Coherent states for the Kepler motion. II

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The coherent states for the quantum Kepler motion proposed in our previous work [Phys. Rev. A **59**, 1021 (1999)], which is based on the dynamical $SU(2)\otimes SU(2)$ symmetry and the Duru-Kleinert auxiliary time, are improved by making use of the theory of the conserved-charge coherent states formulated by Bhaumik *et al.* and by Skagertsam. The expectation values for the angular momentum and the Runge-Lenz-Pauli vector with respect to the improved Kepler coherent states are also discussed.

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I. INTRODUCTION

In our previous paper [1], hereinafter referred to as I, we proposed coherent states for the quantized Kepler motion on the basis of the quantum harmonic oscillator operators corresponding to the dynamical $SU(2)\otimes SU(2)$ symmetry derived by Ravndal and Toyoda [2] as well as the auxiliary time variable introduced by Duru and Kleinert [3], who formulated the quantum Kepler problem in terms of the path integral. In this paper we improve the previously proposed coherent states by making use of the formulation of the conserved-charge coherent states developed by Bhaumik *et al.* [4] and by Skagertsam [5–7]. The conserved-charge coherent states can be lucidly illustrated by considering two one-dimensional harmonic oscillators whose boson operators are a, $a^\dagger$ and b, $b^\dagger$. Their nonvanishing commutators are $[a,a^\dagger]=[b,b^\dagger]=1$. Then the conserved-charge coherent states can be defined as [4,5,8],

$$|z;m\rangle = C\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(n+m)!}\sqrt{n!}}|n+m,n\rangle, \tag{1.1}$$

where C is a normalization factor, z is an arbitrary complex number, m is a fixed non-negative integer, and $|m,n\rangle$ are the eigenkets of the operators $a^\dagger a$ and $b^\dagger b$, i.e., $a^\dagger a|m,n\rangle = m|m,n\rangle$ and $b^\dagger b|m,n\rangle = n|m,n\rangle$. The physical significance of the conserved-charge coherent states is that the states are eigenstates of the Abelian charge $Q=a^\dagger a - b^\dagger b$. By making a superposition of the canonical coherent states

$$|z_1,z_2\rangle = \exp[z_1 a^\dagger - z_1^* a + z_2 b^\dagger - z_2^* b]|0\rangle, \tag{1.2}$$

the conserved-charge coherent states (1.1) can also be constructed as [8]

$$|z;m\rangle = C_1 z_1^{-m}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{im\theta}|e^{-i\theta}z_1, e^{i\theta}z_2\rangle, \tag{1.3}$$

where $z=z_1 z_2$ and C_1 is a normalization factor depending on z_1 and z_2. The above described conserved-charge coherent states are very similar to the Kepler coherent states proposed in I. In this paper we shall show that the basic idea of the conserved-charge coherent states can be applied to the formulation presented in I and leads to improved Kepler coherent states.

II. HARMONIC OSCILLATORS

In order to separate the Schrödinger equation for the Kepler motion and to find the $SU(2)\otimes SU(2)$ dynamical symmetry [2] we first introduce the squared parabolic coordinates μ, ν, and ϕ defined by

$$x=\mu\nu\cos\phi, \quad y=\mu\nu\sin\phi, \quad z=\frac{1}{2}(\mu^2-\nu^2). \tag{2.1}$$

Then, the angle variable ϕ is split into two independent angle variables ϕ_μ and ϕ_ν. This procedure introduces an unphysical degree of freedom to the system. With this additional degree of freedom the Schrödinger equation for the Kepler motion can be written as [1]

$$\begin{aligned} i\hbar\frac{\partial}{\partial\tau}\Psi(\mu,\nu,\phi_\mu,\phi_\nu,\tau) &= \Bigg[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\mu^2}+\frac{1}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{\mu^2}\frac{\partial^2}{\partial\phi_\mu^2}-\frac{2m}{\hbar^2}\mu^2 E_0\right) \\ &\quad + \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\nu^2}+\frac{1}{\nu}\frac{\partial}{\partial\nu}+\frac{1}{\nu^2}\frac{\partial^2}{\partial\phi_\nu^2}-\frac{2m}{\hbar^2}\nu^2 E_0\right)-2e^2\Bigg] \\ &\quad \times\Psi(\mu,\nu,\phi_\mu,\phi_\nu,\tau), \end{aligned} \tag{2.2}$$

where the Duru-Kleinert auxiliary time variable [3]

$$\tau = \frac{t}{\mu^2+\nu^2} \tag{2.3}$$

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has been introduced. Because the Schrödinger equation (2.2) contains an extra degree of freedom due to the angle variables ϕ_μ and ϕ_ν, the condition

$$\frac{\partial}{\partial\phi_\mu}\Psi(\mu,\nu,\phi_\mu,\phi_\nu,\tau)=\frac{\partial}{\partial\phi_\nu}\Psi(\mu,\nu,\phi_\mu,\phi_\nu,\tau) \tag{2.4}$$

must be imposed on physical states.

The Schrödinger equation (2.2) can be mapped on to a dynamical system of four one-dimensional quantum harmonic oscillators, whose operators are defined as

$$A_\pm=\frac{1}{2\sqrt{m\hbar\omega}}\left[-i\hbar\left(\frac{\partial}{\partial\xi_\mu}\pm i\frac{\partial}{\partial\eta_\mu}\right)-im\omega(\xi_\mu\pm i\eta_\mu)\right], \tag{2.5}$$

$$A_\pm^\dagger=\frac{1}{2\sqrt{m\hbar\omega}}\left[-i\hbar\left(\frac{\partial}{\partial\xi_\mu}\mp i\frac{\partial}{\partial\eta_\mu}\right)+im\omega(\xi_\mu\mp i\eta_\mu)\right], \tag{2.6}$$

$$B_\pm=\frac{1}{2\sqrt{m\hbar\omega}}\left[-i\hbar\left(\frac{\partial}{\partial\xi_\nu}\pm i\frac{\partial}{\partial\eta_\nu}\right)-im\omega(\xi_\nu\pm i\eta_\nu)\right], \tag{2.7}$$

and

$$B_\pm^\dagger=\frac{1}{2\sqrt{m\hbar\omega}}\left[-i\hbar\left(\frac{\partial}{\partial\xi_\nu}\mp i\frac{\partial}{\partial\eta_\nu}\right)+im\omega(\xi_\nu\mp i\eta_\nu)\right]. \tag{2.8}$$

These operators satisfy the following commutation relations:

$$[A_+,A_+^\dagger]=[A_-,A_-^\dagger]=[B_+,B_+^\dagger]=[B_-,B_-^\dagger]=1. \tag{2.9}$$

The four harmonic oscillators are not independent due to the condition (2.4). The physical states must satisfy

$$(A_+^\dagger A_+-A_-^\dagger A_--B_+^\dagger B_++B_-^\dagger B_-)|\mathrm{physical}\rangle=0. \tag{2.10}$$

If we define the basic eigenkets for the system of four harmonic oscillators such that

$$A_\pm^\dagger A_\pm|m_+,m_-,n_+,n_-\rangle=|m_+,m_-,n_+,n_-\rangle m_\pm, \tag{2.11}$$

$$B_\pm^\dagger B_\pm|m_+,m_-,n_+,n_-\rangle=|m_+,m_-,n_+,n_-\rangle n_\pm, \tag{2.12}$$

then the condition (2.10) can be expressed as $n_+=m_++n_--m_-$. Therefore the physical sector of the Hilbert space is spanned by $|m_+,m_-,m_++n_--m_-,n_-\rangle$.

III. IMPROVED KEPLER COHERENT STATES

Canonical coherent states for the four one-dimensional harmonic oscillators defined in the previous section can be straightforwardly constructed by introducing the Weyl operator

$$\begin{aligned}W(\alpha_+,\alpha_-,\beta_+,\beta_-)&=\exp(\alpha_+A_+^\dagger-\alpha_+^*A_++\alpha_-A_-^\dagger-\alpha_-^*A_-+\beta_+B_+^\dagger\\&\quad-\beta_+^*B_++\beta_-B_-^\dagger-\beta_-^*B_-)\end{aligned} \tag{3.1}$$

where $\alpha_\pm$ and $\beta_\pm$ are arbitrary complex numbers. Imposing this Weyl operator on the ground state of the harmonic oscillators $|0,0,0,0\rangle$, we obtain

$$\begin{aligned}W(\alpha_+,\alpha_-,\beta_+,\beta_-)|0,0,0,0\rangle&=N_0\sum_{m_+=0}^\infty\sum_{m_-=0}^\infty\sum_{n_+=0}^\infty\sum_{n_-=0}^\infty\frac{\alpha_+^{m_+}}{\sqrt{m_+!}}\frac{\alpha_-^{m_-}}{\sqrt{m_-!}}\frac{\beta_+^{n_+}}{\sqrt{n_+!}}\frac{\beta_-^{n_-}}{\sqrt{n_-!}}\\&\quad\times|m_+,m_-,n_+,n_-\rangle\\&\equiv|\Psi_C(\alpha_+,\alpha_-,\beta_+,\beta_-)\rangle,\end{aligned} \tag{3.2}$$

where N_0 is the normalization constant,

$$\begin{aligned}N_0&=N_0(\alpha_+,\alpha_-,\beta_+,\beta_-)\\&=\exp\left[-\frac{1}{2}(|\alpha_+|^2+|\alpha_-|^2+|\beta_+|^2+|\beta_-|^2)\right].\end{aligned} \tag{3.3}$$

However, this canonical coherent state does not satisfy the physical state condition given by Eq. (2.10). We now apply the general formulation of the conserved-charge coherent states shown in Eqs. (1.1)–(1.3) to modify the above canonical coherent states so that the physical state condition (2.10) can be satisfied. Constructing a linear superposition of the canonical coherent states similar to Eq. (1.3) we can make coherent states that satisfy the physical state condition,

$$\begin{aligned}&N_1\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}|\Psi_C(e^{-i\theta}\alpha_+,e^{i\theta}\alpha_-,e^{i\theta}\beta_+,e^{-i\theta}\beta_-)\rangle\\&=N_1N_0\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\sum_{m_+=0}^\infty\sum_{m_-=0}^\infty\sum_{n_+=0}^\infty\sum_{n_-=0}^\infty\\&\quad\times e^{-i(m_+-m_--n_++n_-)\theta}\frac{\alpha_+^{m_+}}{\sqrt{m_+!}}\frac{\alpha_-^{m_-}}{\sqrt{m_-!}}\frac{\beta_+^{n_+}}{\sqrt{n_+!}}\frac{\beta_-^{n_-}}{\sqrt{n_-!}}\\&\quad\times|m_+,m_-,n_+,n_-\rangle\\&=N_1N_0\sum_{m_+=0}^\infty\sum_{m_-=0}^\infty\sum_{n_-=0}^\infty\frac{\alpha_+^{m_+}}{\sqrt{m_+!}}\frac{\alpha_-^{m_-}}{\sqrt{m_-!}}\\&\quad\times\frac{\beta_+^{(m_++n_--m_-)}}{\sqrt{(m_++n_--m_-)!}}\frac{\beta_-^{n_-}}{\sqrt{n_-!}}\\&\quad\times|m_+,m_-,m_++n_--m_-,n_-\rangle,\end{aligned} \tag{3.4}$$

where N_1 is a normalization constant to be determined later. To avoid redundancy of parameters, let us define

$$\alpha_+\beta_+=\alpha,\quad \beta_+\beta_-=\beta,\quad \alpha_-\beta_+^{-1}=\gamma,\quad N_1N_0=N. \tag{3.5}$$

Then Kepler coherent states can be defined such that

$$|\Psi(\alpha,\beta;\gamma)\rangle=N\sum_{m_+=0}^{\infty}\sum_{n_-=0}^{\infty}\sum_{m_-=0}^{\infty}\frac{1}{\sqrt{(m_++n_--m_-)!}}\times\frac{\alpha^{m_+}}{\sqrt{m_+!}}\frac{\beta^{n_-}}{\sqrt{n_-!}}\frac{\gamma^{m_-}}{\sqrt{m_-!}}\times|m_+,m_-,m_++n_--m_-,n_-\rangle. \tag{3.6}$$

The normalization factor can be straightforwardly calculated:

$$N^{-2}=\sum_{m_+=0}^{\infty}\sum_{n_-=0}^{\infty}\sum_{m_-=0}^{\infty}\frac{1}{(m_++n_--m_-)!}\times\frac{|\alpha|^{2m_+}}{m_+!}\frac{|\beta|^{2n_-}}{n_-!}\frac{|\gamma|^{2m_-}}{m_-!}=I_0(\kappa), \tag{3.7}$$

where I_0 is the modified Bessel function of the first kind and κ is defined as

$$\kappa=2\sqrt{(|\alpha|^2+|\beta|^2)(|\gamma|^2+1)}. \tag{3.8}$$

Equation (3.6) defines the improved Kepler coherent states. The differences between the improved Kepler coherent states and the coherent states introduced in I are the extra factor $[(m_++n_--m_-)!]^{-1/2}$ and the normalization factor. Clearly the improved Kepler coherent states satisfy the physical state condition (2.10). It should be remarked that the improved coherent states are the eigenkets of A_-A_+ and of B_-B_+,

$$A_-A_+|\Psi(\alpha,\beta;\gamma)\rangle=|\Psi(\alpha,\beta;\gamma)\rangle\alpha\gamma \tag{3.9}$$

and

$$B_-B_+|\Psi(\alpha,\beta;\gamma)\rangle=|\Psi(\alpha,\beta;\gamma)\rangle\beta. \tag{3.10}$$

These operators actually belong to the Lie algebra of SU(2). If we define

$$E_+=\frac{i}{\sqrt{2}}A_+^\dagger A_-^\dagger,\quad E_-=\frac{i}{\sqrt{2}}A_-A_+,$$

$$E_0=\frac{1}{2}(A_+^\dagger A_++A_-^\dagger A_-+1) \tag{3.11}$$

and

$$G_+=\frac{i}{\sqrt{2}}B_+^\dagger B_-^\dagger,\quad G_-=\frac{i}{\sqrt{2}}B_-B_+,$$

$$G_0=\frac{1}{2}(B_+^\dagger B_++B_-^\dagger B_-+1), \tag{3.12}$$

then these operators satisfy

$$[E_+,E_-]=E_0,\quad [E_0,E_\pm]=\pm E_\pm \tag{3.13}$$

and

$$[G_+,G_-]=G_0,\quad [G_0,G_\pm]=\pm G_\pm. \tag{3.14}$$

These equations (3.13) and (3.14) show that each set of operators $\{E_\pm,E_0\}$ and $\{G_\pm,G_0\}$ forms the Lie algebra of SU(2). They commute with each other:

$$[E_\pm,G_\pm]=[E_\pm,G_0]=[G_\pm,E_0]=[E_0,G_0]=0. \tag{3.15}$$

Therefore, it can be stated that the improved Kepler coherent state defined by Eq. (3.6) are associated with the Lie algebra of $SU(2)\otimes SU(2)$ in the sense of Ref. [9]. The time evolution of the improved Kepler coherent states is almost equivalent to that of the Kepler coherent states discussed in I. Using the time evolution operator defined in I, we obtain

$$\begin{aligned}\exp\left[-\frac{i}{\hbar}\tau F\right]|\Psi(\alpha,\beta;\gamma)\rangle&=\exp\left[-\frac{i}{\hbar}\tau\{\hbar\omega(A_+^\dagger A_++A_-^\dagger A_-+B_+^\dagger B_++B_-^\dagger B_-+2)-2e^2\}\right]|\Psi(\alpha,\beta;\gamma)\rangle\\&=\exp\left[-\frac{i}{\hbar}\tau\{2\hbar\omega(A_+^\dagger A_++B_-^\dagger B_-+1)-2e^2\}\right]\times|\Psi(\alpha,\beta;\gamma)\rangle.\end{aligned} \tag{3.16}$$

Following the same procedure given in I, it can be readily shown that the improved Kepler coherent states are labeled by two complex numbers which have the time dependence

$$\alpha(\tau)=\alpha e^{-i2\omega\tau} \tag{3.17}$$

and

$$\beta(\tau)=\beta e^{-i2\omega\tau}. \tag{3.18}$$

This result shows clearly that the coherent state defined by Eq. (3.6) does not change its shape during time evolution with respect to the Duru-Kleinert auxiliary time variable.

IV. ANGULAR MOMENTUM AND RUNGE-LENZ-PAULI VECTORS

The complex eigenvalues of the improved Kepler coherent states, α, β, and γ, are related to the physical quantities that characterize the classical Kepler motion. Among those quantities here we consider the angular momentum and the Runge-Lenz-Pauli vectors, which are both constants of motion. Their symmetry properties are closely connected to the harmonic oscillators defined in Eqs. (2.5)–(2.8) [2]. In fact the angular momentum and the Runge-Lenz-Pauli vectors can be explicitly expressed in terms of the harmonic oscillator operators [2]. If we put $\hbar=1$, then angular momentum operators L_x, L_y, and L_z can be expressed as

$$L_x=\frac{1}{2}\{A_+^\dagger B_- + A_+ B_-^\dagger - B_+^\dagger A_- - B_+ A_-^\dagger\}, \tag{4.1}$$

$$L_y=\frac{1}{2i}\{A_+^\dagger B_- - A_+ B_-^\dagger - B_+^\dagger A_- + B_+ A_-^\dagger\}, \tag{4.2}$$

$$L_z=\frac{1}{2}\{A_+^\dagger A_+ - B_-^\dagger B_- + B_+^\dagger B_+ - A_-^\dagger A_-\}, \tag{4.3}$$

and the Runge-Lenz-Pauli vectors A_x, A_y, and A_z can be expressed as

$$A_x=\frac{1}{2}\{A_+^\dagger B_- + A_+ B_-^\dagger + B_+^\dagger A_- + B_+ A_-^\dagger\}, \tag{4.4}$$

$$A_y=\frac{1}{2i}\{A_+^\dagger B_- - A_+ B_-^\dagger + B_+^\dagger A_- - B_+ A_-^\dagger\}, \tag{4.5}$$

$$A_z=\frac{1}{2}\{A_+^\dagger A_+ - B_-^\dagger B_- - B_+^\dagger B_+ + A_-^\dagger A_-\}. \tag{4.6}$$

The vector $\mathbf{A}=(A_x,A_y,A_z)$ corresponds to the Runge-Lenz-Pauli vector [2]

$$\mathbf{A}=\sqrt{\frac{m}{-2E}}\left\{\frac{1}{2m}(\mathbf{L}\times\mathbf{p}-\mathbf{p}\times\mathbf{L})+\frac{e^2\mathbf{r}}{r}\right\}. \tag{4.7}$$

Let us now calculate the expectation values of the z components of these vectors with respect to the improved Kepler coherent states. We have obtained

$$\langle\Psi(\alpha,\beta;\gamma)|A_z|\Psi(\alpha,\beta;\gamma)\rangle=\frac{2I_1(\kappa)}{\kappa I_0(\kappa)}(|\alpha|^2|\gamma|^2-|\beta|^2) \tag{4.8}$$

and

$$\langle\Psi(\alpha,\beta;\gamma)|L_z|\Psi(\alpha,\beta;\gamma)\rangle=\frac{2I_1(\kappa)}{\kappa I_0(\kappa)}(|\alpha|^2-|\beta|^2|\gamma|^2), \tag{4.9}$$

where I_1 and I_0 are the modified Bessel functions of the first kind and κ has been defined by Eq. (3.8). The calculations are given in the Appendix. If we define

$$M\equiv\langle\Psi(\alpha,\beta;\gamma)|F_0|\Psi(\alpha,\beta;\gamma)\rangle=\frac{\kappa I_1(\kappa)}{I_0(\kappa)} \tag{4.10}$$

with

$$F_0=A_+^\dagger A_+ + A_-^\dagger A_- + B_+^\dagger B_+ + B_-^\dagger B_-, \tag{4.11}$$

then the expectation values can be written as

$$\langle\Psi(\alpha,\beta;\gamma)|L_z|\Psi(\alpha,\beta;\gamma)\rangle=\frac{M}{2}\sin(\theta+\phi)\sin(\theta-\phi) \tag{4.12}$$

and

$$\langle\Psi(\alpha,\beta;\gamma)|A_z|\Psi(\alpha,\beta;\gamma)\rangle=\frac{M}{2}\cos(\phi-\theta)\cos(\phi+\theta), \tag{4.13}$$

where the angle parameters ϕ and θ are defined as

$$\tan\phi=\frac{|\beta|}{|\alpha|},\quad \tan\theta=\frac{1}{|\gamma|}. \tag{4.14}$$

If $\langle A_z\rangle=0$, then the expectation value $\langle L_z\rangle$ has the simple form

$$\langle\Psi(\alpha,\beta;\gamma)|L_z|\Psi(\alpha,\beta;\gamma)\rangle=-\frac{M}{2}\cos(2\theta). \tag{4.15}$$

The other components of the angular momentum as well as the Runge-Lenz-Pauli vectors can also be calculated similarly. Here we have demonstrated the method of calculating the expectation values by explicitly showing the calculation of $\langle A_z\rangle$ and $\langle L_z\rangle$.

V. CONCLUDING REMARKS

We have shown that the theory of conserved-charge coherent states can be applied to the quantum Kepler problem and leads to the improved Kepler coherent states defined by Eq. (3.6). Equations (3.9) and (3.10) clearly show that the improved Kepler coherent states are indeed the eigenkets of the operators E_- and G_- belonging to the Lie algebras of SU(2). That is, they are the coherent states associated with the Lie algebra of $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ in the sense discussed by Barut and Girardello [9]. The Kepler coherent states proposed in I do not have such properties. It is desirable to define similar coherent states associated directly with the angular momentum and the Runge-Lenz-Pauli vectors because of their physical significance. However, it seems impossible because the operators associated with them do not commute as shown in Ref. [2].

It has been shown that the operators, $E_\pm$, E_0, $G_\pm$, G_0, $\mathbf{L}$, and $\mathbf{A}$ can be simply expressed in terms of the harmonic oscillator operators defined by Eqs. (2.5)–(2.8). This fact seems to imply the fundamental significance of these harmonic oscillators in the quantum Kepler problem [10]. The relevant physical quantities in the Kepler motion can be expressed in terms of these harmonic oscillator operators. Thus the calculation procedures for the expectation values illustrated in the last section can be applied to various physical quantities. Numerical analysis of the improved Kepler coherent states based on such expectation values to analyze experiments on the Rydberg atom [11] will be discussed in a forthcoming paper.

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APPENDIX

The expectation values for the operator products can be straightforwardly calculated:

$$\langle\Psi(\alpha,\beta;\gamma)|A_+^\dagger A_+|\Psi(\alpha,\beta;\gamma)\rangle=|\alpha|^2|N|^2 Z(\alpha,\beta,\gamma;+), \tag{A1}$$

$$\langle\Psi(\alpha,\beta;\gamma)|B_-^\dagger B_-|\Psi(\alpha,\beta;\gamma)\rangle=|\beta|^2|N|^2 Z(\alpha,\beta,\gamma;+), \tag{A2}$$

$$\langle\Psi(\alpha,\beta;\gamma)|A_-^\dagger A_-|\Psi(\alpha,\beta;\gamma)\rangle=|\gamma|^2|N|^2 Z(\alpha,\beta,\gamma;-), \tag{A3}$$

$$\langle\Psi(\alpha,\beta;\gamma)|B_+^\dagger B_+|\Psi(\alpha,\beta;\gamma)\rangle=|N|^2 Z(\alpha,\beta,\gamma;-), \tag{A4}$$

where Z is defined as

$$Z(\alpha,\beta,\gamma;\pm) \equiv \sum_{m_+=0}^{\infty}\sum_{m_-=0}^{\infty}\sum_{n_-=0}^{\infty}\frac{|\alpha|^{2m_+}|\beta|^{2n_-}|\gamma|^{2m_-}}{(m_+-m_-+n_-\pm 1)!\,m_+!\,n_-!\,m_-!}. \tag{A5}$$

This sum can be straightforwardly calculated. We obtain

$$Z(\alpha,\beta,\gamma;+)=\frac{2}{\kappa}(|\gamma|^2+1)I_1(\kappa) \tag{A6}$$

and

$$Z(\alpha,\beta,\gamma;-)=\frac{2}{\kappa}(|\alpha|^2+|\beta|^2)I_1(\kappa), \tag{A7}$$

where κ has been defined by Eq. (3.8). Combining these results and using Eqs. (4.3) and (4.6), we obtain Eqs. (4.8) and (4.9).

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