Nonlocal content of quantum operations

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We show that quantum operations on multiparticle systems have a nonlocal content; this mirrors the nonlocal content of quantum states. We introduce a general framework for discussing the nonlocal content of quantum operations, and give a number of examples. Quantitative relations between quantum actions and the entanglement and classical communication resources needed to implement these actions are also described. We also show how entanglement can catalyze classical communication from a quantum action.

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I. INTRODUCTION

In the past, most of the research on quantum nonlocality has been devoted to the issue of nonlocality of *quantum states*. However, we feel that an equally important issue is that of nonlocality of *quantum evolutions*. That is, in parallel with the understanding of nonlocality of quantum kinematics one should also develop an understanding of the nonlocality of quantum dynamics.

Let us start with a simple example. Consider two qubits situated far from each other, one held by Alice and the other one by Bob. Suppose Alice and Bob would like to implement a two-qubit quantum evolution described by the unitary operator U. (We wish to be able to apply U on any initial state of the two qubits.) With the exception of the case when U is a product of two local unitary operators, $U = U_A \otimes U_B$, U is nonlocal.

Implementing a unitary operation U that can modify the degree of entanglement between two remote quantum systems requires that the two systems interact with each other. This also means that the implementation takes time (since the systems are far apart and signals cannot propagate faster than light). It is our aim to give a quantitative description of the relevant aspects of this interaction.

The interaction can take place in many different ways, and may be implemented by using different resources. We will restrict ourselves to two resources—classical communication and entanglement. The reason is that these two resources are a minimal and irreducible set of resources—any unitary evolution can be implemented using them, and having only one of them is insufficient. This framework has also been put forward by Chefles, Gilson, and Barnett [2].

The fact that we allow enough time for the classical communication suggests that we could equally well use this time to send quantum bits. But sending qubits is not a resource independent of using classical communication and entanglement. Indeed, the exact quantitative relationship between these different resources is an important question which was solved in the original teleportation paper [1].

We emphasize that, although we have largely discussed the role of quantum entanglement above, the role of the classical communication is equally important. Understanding the character of a quantum evolution requires knowing both the amount of entanglement and the amount of classical communication needed.

II. GENERAL SUFFICIENCY CONDITIONS

First of all, it is important to note that any unitary evolution can be implemented given enough shared entanglement and classical communication. Indeed, consider the case of two qubits, one held by Alice and one by Bob. Any unitary transformation U on these two qubits can be accomplished by having Alice teleport her qubit to Bob, Bob perform U locally, and finally Bob teleport Alice's qubit back to Alice. The resources needed for the two teleportation actions are [one e-bit (an e-bit is the entangelement of one singlet) plus two classical bits transmitted from Alice to Bob for the Alice to Bob teleportation] plus (one e-bit plus two classical bits transmitted from Bob to Alice for the Bob to Alice teleportation). It is obvious now that any unitary operation involving any number of parties and any number of qubits can be accomplished by a similar procedure (teleporting all states to a single location, performing U locally, and teleporting the qubits back to their original locations).

The "double teleportation" procedure shown above is sufficient to implement any quantum evolution. The question is, however, whether so many resources are actually needed. We will discuss a number of specific examples below.

III. THE SWAP OPERATION ON TWO QUBITS

The SWAP operation defined by

$$U_{\mathrm{SWAP}} |\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle \tag{1}$$

is a particularly intriguing case, since, although it takes product states to product states, it is, as we now show, the most nonlocal operation possible in the sense described above. That is, we will prove that in order to implement a SWAP operation on two qubits it is not only sufficient but also *necessary* to use two e-bits plus two bits of classical communication from Alice to Bob plus two bits of classical communication from Bob to Alice.

Proof: To prove that the SWAP operation needs as nonlocal resources two e-bits, we will show that if we have an apparatus able to implement the SWAP operation we can use it in order to create two e-bits. Thus, since entanglement cannot

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be created *ex nihilo*, the apparatus that implements the SWAP must use two e-bits as an internal nonlocal resource.

Let us show how to generate two singlets using the SWAP operation. First Alice and Bob prepare singlets locally,

$$\uparrow_A\uparrow_a+\downarrow_A\downarrow_a \quad \text{and} \quad \uparrow_B\uparrow_b+\downarrow_B\downarrow_b. \tag{2}$$

Alice's spins are labeled A and a and Bob's B and b and $\uparrow$ ($\downarrow$) represents a spin polarized in the positive (negative) direction along the z axis (here and in what follows we will leave out normalization factors for states). Now perform the SWAP operation on spins A and B:

$$(\uparrow_A\uparrow_a+\downarrow_A\downarrow_a)(\uparrow_B\uparrow_b+\downarrow_B\downarrow_b)$$
$$\mapsto(\uparrow_B\uparrow_a+\downarrow_B\downarrow_a)(\uparrow_A\uparrow_b+\downarrow_A\downarrow_b). \tag{3}$$

This state contains two singlets held between Alice and Bob.

To find the classical communication resources needed to implement the SWAP operation we will adapt an argument first given in [1]. We show that if we have an apparatus able to implement the SWAP operation we can use it in order to communicate two bits from Alice to Bob plus two bits from Bob to Alice. From this it follows that it must be the case that the SWAP apparatus uses two bits of classical communication from Alice to Bob plus two bits of classical communication from Bob to Alice as an internal resource, otherwise Alice could receive information from Bob transmitted faster than light.

Suppose that there is a SWAP protocol that requires fewer than four bits of classical communication (two bits each way). Alice and Bob can produce an instantaneous SWAP operation that works correctly with probability greater than one-sixteenth in the following way. Alice and Bob run the supposed SWAP protocol, but, instead of waiting for classical communication from each other, they simply guess the bits that they would have received and perform the associated actions immediately. Since we have assumed that the given SWAP protocol requires fewer than four bits, the probability that Alice and Bob guess correctly is greater than one-sixteenth and hence the SWAP operation also succeeds with probability greater than one-sixteenth.

Thus using the protocol described previously we can now use this imperfect, but instantaneous SWAP to communicate four bits instantaneously. The bits arrive correctly when the SWAP is implemented correctly. Hence the probability that four bits arrive correctly is larger than one-sixteenth; four bits communicated correctly with probability greater than one-sixteenth represents a nonzero amount of information. Thus Alice and Bob have managed to convey some information to each other instantaneously. We conclude therefore that the SWAP operation cannot be done with fewer that four bits of classical communication; otherwise it allows communication faster than the speed of light.

Earlier in this section we showed that the SWAP operation can be used to generate two singlets. We now show that the SWAP operation can also be used to perform four bits of classical communication (two bits each way): the main idea is that of ‘‘superdense coding’’ [3]. Suppose that initially Alice and Bob share two singlets:

$$\uparrow_A\uparrow_B+\downarrow_A\downarrow_B \quad \text{and} \quad \uparrow_a\uparrow_b+\downarrow_a\downarrow_b. \tag{4}$$

Now Alice chooses one of four local unitary operations 1 (identity), σ_x, σ_y, σ_z and performs it on her spin A. This causes the first singlet to be in one of the four Bell states. Bob also, independently, chooses one of these four local unitaries and performs it on his spin b, putting the second singlet into one of the Bell states. Then the SWAP operation is performed on spins A and b. Now both Bob and Alice have one of the Bell states locally; which one they have depends on which operation the other performed. By measurement, they can work out which of the four unitaries the other performed. Thus the SWAP operation has enabled two bits of classical communication to be performed each way.

IV. THE CONTROLLED-NOT OPERATION ON TWO QUBITS

Another important quantum operation is controlled NOT (CNOT), defined as

$$\uparrow\uparrow\mapsto\uparrow\uparrow, \tag{5}$$

$$\uparrow\downarrow\mapsto\uparrow\downarrow, \tag{6}$$

$$\downarrow\uparrow\mapsto\downarrow\downarrow, \tag{7}$$

$$\downarrow\downarrow\mapsto\downarrow\uparrow. \tag{8}$$

As we prove below, the necessary and sufficient resources for the CNOT operation are one e-bit plus one bit of classical communication from Alice to Bob plus one bit of classical communication from Bob to Alice.

Proof: Constructing a CNOT operation. We now show how to construct the CNOT operation using one singlet and two bits of classical communication. We then show how to generate one singlet or perform two bits of classical communication using the CNOT.

First we will show how, using one singlet and one bit of classical communication each way, we can perform a CNOT operation on the state

$$(\alpha\uparrow_A+\beta\downarrow_A)(\gamma\uparrow_B+\delta\downarrow_B), \tag{9}$$

i.e., transform it to

$$\alpha\uparrow_A(\gamma\uparrow_B+\delta\downarrow_B)+\beta\downarrow_A(\gamma\downarrow_B+\delta\uparrow_B). \tag{10}$$

Since the operation behaves linearly, the protocol performs the CNOT operation on any input state (i.e., even if the qubits are entangled with each other or with other systems).

Step 1. The first step is to append a singlet held between Alice and Bob to the state (9):

$$(\alpha\uparrow_A+\beta\downarrow_A)(\uparrow_a\uparrow_b+\downarrow_a\downarrow_b)(\gamma\uparrow_B+\delta\downarrow_B); \tag{11}$$

then Alice measures the absolute value of the total spin of her spins A and a along the z axis. If the absolute value of this total spin along the z axis is 1, then the state becomes

$$(\alpha\uparrow_A\uparrow_a\uparrow_b+\beta\downarrow_A\downarrow_a\downarrow_b)(\gamma\uparrow_B+\delta\downarrow_B). \tag{12}$$

Now Alice disentangles the singlet spin by performing the following (local) operation:

$$\uparrow_A\uparrow_a \mapsto \uparrow_A\uparrow_a, \quad \downarrow_A\downarrow_a \mapsto \downarrow_A\uparrow_a, \tag{13}$$

and the state becomes

$$(\alpha\uparrow_A\uparrow_b+\beta\downarrow_A\downarrow_b)\;(\gamma\uparrow_B+\delta\downarrow_B)\uparrow_a. \tag{14}$$

If the absolute value of the total spin along the z axis is zero, then rather than (12) the state becomes

$$(\alpha\uparrow_A\downarrow_a\downarrow_b+\beta\downarrow_A\uparrow_a\uparrow_b)(\gamma\uparrow_B+\delta\downarrow_B). \tag{15}$$

In this case Alice can disentangle the a spin by

$$\uparrow_A\downarrow_a \mapsto \uparrow_A\uparrow_a, \quad \downarrow_A\uparrow_a \mapsto \downarrow_A\uparrow_a, \tag{16}$$

leading to

$$(\alpha\uparrow_A\downarrow_b+\beta\downarrow_A\uparrow_b)(\gamma\uparrow_B+\delta\downarrow_B)\uparrow_a. \tag{17}$$

In order to get this state in the correct form, Bob needs to invert his b spin. Thus Alice must communicate one bit to Bob to tell him whether she found the absolute value of the total spin 1 or zero, and thus whether he needs to invert his spin or not.

After these operations, the state is

$$(\alpha\uparrow_A\uparrow_b+\beta\downarrow_A\downarrow_b)(\gamma\uparrow_B+\delta\downarrow_B)\uparrow_a. \tag{18}$$

Step 2. Now Bob performs a CNOT operation on the b and B spins; thus the total state is

$$[\alpha\uparrow_A\uparrow_b(\gamma\uparrow_B+\delta\downarrow_B)+\beta\downarrow_A\downarrow_b(\gamma\downarrow_B+\delta\uparrow_B)]\uparrow_a. \tag{19}$$

Step 3. Bob now measures σ_x on his part of the singlet b. The state becomes either

$$[\alpha\uparrow_A(\gamma\uparrow_B+\delta\downarrow_B)+\beta\downarrow_A(\gamma\downarrow_B+\delta\uparrow_B)]\otimes\uparrow_a(\uparrow_b+\downarrow_b) \tag{20}$$

or

$$[\alpha\uparrow_A(\gamma\uparrow_B+\delta\downarrow_B)-\beta\downarrow_A(\gamma\downarrow_B+\delta\uparrow_B)]\otimes\uparrow_a(\uparrow_b-\downarrow_b). \tag{21}$$

In the former case (i.e., the x component of spin was $+$) we have performed the protocol as desired. In the latter, Alice needs to perform a σ_z rotation by π. Thus Bob needs to communicate one bit to Alice to tell her whether or not to perform the rotation.

We have thus shown how to perform a CNOT operation using one singlet and one bit of classical communication each way.

Creating entanglement by CNOT operation. We show now that a CNOT apparatus can be used to create one e-bit between Alice and Bob; thus (since entanglement cannot be increased by local operations) one e-bit is a necessary resource for constructing a CNOT operation.

Creating one e-bit by a CNOT operation is straightforward:

$$(\uparrow_A+\downarrow_A)\uparrow_B \mapsto \uparrow_A\uparrow_B+\downarrow_A\downarrow_B. \tag{22}$$

Classical communication by CNOT operation. Suppose that Alice and Bob have an apparatus that implements a CNOT operation and they also share one e-bit. They can use these resources to communicate *at the same time* one classical bit from Alice to Bob and one classical bit from Bob to Alice. This proves (see the preceding section) that communicating one classical bit each way is a necessary resource for constructing a CNOT apparatus.

Suppose the initial state is

$$\uparrow_a\uparrow_b+\downarrow_a\downarrow_b. \tag{23}$$

Alice can encode a "0" by not doing anything to the state and a "1" by flipping her qubit. Bob can encode a "0" by not doing anything to the state and a "1" by changing the phase as follows: $\uparrow\rightarrow\uparrow$ and $\downarrow\rightarrow-\downarrow$. The four states corresponding to the different bit combinations are thus

$$\uparrow_a\uparrow_b+\downarrow_a\downarrow_b \quad \text{corresponds to} \quad 0_A 0_B, \tag{24}$$

$$\downarrow_a\uparrow_b+\uparrow_a\downarrow_b \quad \text{corresponds to} \quad 1_A 0_B, \tag{25}$$

$$\uparrow_a\uparrow_b-\downarrow_a\downarrow_b \quad \text{corresponds to} \quad 0_A 1_B, \tag{26}$$

$$\downarrow_a\uparrow_b-\uparrow_a\downarrow_b \quad \text{corresponds to} \quad 1_A 1_B, \tag{27}$$

After encoding their bits, Alice and Bob apply the CNOT operation. This results in the corresponding four states:

$$\uparrow_a\uparrow_b+\downarrow_a\uparrow_b=(\uparrow_a+\downarrow_a)\uparrow_b \quad \text{corresponds to} \quad 0_A 0_B, \tag{28}$$

$$\downarrow_a\downarrow_b+\uparrow_a\downarrow_b=(\uparrow_a+\downarrow_a)\downarrow_b \quad \text{corresponds to} \quad 1_A 0_B, \tag{29}$$

$$\uparrow_a\uparrow_b-\downarrow_a\uparrow_b=(\uparrow_a-\downarrow_a)\uparrow_b \quad \text{corresponds to} \quad 0_A 1_B, \tag{30}$$

$$\downarrow_a\downarrow_b-\uparrow_a\downarrow_b=(\downarrow_a-\uparrow_a)\downarrow_b \quad \text{corresponds to} \quad 1_A 1_B. \tag{31}$$

Bob can now find out Alice's bit by measuring his qubit in the $\{\uparrow_b, \downarrow_b\}$ basis while Alice can find out Bob's bit by measuring her qubit in the $\{\uparrow_a+\downarrow_a, \uparrow_a-\downarrow_a\}$ basis.

V. THE DOUBLE CONTROLLED-NOT OPERATION ON TWO QUBITS

One might have thought that the SWAP operation was the *unique* maximally nonlocal operation, at least in the terms used in this paper. We here demonstrate that there is another maximally nonlocal operator, which is the double CNOT or DCNOT gate, formed by performing a CNOT operation from particle 1 onto particle 2, and then a second CNOT operation from particle 2 onto particle 1. It is defined by

$$\uparrow\uparrow\mapsto\uparrow\uparrow, \tag{32}$$

$$\uparrow\downarrow\mapsto\downarrow\downarrow, \tag{33}$$

$$\downarrow\uparrow\mapsto\uparrow\downarrow, \tag{34}$$

$$\downarrow\downarrow\mapsto\downarrow\uparrow. \tag{35}$$

First we note that that the DCNOT operation is genuinely different from the SWAP operation even under local transformations. This is because, although the SWAP operation takes all product states to product states, it is easy to find product states that the DCNOT operation takes to entangled states.

To show that the DCNOT operation is maximally nonlocal, we shall first demonstrate that it can be used to create two e-bits. We shall then show that it can be used to communicate two bits of information from Alice to Bob, and simultaneously to send two bits from Bob to Alice. The argument used for the SWAP operation then proves that to build a DCNOT operation we need two e-bits plus two bits of classical communication from Alice to Bob plus two bits of classical communication from Bob to Alice. Since any transformation on two qubits can be performed using these resources via teleportation, we will then have shown that the DCNOT operation is maximally nonlocal, in terms of resources.

Creating two e-bits is easy. Alice and Bob prepare singlets locally, and then perform the DCNOT operation on spins A and B:

$$(\uparrow_A\uparrow_a+\downarrow_A\downarrow_a)(\uparrow_B\uparrow_b+\downarrow_B\downarrow_b)\mapsto\uparrow_A\uparrow_a\uparrow_B\uparrow_b+\downarrow_A\uparrow_a\downarrow_B\downarrow_b+\uparrow_A\downarrow_a\downarrow_B\uparrow_b+\downarrow_A\downarrow_a\uparrow_B\downarrow_b. \tag{36}$$

We now have a Schmidt decomposition of rank 4, i.e., a two party state that is locally equivalent to two e-bits transmitted between Alice and Bob.

Transmitting two bits of information in both directions at the same time is a little more tricky. Alice and Bob need to have two e-bits in addition to the DCNOT operation. They first transform their e-bits (locally) into the state

$$\uparrow_A\uparrow_a\uparrow_B\uparrow_b+\downarrow_A\uparrow_a\uparrow_B\downarrow_b+\downarrow_A\downarrow_a\downarrow_B\uparrow_b+\uparrow_A\downarrow_a\downarrow_B\downarrow_b. \tag{37}$$

Alice now encodes one bit of information in the state by either applying or not applying $\sigma_z\otimes\sigma_z$ to her two spins. She encodes a second bit of information by applying or not applying σ_x to her first spin A. Bob similarly encodes two bits of information, using the transformation σ_z on spin B to encode his first bit, and $\sigma_x\otimes\sigma_x$ to encode his second bit.

Having encoded the information, they make it locally accessible by applying the DCNOT operation to spins A and B. It is not obvious, but simple to check, that Alice and Bob now each have one of the four Bell states locally, and that Alice's particular state corresponds to Bob's encoded bits, and vice versa.

VI. MULTIPARTITE OPERATIONS

In the previous sections we studied different bipartite operations. What about multipartite operations, such as the Toffoli or Fredkin gates on three qubits? As we showed in Sec. II, they can all be implemented by using the ''double teleportation'' method. On the other hand, finding the necessary resources is far more difficult than in the bipartite case; indeed it is not possible at present. The reason is that there exist different inequivalent types of multipartite entanglement [4,5]. For example, it is known that singlets and GHZ states are inequivalent in the sense that they cannot be reversibly transformed into each other, not even in the asymptotic limit. Although GHZ states (like all other entangled states) can be built out of singlets, such a procedure is wasteful. Hence, when investigating the minimal entanglement resources needed to implement multipartite quantum operations, we have to use the different inequivalent types of entanglement. Unfortunately, at present multipartite entanglement is far from being fully understood.

A further issue is that certain multipartite operations can be performed with different nonlocal resources. For example, the controlled controlled NOT gate (the Toffoli gate) acting on three parties can be performed using two singlets between the three parties in three different configurations, one from A to C and another from B to C; it can also be performed using two singlets in a different configuration, one from A to C and one from A to B; or with singlets from A to B and from B to C. To see this, simply note that the controlled spin can be either A, B, or C depending on which basis the gate is described in. The Toffoli gate is, however, impossible to produce with a single singlet.

VII. ''CONSERVATION'' RELATIONS

In studying the nonlocality of quantum states a most important issue is that of ''manipulating'' entanglement, i.e., of transforming some states into others [6]. Similarly we can ask: Given a unitary evolution, can we use it to implement some other unitary evolution?

In particular, for pure quantum states we have *conservation* relations [6,7]. For example, when Alice and Bob share a large number n of pairs of particles, each pair in the same state Ψ, they could use these pairs to generate some other number k of pairs in some other state Φ. In the limit of large n, this transformation can be performed reversibly, meaning that the total amount of nonlocality contained in the n copies of the state Ψ is the same as the total amount of nonlocality contained in the k copies of the state Φ. Is something similar taking place for unitary transformations?

For unitary transformations we have not yet studied the case of the asymptotic limit, i.e., performing the same transformation U on many pairs of particles. However, an interesting pattern emerges even at the level of a single copy.

Consider first the case of the SWAP operation. We know what the minimal resources needed to implement a SWAP operation are. But suppose now that we are given a device that implements a SWAP operation. Could we could use it to get back the original resources needed to create the SWAP device?

The balance of resources needed to implement a SWAP apparatus can be written as

$$2\ e\text{-bits}\ +\ 2\ \text{bits}_{A\rightarrow B}\ +\ 2\ \text{bits}_{B\rightarrow A}\Rightarrow \text{SWAP}. \tag{38}$$

The question is whether

$$\text{SWAP}\Rightarrow 2\ \text{e-bits}\ +\ 2\ \text{bits}_{A\rightarrow B}\ +\ 2\ \text{bits}_{B\rightarrow A}? \tag{39}$$

The answer is ''No.'' That is, combining entanglement and classical communication resources to yield a SWAP operation is an irreversible process—we cannot use the SWAP operation to get the resources back.

To work out what we can produce from a SWAP operation, we consider it to be composed of the two operations of Alice sending a qubit to Bob and Bob sending a qubit to Alice. We know that we cannot recover more than one e-bit from sending a qubit, and so to regain all the resources needed to implement a SWAP operation we need to use the qubit to create one e-bit, and also to send some classical communication. Now, suppose we can use sending a qubit to do the following (we allow for catalysis by z e-bits):

$$1 \text{ qubit} + z \text{ e-bits} \Rightarrow x \text{ bits}_{A\rightarrow B} + (1+z) \text{ e-bits}. \tag{40}$$

Then, using superdense coding, we could do the following:

$$z \text{ e-bits} + (1+1+z) \text{ qubits} \Rightarrow [x+2(1+z)] \text{ bits}_{A\rightarrow B}. \tag{41}$$

Now the final mutual information between Alice and Bob is at most $z+(2+z)$: the initial entanglement between them (contributing z) plus the $(2+z)$ transmitted qubits. By Holevo's theorem, the mutual information is an upper bound for the number of classical bits that can be transmitted. Thus

$$2z+2+x \leqslant z+(2+z), \tag{42}$$

and so

$$x \leqslant 0. \tag{43}$$

Thus if we use the SWAP operation to produce two e-bits, we cannot use it to send any classical communication.

On the other hand, looking back to the proof of the resources needed for the SWAP operation, we see that we can write the following tight "implications:"

$$2 \text{ e-bits} + 2 \text{ bits}_{A\rightarrow B} + 2 \text{ bits}_{B\rightarrow A} \Rightarrow 1 \text{ SWAP}, \tag{44}$$

$$2 \text{ e-bits} + 1 \text{ SWAP} \Rightarrow 2 \text{ bits}_{A\rightarrow B} + 2 \text{ bits}_{B\rightarrow A}, \tag{45}$$

$$1 \text{ SWAP} \Rightarrow 2 \text{ e-bits}. \tag{46}$$

The first of these three implications is to be read as "given two e-bits and two bits$_{A\rightarrow B}$ and two bits$_{A\rightarrow B}$ we can produce the SWAP operation; also, if we wish to produce the SWAP operation with e-bits and bits communicated from Alice to Bob and vice versa, we cannot do so with fewer than two e-bits and two bits$_{A\rightarrow B}$ and 2 bits$_{A\rightarrow B}$."

The second and third implications have a slightly different meaning. For example, we read the second implication as "given one SWAP operation and two e-bits, we can communicate four classical bits (two each way); also, we cannot communicate more than four classical bits (two each way)." On the other hand, it does not mean that "one SWAP operation and two e-bits are necessary for communicating four classical bits (two each way)"—for example, we can implement this classical communication with two SWAP operations.

Exactly the same implications apply for the DCNOT operation:

$$2 \text{ e-bits} + 2 \text{ bits}_{A\rightarrow B} + 2 \text{ bits}_{B\rightarrow A} \Rightarrow 1 \text{ DCNOT}, \tag{47}$$

$$2 \text{ e-bits} + 1 \text{ DCNOT} \Rightarrow 2 \text{ bits}_{A\rightarrow B} + 2 \text{ bits}_{B\rightarrow A}, \tag{48}$$

$$1 \text{ DCNOT} \Rightarrow 2 \text{ e-bits}. \tag{49}$$

Furthermore, very similar implications can be written for the CNOT operation:

$$1 \text{ e-bit} + 1 \text{ bit}_{A\rightarrow B} + 1 \text{ bit}_{B\rightarrow A} \Rightarrow 1 \text{ CNOT}, \tag{50}$$

$$1 \text{ e-bit} + 1 \text{ CNOT} \Rightarrow 1 \text{ bit}_{A\rightarrow B} + 1 \text{ bit}_{B\rightarrow A}, \tag{51}$$

$$1 \text{ CNOT} \Rightarrow 1 \text{ e-bit}. \tag{52}$$

In fact these implications are very similar to the implications that describe teleportation and superdense coding, which appear, together with many other similar implications on Bennett's famous transparency presented at almost all early quantum information conferences (see also [8]) as follows:

$$1 \text{ e-bit} + 2 \text{ bits}_{A\rightarrow B} \Rightarrow 1 \text{ qubit}, \tag{53}$$

$$1 \text{ e-bit} + 1 \text{ qubit} \Rightarrow 2 \text{ bits}_{A\rightarrow B}, \tag{54}$$

$$1 \text{ qubit} \Rightarrow 1 \text{ e-bit}. \tag{55}$$

The above three implications (53), (54), and (55) are generally thought to describe relations between classical information, quantum information, and entanglement. However, we would like to argue that their true meaning may be more closely related to dynamics, and that a more illuminating form is probably

$$1 \text{ e-bit} + 2 \text{ bits}_{A\rightarrow B} \Rightarrow 1 \text{ teleportation}_{A\rightarrow B}, \tag{56}$$

$$1 \text{ e-bit} + 1 \text{ teleportation}_{A\rightarrow B} \Rightarrow 2 \text{ bits}_{A\rightarrow B}, \tag{57}$$

$$1 \text{ teleportation}_{A\rightarrow B} \Rightarrow 1 \text{ e-bit}. \tag{58}$$

We conjecture that similar relations hold between any quantum action and the resources needed to implement it, that is,

$$\text{entanglement} + \text{classical communication} \Rightarrow \text{action}, \tag{59}$$

$$\text{entanglement} + \text{action} \Rightarrow \text{classical communication}, \tag{60}$$

$$\text{action} \Rightarrow \text{entanglement}. \tag{61}$$

It may be that these relations hold, in general, only in the asymptotic limit of many copies of the quantum action.

VIII. DIFFERENT WAYS OF ACHIEVING THE SAME TASK

It is interesting to note that, although the transformation from resources to unitary actions is irreversible, sometimes the same end product can be achieved in two different ways. For example, there are two alternative ways to implement

$$2 \text{ CNOT operations} \Rightarrow 1 \text{ bit}_{A\rightarrow B} + 1 \text{ bit}_{B\rightarrow A}. \tag{62}$$

The first way is to use one CNOT operation to transmit one classical bit from Alice to Bob and the other CNOT operation to transmit one classical bit from Bob to Alice, i.e.,

$$1 \text{ CNOT} \Rightarrow 1 \text{ bit}_{A\rightarrow B} \tag{63}$$

and

$$1 \text{ CNOT} \Rightarrow 1 \text{ bit}_{B\rightarrow A}. \tag{64}$$

Another possibility is to use first one CNOT operation to create one e-bit (52) and then the other CNOT operation plus the e-bit to transmit the two classical bits (51), i.e.,

$$2 \text{ CNOT operations} \Rightarrow 1 \text{ e-bit} + 1 \text{ CNOT}$$

$$\Rightarrow 1 \text{ bit}_{A\rightarrow B} + 1 \text{ bit}_{B\rightarrow A}. \tag{65}$$

IX. CATALYZING CLASSICAL COMMUNICATION

A very interesting phenomenon is that of ‘‘catalyzing’’ classical communication. This phenomenon is similar in its spirit to that of ‘‘catalyzing entanglement manipulation’’ [9,4]. An example is the following.

On its own, the SWAP operation can only send one bit in each direction at the same time, and cannot be used for Alice to send two bits to Bob, even if Bob sends no information whatsoever. That is,

$$1 \text{ SWAP} \not\Rightarrow 2 \text{ bits}_{A\rightarrow B}. \tag{66}$$

However, if Alice and Bob share one e-bit, Alice can send two bits to Bob *without destroying* the e-bit, i.e.,

$$1 \text{ SWAP} + 1 \text{ e-bit} \Rightarrow 2 \text{ bits}_{A\rightarrow B} + 1 \text{ e-bit}. \tag{67}$$

This may be done as follows. Initially Alice and Bob share a nonlocal singlet; Bob also prepares a second singlet locally. Alice encodes the two bits she wishes to send to Bob by performing one of the four rotations 1, σ_x, σ_y, σ_z on her half of the nonlocal singlet. By performing the SWAP operation on Alice's particle from the nonlocal singlet and one particle of the singlet that Bob has prepared locally, Alice and Bob end up with a nonlocal singlet held between them; also Bob can find out the two bits by measurements on the local singlet he now holds. Specifically, we begin with the state

$$(\uparrow_A\uparrow_{b1}+\downarrow_A\downarrow_{b1})(\uparrow_B\uparrow_{b2}+\downarrow_B\downarrow_{b2}), \tag{68}$$

where A is Alice's particle, and B, $b1$, and $b2$ are Bob's particles. Alice performs one of the rotations 1, σ_x, σ_y, σ_z on her particle. They then perform the SWAP operation on particles A and B, and get (if Alice performed 1)

$$(\uparrow_B\uparrow_{b1}+\downarrow_B\downarrow_{b1})(\uparrow_A\uparrow_{b2}+\downarrow_A\downarrow_{b2}). \tag{69}$$

If Alice performed one of the other rotations, Bob will get one of the other Bell states in system $(B, b1)$. Bob now measures that system in the Bell basis to extract the information, and Alice and Bob are left with a singlet between systems A and $b2$.

In effect the SWAP operation acts as a double teleportation; one from Alice to Bob and one from Bob to Alice. Teleporting Alice's qubit, in conjunction with the e-bit, implements a transmission of two bits from Alice to Bob using superdense coding; it destroys the e-bit in the process. Simultaneously, the Bob to Alice teleportation restores the e-bit.

X. TRADING ONE TYPE OF ACTION FOR ANOTHER

An interesting question is the following. There are cases in which two different actions require the same resources. For example the resources needed for one SWAP operation are the same as for two CNOT operations, i.e., 2 e-bits $+2$ bits$_{A\rightarrow B}+2$ bits$_{B\rightarrow A}$. Now, suppose we had already used the resources to build two CNOT operations, but we wanted to change our mind and do one SWAP operation instead. Due to the irreversibility discussed above, we cannot simply get back the original resources and use them to construct the SWAP operation. Is it possible, however, to go *directly* from two CNOT operations to one SWAP operation, without going back to the original resources? As far as we are aware, the answer is ‘‘No.’’

It turns out, however, that if we have many CNOT operation it is nevertheless useful to build a SWAP apparatus from CNOT operations directly rather than going back to the original resources. Indeed, to obtain the entanglement and classical communication resources needed for one SWAP operation, i.e., 2 e-bits$+2$ bits$_{A\rightarrow B}+2$ bits$_{B\rightarrow A}$, we need four CNOT operations. However, it is well known that one can construct one SWAP operation directly from three CNOT operations. Indeed, we do not even need three CNOT operations, but can realize a SWAP operation by

$$2 \text{ CNOT operations} + 1 \text{ bit}_{A\rightarrow B} + 1 \text{ bit}_{B\rightarrow A} \Rightarrow 1 \text{ SWAP}, \tag{70}$$

which uses fewer nonlocal resources than three CNOT operations. To see this, it suffices to note that

$$1 \text{ CNOT} + 1 \text{ bit}_{A\rightarrow B} \Rightarrow 1 \text{ teleportation}_{A\rightarrow B} \tag{71}$$

and similarly

$$1 \text{ CNOT} + 1 \text{ bit}_{B\rightarrow A} \Rightarrow 1 \text{ teleportation}_{B\rightarrow A}. \tag{72}$$

To implement (71) Alice starts with her qubit in the state $\Psi=\alpha\uparrow+\beta\downarrow$ which has to be teleported and Bob with his qubit in the state $\uparrow$. After the CNOT operation the state becomes

$$\Psi\uparrow=(\alpha\uparrow+\beta\downarrow)\uparrow\mapsto\alpha\uparrow\uparrow+\beta\downarrow\downarrow. \tag{73}$$

Alice then measures her qubit in the $|+\rangle=(1/\sqrt{2})(\uparrow+\downarrow)$ and $|-\rangle=(1/\sqrt{2})(\uparrow-\downarrow)$ basis and communicates the result to Bob. If $(+)$ then Bob's qubit is already in the required state $\Psi=\alpha\uparrow+\beta\downarrow$; if $(-)$ then Bob's qubit is in the state $\Psi' = \alpha\uparrow - \beta\downarrow$ and Bob can obtain Ψ by changing the relative phase between $\uparrow$ and $\downarrow$ by π.

While completing this work, we became aware of closely related work by Eisert, Jacobs, Papadopoulos, and Plenio [10]. Also we became aware of [11] in which a protocol for creating a CNOT gate using a singlet was presented that is similar to that given in Sec. IV; however, its optimality was not discussed.

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