the following generalization of Eq. (2.43) to the case of arbitrary spins:

$$\delta\alpha_{\bigcirc B}=\frac{62\mu}{3g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left[\frac{1+2\kappa_1}{m_1^2}(s_1+\zeta_1)+\frac{1+2\kappa_2}{m_2^2}\times(s_2+\zeta_2)+\frac{2(1+\kappa_1)(1+\kappa_2)}{m_1m_2}(\Lambda-\lambda_1-\lambda_2)\right]. \tag{2.56}$$

The generalization of Eq. (2.46) is

$$\alpha_T=-\frac{47\mu}{40g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left[\frac{2(1+\kappa_1)(1+\kappa_2)}{m_1m_2}A(S,s_1,s_2)+\frac{1+2\kappa_1}{m_1^2}\xi_1B(S,s_1,s_2)+\frac{1+2\kappa_2}{m_2^2}\xi_2B(S,s_2,s_1)\right]. \tag{2.57}$$

Let us now take into account the q dependence of the electromagnetic form factors of the constituents defined in Eq. (2.50). We assume, that the scale of variation of these form factors are much larger than the typical momentum transfer $\sim\mu g$. In other words, the characteristic size of each constituent is much smaller than the size of the whole system $a=1/\mu g$. In this case it is sufficient, within our accuracy, to take $G_m=1+\kappa$ and to expand the form factor F_e up to quadratic in q terms,

$$F_e(q^2,(s^\mu q_\mu)^2)\approx 1-\frac{r_e^2\mathbf{q}^2}{6}+\frac{r_s^2(\mathbf{s}\cdot\mathbf{q})^2}{2}, \tag{2.58}$$

where $r_{e,s}^2$ are some constants. Multiplying the $O(\mathbf{q}^2)$ terms in this expression by $-4\pi g/\mathbf{q}^2$ and performing the Fourier transform, we obtain the additional terms in the Hamiltonian

$$\delta_f H_B=\sum_{a=1,2}\left[\frac{2\pi g}{3}(r_{ea}^2-r_{sa}^2\mathbf{s}_a^2)\delta(\mathbf{r})+gr_{sa}^2\frac{3(\mathbf{s}_a\cdot\mathbf{n})^2-\mathbf{s}_a^2}{2r^3}\right]. \tag{2.59}$$

Since the terms in this Hamiltonian have the same structure as above, it is easy to write down the result for the corresponding corrections to polarizabilities:

$$\delta_f\alpha_{\bigcirc B}=\frac{62\mu}{3g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2(r_{e1}^2-r_{s1}^2\lambda_1+r_{e2}^2-r_{s2}^2\lambda_2),$$
$$\delta_f\alpha_T=-\frac{47\mu}{40g^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2[r_{s1}^2B(S,s_1,s_2)+r_{s2}^2B(S,s_2,s_1)]. \tag{2.60}$$

If the parameters of the form factors $r_{e,s}^2\sim 1/m^2\ll a^2$, then the contributions Eq. (2.60) to the polarizabilities are of the same order as $\alpha_{\bigcirc B}$. The first relativistic correction to the Compton scattering amplitude at $\omega=0$, Eq. (2.33), is proportional to $\varepsilon_0=-\mu g^2/2$ and is independent of the spins of the constituents. Then, the correction to the amplitude at $\omega=0$ connected with spin-dependent terms in Breit Hamiltonian as well as the Darwin terms (also having the spin origin) should vanish. This statement was checked explicitly (see Appendix B).

Let us consider now the paramagnetic contribution to the magnetic polarizability from the first term in Eq. (1.5). Let $s_1\geqslant s_2$. Then, the total spin of the ground state is $S=s_1-s_2$, and the main contribution corresponds to the transition from the ground state to the state with $S=s_1-s_2+1$, with both states having the same angular momentum, $l=0$, and radial quantum number $n_r=0$ (hyperfine splitting). A simple explicit calculation leads to

$$\beta_1=-\frac{(f_1-f_2)^2s_2(s_1+1)a^3}{4f_1f_2(s_1-s_2+1)^2}. \tag{2.61}$$

This term should be added to the diamagnetic contribution β_{dia} [see Eq. (2.12)]:

$$\beta_{dia}=-\frac{1}{2g^2}\left(\frac{e_1^2}{m_1^3}+\frac{e_2^2}{m_2^3}\right)-\frac{3}{2Mg^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2. \tag{2.62}$$

III. CONCLUSION

We have obtained the complete result for the first relativistic corrections to the electromagnetic polarizabilities, including the tensor part that exists for the total spin $S\geqslant 1$. We demonstrated that, within our accuracy, this tensor part contains the quadrupole moment of the system and not any higher multipoles. For the system of two spinless particles it is easy to check that the total relativistic correction Eqs. (2.13) and (2.31) is negative at arbitrary masses and charges. In the general case of nonzero spins and arbitrary anomalous magnetic moments, the relativistic correction $\Delta\alpha+\alpha_{\bigcirc B}+\delta\alpha_{\bigcirc B}$, where $\delta\alpha_{\bigcirc B}$ is given by Eq. (2.56), can be positive. It is interesting to consider some special cases. The first of them is a hydrogenlike ion. In this case, $e_1=e$, $e_2=-Ze$, and $m_2\gg m_1$. In the limit $m_2\to\infty$ the result for electric polarizabilities is independent of the spin and magnetic moment of the nucleus. Neglecting also the anomalous magnetic moment κ_1 of the electron, we obtain from Eqs. (2.12), (2.13), (2.31), and (2.43)

$$\bar{\alpha}=\frac{9}{2m^3\alpha_{em}^3Z^4}-\frac{14}{3m^3\alpha_{em}Z^2}, \tag{3.1}$$

where $\alpha_{em}=e^2=1/137$ is the fine-structure constant. Note that in this limit, the correction $\Delta\alpha$, Eq. (1.3), vanishes. The result Eq. (3.1) is in agreement with that obtained in [34,35] with the use of the reduced Green function of the Dirac equation for an electron in a Coulomb field. For the magnetic polarizability at $s_2=0$ we have

$$\bar{\beta}=\beta_{dia}=-\frac{1}{2m^3\alpha_{em}Z^2}. \tag{3.2}$$

For $s_2=1/2$ in the limit $m_2\gg m_1$ there is a very big contribution from the paramagnetic part of the magnetic polarizability, Eq. (2.47). In the Compton scattering amplitude this contribution should be taken into account only for photon energies ω much smaller than the energy $E_{hf}\sim\alpha_{em}^4m_1^2/m_2$ of the hyperfine splitting. For $\mu g^2\gg\omega\gg E_{hf}$ the paramagnetic contribution should be omitted.

Another interesting example is positronium. As we mentioned above, in this case, it is necessary to replace $(2\mathbf{S}^2/3-1)\to(7\mathbf{S}^2/6-1)$ in the coefficient of the δ-function in Eq. (2.41) due to the contribution of the annihilation diagram. Putting $m_1=m_2=m$, $e_1=-e_2=e$, and $\kappa_1=\kappa_2=0$, we obtain $\Delta\alpha=0$ and the complete result for the polarizabilities

$$\bar{\alpha}=\frac{36}{(m\alpha_{em})^3}+\frac{1}{6m^3\alpha_{em}}\begin{cases}-1001 & \text{for } S=0\\ 735 & \text{for } S=1\end{cases},$$
$$\bar{\beta}=(-1)^S\frac{24}{7m^3\alpha_{em}^3}-\frac{4}{m^3\alpha_{em}}. \tag{3.3}$$

As in the previous case, for photon energy $\omega\gg\alpha_{em}^4m$ the paramagnetic contribution should be omitted in the Compton scattering amplitude.

For $S=1$ (orthopositronium) we also have the tensor polarizability

$$\alpha_T=-\frac{47}{20m^3\alpha_{em}}. \tag{3.4}$$

Thus, we have shown that the complete set of the first relativistic corrections differs essentially from the commonly used term $\Delta\alpha$. We suppose that for the electromagnetic polarizabilities of hadrons investigated within the constituent quark model an analogous situation may be found.

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APPENDIX A

In this appendix, we derive the formulas for the result of the action of the operator G_0, Eq. (2.14), on the wave function $|\psi_0\rangle$, multiplied by some polynomial of $\mathbf{r}$. More precisely, we obtain the expression for $G_0Y_{lm}(\mathbf{r}/r)r^n|\psi_0\rangle$ in the form of the product $Y_{lm}(\mathbf{r}/r)P(r)|\psi_0\rangle$, where $P(r)$ is some polynomial. Since the Hamiltonian H_{nr} commutes with the operator of angular momentum $\mathbf{l}=\mathbf{r}\times\mathbf{p}$, we can make the following transformation:

$$G_0Y_{lm}(\mathbf{r}/r)r^n|\psi_0\rangle=Y_{lm}(\mathbf{r}/r)G_0^{(l)}(r^n-\delta_{l0}\langle r^n\rangle)|\psi_0\rangle, \tag{A1}$$

where $G_0^{(l)}=[\varepsilon_0-H_{nr}^{(l)}+i0]^{-1}$, $H_{nr}^{(l)}=-(2\mu r)^{-1}\partial_r^2r+l(l+1)/(2\mu r^2)-g/r$ is the radial Hamiltonian with the angular momentum l, and

$$\langle r^n\rangle=\langle\psi_0|r^n|\psi_0\rangle=\frac{(n+2)!}{2^{n+1}}a^n, \tag{A2}$$

where $a=1/\mu g$. In the derivation of Eq. (A1) we used the identity

$$(1-|\psi_0\rangle\langle\psi_0|)Y_{lm}(\mathbf{r}/r)r^n|\psi_0\rangle=Y_{lm}(\mathbf{r}/r)(r^n-\delta_{l0}\langle r^n\rangle)|\psi_0\rangle.$$

For our purposes, it is sufficient to consider the cases $n\geq l-1$ for $l\neq0$ and $n\geq1$ for $l=0$. It is easy to check that in these cases one can represent the result of action of $G_0^{(l)}$ in rhs of Eq. (A1) in the form

$$G_0^{(l)}(r^n-\delta_{l0}\langle r^n\rangle)|\psi_0\rangle=\sum_{k=0}^{\infty}C_kr^k|\psi_0\rangle, \tag{A3}$$

where C_k are some constants to be found. Acting on both sides of this equation with the operator $\varepsilon_0-H_{nr}^{(l)}$ and collecting the coefficients with different powers of r, we obtain

$$r^n-\delta_{l0}\langle r^n\rangle=-\frac{l(l+1)}{2\mu}C_0r^{-2}-\frac{(l-1)(l+2)}{2\mu}C_1r^{-1}+\sum_{k=0}^{\infty}\left(\frac{(k-l+2)(k+l+3)}{2\mu}C_{k+2}-g(k+1)C_{k+1}\right)r^k. \tag{A4}$$

From this relation, we can find the coefficients C_i. For the case $n\geq l-1$, $l\neq0$, we finally obtain

$$G_0 Y_{lm}(\mathbf{r}/r) r^n |\psi_0\rangle = -Y_{lm}(\mathbf{r}/r)\frac{(n-l+1)!(n+l+2)!}{g(2/a)^{n+1}(n+1)!} \times \sum_{k=l}^{n+1} \frac{(k-1)!(2r/a)^k}{(k-l)!(k+l+1)!}|\psi_0\rangle. \tag{A5}$$

For the case $n \geqslant 1$, $l=0$ we have

$$G_0 r^n |\psi_0\rangle = -\frac{(n+2)!}{g(2/a)^{n+1}} \sum_{k=2}^{n+1} \frac{1}{k}\left(\frac{(2r/a)^k}{(k+1)!} - \frac{k+2}{2}\right)|\psi_0\rangle. \tag{A6}$$

Using the formulas (A4) and (A5), one can easily calculate all matrix elements needed.

APPENDIX B

In this appendix, we check the fulfillment of the low-energy theorem. Namely, we reproduce the two first terms of the expansion with respect to ε_0/M of the Compton scattering amplitude at $\omega=0$:

$$T(\omega=0) \approx -\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^* \frac{(e_1+e_2)^2}{M}\left(1-\frac{\varepsilon_0}{M}\right). \tag{B1}$$

In fact, the first term is contained in Eq. (2.12). In order to obtain the second term, we have to take into account the corrections to the current, seagull, wave function, propagator, and energy due to the Breit Hamiltonian $\widetilde{H}_B[\mathbf{A}]$. The contribution to the amplitude at $\omega=0$ due to $\mathbf{J}_B$ reads

$$T_c = -\langle \psi_0 | [\boldsymbol{\epsilon}_2^* \cdot \mathbf{J}_B G_0 \boldsymbol{\epsilon}_1 \cdot \mathbf{J}_{nr} + \boldsymbol{\epsilon}_2^* \cdot \mathbf{J}_{nr} G_0 \boldsymbol{\epsilon}_1 \cdot \mathbf{J}_B] | \psi_0\rangle + (\boldsymbol{\epsilon}_1 \leftrightarrow \boldsymbol{\epsilon}_2^*). \tag{B2}$$

The contribution due to the correction to seagull (the terms in $\widetilde{H}_B[\mathbf{A}]$ being quadratic in $\mathbf{A}$) reads

$$T_s = \langle \psi_0 | \left\{ \left(\frac{e_1^2}{m_1^3} + \frac{e_2^2}{m_2^3}\right)\left[(\boldsymbol{\epsilon}_1 \cdot \mathbf{p})(\boldsymbol{\epsilon}_2^* \cdot \mathbf{p}) + (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*)\frac{\mathbf{p}^2}{2}\right] + \frac{g^2}{m_1 m_2}\left[\frac{\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*}{r} + \frac{(\boldsymbol{\epsilon}_1 \cdot \mathbf{r})(\boldsymbol{\epsilon}_2^* \cdot \mathbf{r})}{r^3}\right]\right\} |\psi_0\rangle. \tag{B3}$$

The contribution connected with the expansion of propagator with respect to H_B has the form

$$T_p = -\langle \psi_0 | (\boldsymbol{\epsilon}_2^* \cdot \mathbf{J}_{nr} G_0 H_B G_0 \boldsymbol{\epsilon}_1 \cdot \mathbf{J}_{nr}) | \psi_0\rangle + (\boldsymbol{\epsilon}_1 \leftrightarrow \boldsymbol{\epsilon}_2^*). \tag{B4}$$

The contribution due to the correction to wave function is

$$T_w = -\langle \psi_0 | [\boldsymbol{\epsilon}_2^* \cdot \mathbf{J}_{nr} G_0 \boldsymbol{\epsilon}_1 \cdot \mathbf{J}_{nr} G_0 H_B + H_B G_0 \boldsymbol{\epsilon}_2^* \cdot \mathbf{J}_{nr} G_0 \boldsymbol{\epsilon}_1 \cdot \mathbf{J}_{nr}] | \psi_0\rangle + (\boldsymbol{\epsilon}_1 \leftrightarrow \boldsymbol{\epsilon}_2^*). \tag{B5}$$

At last, the contribution corresponding to the correction to the ground-state energy reads:

$$T_e = \delta\varepsilon_0 \langle \psi_0 | \boldsymbol{\epsilon}_2^* \cdot \mathbf{J}_{nr} G_0^2 \boldsymbol{\epsilon}_1 \cdot \mathbf{J}_{nr} | \psi_0\rangle + (\boldsymbol{\epsilon}_1 \leftrightarrow \boldsymbol{\epsilon}_2^*). \tag{B6}$$

Using the results of Appendix A, we obtain the following expressions for the corrections:

$$T_c = -\frac{\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*}{3} g^2 \mu^2 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\left[5\mu\left(\frac{e_1}{m_1^3} - \frac{e_2}{m_2^3}\right) + 4\frac{e_1 - e_2}{m_1 m_2}\right], \tag{B7}$$

$$T_s = \frac{\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*}{6} g^2 \mu \left[5\mu\left(\frac{e_1^2}{m_1^3} + \frac{e_2^2}{m_2^3}\right) + \frac{8g}{m_1 m_2}\right],$$

$$T_p = \frac{\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*}{12} g^2 \mu^3 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \left[7\mu\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{12}{m_1 m_2}\right],$$

$$T_w = \frac{\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*}{6} g^2 \mu^3 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \left[9\mu\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{14}{m_1 m_2}\right],$$

$$T_e = -\frac{\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^*}{4} g^2 \mu^3 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \left[5\mu\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{8}{m_1 m_2}\right].$$

Summing up these contributions, we get

$$T_B = -\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^* \frac{(e_1+e_2)^2}{2M^2} \mu g^2, \tag{B8}$$

which is the second term in Eq. (B1). Let us consider now the contribution to the Compton amplitude at $\omega=0$, connected with the spin-dependent terms and the Darwin terms in Breit Hamiltonian. Note that all these terms are proportional to either $\delta(\mathbf{r})$ or to the operator $(3n_i n_j - \delta_{ij})/r^3$. The terms $\propto \delta(\mathbf{r})$ give the contributions to T_w and T_e. Using the results of Appendix A, it is easy to show that the sum of these two contributions is zero. The terms $\propto (3n_i n_j - \delta_{ij})/r^3$ give the contributions to T_w and T_p. Again, direct calculations show that they also cancel each other. Therefore, we proved that the first relativistic correction to the Compton amplitude at $\omega=0$ is spin independent, which is in agreement with the low-energy theorem.

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