Quantum measurement as a driven phase transition: An exactly solvable model

Armen E. Allahverdyan,[1,3] Roger Balian,[1] and Theo M. Nieuwenhuizen[2]
[1] *S.Ph.T., CEA Saclay, 91191 Gif-sur-Yvette cedex, France*
[2]*Department of Physics and Astronomy, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[3]*Yerevan Physics Institute, Alikhanian Brothers St. 2, Yerevan 375036, Armenia*
(Received 23 February 2001; published 16 August 2001)

A model of quantum measurement is proposed, which aims to describe statistical mechanical aspects of this phenomenon, starting from a purely Hamiltonian formulation. The macroscopic measurement apparatus is modeled as an ideal Bose gas, the order parameter of which, that is, the amplitude of the condensate, is the pointer variable. It is shown that properties of irreversibility and ergodicity breaking, which are inherent in the model apparatus, ensure the appearance of definite results of the measurement, and provide a dynamical realization of wave-function reduction or collapse. The measurement process takes place in two steps: First, the reduction of the state of the tested system occurs over a time of order $\hbar/(TN^{1/4})$, where T is the temperature of the apparatus, and N is the number of its degrees of freedom. This decoherence process is governed by the apparatus-system interaction. During the second step classical correlations are established between the apparatus and the tested system over the much longer time scale of equilibration of the apparatus. The influence of the parameters of the model on nonideality of the measurement is discussed. Schrödinger kittens, EPR setups, and information transfer are analyzed.

DOI: 10.1103/PhysRevA.64.032108 PACS number(s): 03.65.Ta, 03.65.Yz, 05.30.−d

I. INTRODUCTION

Understanding the specific features of quantum measurements has been a long-standing question [1–8]. Discussions on this subject and its interpretations started in the early days of the quantum theory [1–5]. Physical and philosophical reflections on the problem and consideration of its different aspects became a source for deep conclusions about the quantum world and its classical counterpart [2,4]. Also the current activity in this field [6–11] clearly displays its complexity and multifariousness, and witnesses that its final settling is not yet close.

The main purpose of this paper is to exhibit the complete solution of a model for a measurement process, which will show that two main paradigms of modern statistical physics, irreversibility and ergodicity breaking in phase transitions, are crucially relevant for the quantum measurement problem. Together with some other conditions they provide all necessary ingredients for the realization of ideal quantum measurements. No additional postulates need to be posed, since standard quantum statistical physics completely suffices for the self-consistent explanation of this phenomenon.

We will start with a discussion on the problem of quantum measurement and the main steps which were made in its interpretation and understanding.

A. General measurement

Let us recall the general requirements that a measurement should satisfy. The quantity to be measured in the system S under study is represented by a Hermitian operator x with eigenvalues x_k. The so-called pointer variable X in the measuring apparatus A may take values X_k in correspondence with x_k. The observation of X_k provides statistical information about the state of the system. Moreover, some measurements termed as ideal [7] can be used as filters to prepare the system in a state where x takes a well-defined value x_k.

More precisely, we denote by ρ the density operator of the system under study, by R that of the apparatus, and by $\mathcal{R}$ the global density operator. To include the possibility of the most general descriptions of the system and the apparatus, we analyze the situation in terms of density operators rather than pure states. For simplicity we will operate in this section with discrete spectra. At the initial time $t=0$ just before measurement, the system is in an *unknown* state $\rho(0)$, the apparatus is in a fixed state $R(0)$ and these states are uncorrelated, so that the complete state is described by

$$\mathcal{R}(0)=\rho(0)\otimes R(0). \tag{1.1}$$

The uncorrelated character of this state simply reflects the fact that for $t<0$ the tested system and the apparatus were not interacting. An evolution operator $\mathcal{T}$ transforms this initial density operator into the overall density operator $\mathcal{R}(\theta)$ at the final time θ of the measurement, which should have the form

$$\mathcal{T}\,[\rho(0)\otimes R(0)]\equiv\mathcal{R}(\theta)\simeq\sum_k p_k\mathcal{R}_k\,. \tag{1.2}$$

Here $\mathcal{T}$ does not depend on the initial state of the system, which is arbitrary and unknown, though it may depend on the initial state of the apparatus $R(0)$. By $\simeq$ we mean "as precise as desired provided that the parameters of the apparatus are tuned suitably."

In a precise and unbiased measurement the final possible states $\mathcal{R}_k$ entering Eq. (1.2) should constitute an orthogonal set,

$$\mathrm{tr}(\mathcal{R}_k\mathcal{R}_l)\simeq\delta_{kl}\,, \tag{1.3}$$

so as to ensure that they represent *exclusive* events to which ordinary (nonquantum) probabilities can be assigned. They may be distinguished from one another by observing the pointer variable X, which takes in each $\mathcal{R}_k$ the value X_k with a negligible statistical fluctuation, namely

$$\mathrm{tr}(\mathcal{R}_k X) = X_k, \quad \mathrm{tr}(\mathcal{R}_k X^2) \simeq X_k^2. \tag{1.4}$$

Hereafter tr_A and tr_S indicate traces in the subspaces of the apparatus and the system respectively, while tr is reserved for the complete trace.

Each X_k occurs in Eq. (1.2) with a probability p_k, which is determined by the initial state of the system in the form

$$p_k = \mathrm{tr}_S(\rho(0)\Pi_k), \tag{1.5}$$

where $\Pi_k = |x_k\rangle\langle x_k|$ denotes the projection operator onto the eigenvalue x_k of x in the Hilbert space of the system. Although the states $\mathcal{R}_k$ may depend on the initial state $\rho(0)$ of the particle, the quantities X_k do not depend on $\rho(0)$ for precise and unbiased measurements, but are determined by the structure of the apparatus only. On the other hand, the probabilities p_k are determined by the initial state $\rho(0)$ only.

The fact that the process which leads from $\mathcal{R}(0)$ to $\mathcal{R}(\theta)$ describes a measurement is reflected in a specific feature. Due to the exclusiveness property (1.3), Eq. (1.2) represents the occurrence of a *classical* random quantity k, with probability p_k. It expresses that after the observation has taken place, the overall system is left in a state $\mathcal{R}_k$ with probability p_k. Indeed, the density operator $\mathcal{R}(\theta)$ describes a statistical *ensemble of measurements* all performed under similar conditions rather than an individual experiment. Since the variable X has a definite value in each state $\mathcal{R}_k$, one can count the frequencies of the different X_k's and thus recover the unknown probability distribution p_k. It is the specific type of correlation exhibited for each k in Eq. (1.2), between the properties (1.4) pertaining to the apparatus and the expression (1.5) for the probabilities p_k in terms of $\rho(0)$, which allows us to gain information about the tested system.

The crucial problem of quantum measurement is therefore to explain how the evolution process of the coupled state $\mathcal{R}(t)$ from $\mathcal{R}(0)$ to $\mathcal{R}(\theta)$ can produce a transformation $\mathcal{T}$ which ensures Eq. (1.2). The subsequent process of *observation* then merely amounts to the *selection* of a single term of Eq. (1.2) characterized by the value X_k of the pointer. This last step is by no means different from the analogous process in the *classical probability* theory [6,8], as was stressed recently by van Kampen and one of us [6]. It should be stressed additionally that observation and selection refer to the specific type of measurement process which has been previously performed.

B. Ideal measurement

The above measurement scheme is rather general, and in particular describes situations where the system itself does not have a definite state after measurement (e.g., a photon is destroyed when detected by a photomultiplier).

A theoretically and practically important class of measurements are the ideal ones, which leave the tested system as weakly perturbed as possible after selection of the result. In an ideal measurement, the possible final states $\mathcal{R}_k$ factorize as

$$\mathcal{R}_k(\theta) = \rho_k \otimes R_k, \tag{1.6}$$

where ρ_k is expressed in terms of the initial state $\rho(0)$ of the system as

$$\rho_k = \frac{1}{p_k}\Pi_k \rho(0) \Pi_k, \tag{1.7}$$

and where the possible final states of the apparatus, characterized by the value X_k of the pointer variable as in Eq. (1.4), are orthogonal as in Eq. (1.3),

$$\mathrm{tr}(R_k R_l) \simeq \delta_{kl}, \tag{1.8}$$

and are independent of the initial state $\mathcal{R}(0)$. Notice that the condition (1.8) is stronger than that given by Eq. (1.3). Later we will connect it with *robustness* of the apparatus as an information-storing device.

Thus, after observation of the apparatus, and sorting of a given outcome X_k, the system is *prepared* in the state (1.7) by means of the so-called ''reduction of the wave packet'' or ''collapse of the wave function.'' Apart from bringing the system into an eigenstate of the observable x associated with the eigenvalue x_k, the projection (1.7) does not affect its other degrees of freedom.

A complete measurement theory should provide conditions under which an apparatus interacting with the tested system brings it into one among the reduced states ρ_k.

C. The standard approach

When they deal with quantum measurements, textbooks usually justify the above properties by relying on the conventional arguments initiated by Bohr [1,3,12]. Somewhat qualitative and incomplete, this general line of reasoning became known as the Copenhagen interpretation. Different, though closely connected versions were summarized by Rosenfeld [12] and more recently by van Kampen [8] among others. The first precise discussion on the measurement problem was given by von Neumann [5], who clarified the issues, but was led to consider the properties of quantum measurements, in particular the reduction (1.7) as a *postulate*, which complements the standard principles of quantum mechanics. This additional postulate is, however, not needed as one can show, using consistency arguments, that the reduction of the wave packet can be derived from the standard principles and from natural properties attributed to measurements, in particular, repeatability [6].

Nevertheless, a complete understanding of a quantum measurement requires its analysis as a *dynamical process*. A crucial point to be explained is the nonexistence in the final state (1.2) of so-called Schrödinger cat terms with respect to the index k. For simplicity let us specialize on an ideal measurement. Starting from any initial state of the system, the final density operator $\mathcal{R}(\theta)$ should commute with both the measured observable x of the system and the pointer observ-

able X of the apparatus. This property ensures that the result of the measurement process can be described in the language of classical probability theory. Otherwise, if $\mathcal{R}(\theta)$ did include off-diagonal contributions in k, the standard interpretation of the measurement process would fail.

The Copenhagen school of thought overcomes this difficulty by saying that any physically acceptable apparatus has to be a classical system [1,3]. Thus it cannot exist in a state with superpositions. More refined Copenhagen-like approaches [12,13,8] state that the apparatus is a macroscopic system, and therefore coherent superpositions may arise, but cannot be detected at least when measuring certain observables. This situation was illustrated by an exactly solvable model [14] (see also [15] in this context), where a class of observables was proposed, which are indeed nonsensitive to superpositions. Nevertheless, a small modification of this class allows to produce superpositions [16].

In the von Neumann–Wigner approach [2,5] the states of the apparatus are pure, as $R(0)=|\Psi\rangle\langle\Psi|$, and $R_k=|\Psi_k\rangle\langle\Psi_k|$, and the evolution is characterized by the mapping

$$|\psi_k\rangle|\Psi\rangle\mapsto|\psi_k\rangle|\Psi_k\rangle \tag{1.9}$$

between initial and final states of the compound system. If this evolution is applied to a coherent initial state of the tested system such as

$$\rho(0)=\left(\sum_k \alpha_k|\psi_k\rangle\right)\left(\sum_{k'} \alpha_{k'}^*\langle\psi_{k'}|\right), \tag{1.10}$$

then one finally gets the following state $\tilde{\mathcal{R}}(\theta)$ associated with $\rho(0)$:

$$\tilde{\mathcal{R}}(\theta)=\sum_{k,k'} \alpha_k|\psi_k\rangle|\Psi_k\rangle\alpha_{k'}^*\langle\psi_{k'}|\langle\Psi_{k'}|, \tag{1.11}$$

whereas the desired state $\mathcal{R}(\theta)$ in Eq. (1.2) involves only the diagonal elements of $\tilde{\mathcal{R}}(\theta)$:

$$\mathcal{R}(\theta)=\sum_k |\alpha_k|^2|\psi_k\rangle|\Psi_k\rangle\langle\psi_k|\langle\Psi_k|. \tag{1.12}$$

Partial traces of $\tilde{\mathcal{R}}$ and $\mathcal{R}$ over either the apparatus or the system subspace are equal, but we should explain why the off-diagonal terms of $\tilde{\mathcal{R}}$ are never seen. Indeed, one has for the partial density matrix of the particle

$$\rho(\theta)=\mathrm{tr}_A\tilde{\mathcal{R}}(\theta)=\mathrm{tr}_A\mathcal{R}(\theta)=\sum_k |\alpha_k|^2\,|\psi_k\rangle\langle\psi_k|. \tag{1.13}$$

However, the situation described by $\tilde{\mathcal{R}}(\theta)$ does not correspond to any measurement, since the apparatus and the tested system cannot be in definite states with definite probabilities. Actually, since with modern experimental equipments one is able to detect mesoscopic and even *macroscopic* superpositions (see, e.g., [17,18] where recent results are reported in the context of charge and flux macroscopic superpositions in Josephson junctions), the question remains open and can be formulated in the following form: What are the concrete properties of a system, that make it usable as a measuring apparatus characterized by the collapsed state Eqs. (1.1)–(1.7)?

D. Irreversibility

An important requirement for the realization of Eqs. (1.1)–(1.7) is the existence of *irreversibility*. Whereas Eq. (1.11) is a pure state, we expect the pointer variable to take some well-defined value X_k with the classical probability p_k, and this implies the final state to be the mixture (1.12). In other words, the von Neumann entropy $S_{\mathrm{vN}}(\mathcal{R}(\theta))=-\mathrm{tr}\mathcal{R}(\theta)\ln\mathcal{R}(\theta)$ of the state $\mathcal{R}(\theta)$ in Eq. (1.12) is positive, while $S_{\mathrm{vN}}(\tilde{\mathcal{R}}(\theta))=0$. More generally the von Neumann entropy of Eq. (1.2) can be shown to be different from that of Eq. (1.1), due to the elimination of the coherent terms in the final state. This implies a *loss of information* about the coherence, a loss which is required to ensure the classical interpretation of the measurement and the reduction of the wave packet.

A measurement process should therefore be analyzed in the same way as an irreversible process in quantum statistical mechanics, a second reason for using density operators. The size of the apparatus should be sufficiently large, so that irreversibility and relaxation emerge from the microscopic reversible evolution. Otherwise measurements could not be ideal. The fact that the condition (1.2) cannot be realized with a unitary transformation from any initial state (1.1) is nowadays well established with different degrees of generalization [2,19]. To understand this in simple terms, let us write down the ideal measurement transformation for two different initial states $\rho^{(s)}(0)$, $s=1,2$ of the system,

$$\mathcal{T}[\rho^{(s)}(0)\otimes R(0)]=\mathcal{R}^{(s)}(\theta)\simeq\sum_k p_k^{(s)}\rho_k\otimes R_k, \tag{1.14}$$

and assume for simplicity that the spectrum of x is nondegenerate: $\Pi_k=|x_k\rangle\langle x_k|$. Then unitarity of $\mathcal{T}$ requires $\mathrm{tr}_S[\rho^{(1)}(0)\rho^{(2)}(0)]=\mathrm{tr}[\mathcal{R}^{(1)}(\theta)\mathcal{R}^{(2)}(\theta)]$, and conditions (1.7) and (1.8) imply

$$\begin{aligned}\mathrm{tr}_S\rho^{(1)}(0)\rho^{(2)}(0)&\equiv\sum_{k,l}\langle x_k|\rho^{(1)}(0)|x_l\rangle\langle x_l|\rho^{(2)}(0)|x_k\rangle\\&\simeq\sum_k\langle x_k|\rho^{(1)}(0)|x_k\rangle\langle x_k|\rho^{(2)}(0)|x_k\rangle\end{aligned} \tag{1.15}$$

and hence

$$\sum_{k\neq l}\langle x_k|\rho^{(1)}(0)|x_l\rangle\langle x_l|\rho^{(2)}(0)|x_k\rangle\simeq 0, \tag{1.16}$$

which cannot be true for arbitrary $\rho^{(1)}(0)$ and $\rho^{(2)}(0)$. Thus, the unitarity of $\mathcal{T}$ has to be disguised. Actually, the situation is the same as in any relaxation process: The overall

system is isolated, and the evolution of $\mathcal{R}(t)$ is in principle governed by Hamiltonian dynamics, but on suitable time scales irreversibility occurs owing to the presence of a large number of degrees of freedom, which act as an external bath. Statistical physics is needed to explain this behavior, in which microscopic reversible equations of motion result in macroscopic irreversible ones. Just the same approach was followed recently by two of us, to discover that the standard issue of Brownian motion leads to incompatibilities with thermodynamics in the regime of quantum entanglement [20].

E. Ergodic and decoherence approaches to quantum measurements

A first application of statistical physics in support of the Copenhagen interpretation was given by Daneri, Longier, and Prosperi [13,12]. Somewhat related (but not equivalent) approaches are reviewed in Ref. [10]. After considering the measuring apparatus and measured system together as an *isolated* system, one attributes the absence of macroscopic superpositions to inevitable statistical uncertainties, which are present in macroscopic bodies. Mathematically this is reflected in different kinds of ergodic assumptions, which are reasonably creditable for those systems. However, these approaches have several drawbacks, which, in particular, originate from the fact that they do not provide dynamical mechanisms for the realization of quantum measurements. Extensive criticism of them can be found in [9].

There is another, nowadays not less influential, school of thought which we shall follow. It attempts to handle the problem also involving certain arguments from statistical physics [21,22,9,11]. In this *decoherence* approach the loss of coherence is viewed as a process established by an external environment, which is generally understood as a collection of uncontrollable and unobservable degrees of freedom. A decoherence process suppresses superpositions of some special states, which are determined by the interaction between the environment and the system. However, in an ideal quantum measurement, the coherence associated with both the observable x of the tested system, and the observable X of the apparatus should disappear *independently* of the concrete form of environment-system interaction, whereas the other coherences which exists in $\rho(0)$ should remain present in the reduced states ρ_k defined by Eq. (1.7). The type of decoherence occurring in quantum measurements is thus very special and we shall relate its features to the apparatus-system interaction rather than to an environment-system interaction.

F. Requirements on quantum measurement models

Keeping in mind both the successes and shortcomings of the various existing ideas, we tackle in this paper the quantum measurement problem by investigating a specific model. The coupled evolution of the system and apparatus is treated as a dynamical process of quantum statistical mechanics. By deriving an explicit solution we wish to show how the various features of an ideal measurement, expressed by Eqs. (1.1)–(1.7), can emerge from the microscopic dynamics generated by the Hamiltonian of our model.

When choosing the model we have been guided by various conditions that an apparatus should satisfy.

(1) It should have a degree of freedom X which may relax towards definite values X_k.

(2) It should be macroscopic so as to ensure an irreversible relaxation.

(3) This relaxation should be selectively triggered by the interaction of X with the variable x of the measured system.

(4) The various values X_k should a priori be equally probable, so as to avoid any bias produced by the apparatus. Thus, the various final states R_k, characterized by the value of X_k, should have the same entropy.

(5) The apparatus should be a stable and robust information storing device, which implies that the states R_k are nearly in equilibrium and that after the measurement has been completed, X is a nearly conserved collective variable.

These properties suggest to take for the apparatus a suitably chosen macroscopic system which is able to undergo a phase transition, with X_k as an *order parameter*. Indeed, a phase transition is a macroscopic process with robust and stable (or at least metastable) outcomes. Notice that the existence of an order parameter implies ergodicity breaking in contrast to the purely ergodic view at measurement [13].

(6) The measured quantity x should be coupled to the order parameter, and the apparatus should *amplify* this signal received during its interaction with the system. This is achieved by noting that the value of an order parameter can be controlled by an infinitesimally small source. The microscopic variable will play the role of such a source, which controls X but otherwise does not affect the apparatus.

(7) The relaxation of the order parameter is ensured by its coupling with other degrees of freedom of the apparatus, referred to as a thermal bath. It is this coupling which, together with the thermodynamical limit for the apparatus, will ensure the specific type of relaxation discussed above.

We shall work out a model, as simple as possible, subject to the above requirements. The tested system is a one-dimensional particle, and the quantity to be measured is its position. The apparatus is a noninteracting Bose gas, which has an easily tractable phase transition. The variable X is the amplitude of the condensate. This situation will be shown to be generalizable for an arbitrary tested system and an arbitrary measured observable (Appendix A). Although specific and not realistic, the model is thus suggestive for more general measurements. The possibility of tuning the parameters will help us to find the limit in which the measurement is ideal and to explore some imperfections of the measurement.

This paper is organized as follows. In Sec. II we present the model and discuss its equations of motion. Limits which are especially relevant for the quantum measurement problem, as well as exact solutions of the equations of motion in the Schrödinger and Heisenberg pictures are considered in Sec. III. In Sec. IV we show that the present model realizes the conditions of ideal measurements discussed above. There we also consider characteristic times of this realization and discuss imperfections which arise due to an incomplete thermodynamical limit for the apparatus. Our conclusions are presented in the last section. Several technical questions are considered in Appendices A, B, C, and D.

II. THE MODEL

A. The apparatus and its bath

As a model for our apparatus we will choose a system of N noninteracting bosons in a three-dimensional cubic box with volume V and periodic boundary conditions. Its Hamiltonian reads

$$H_{\mathrm{A}}=\sum_i \varepsilon_i a_i^\dagger a_i, \tag{2.1}$$

where $a_i^\dagger$, a_i are the creation and annihilation operators of each single-boson state, and ε_i is its energy, given in terms of its wave vector k and of the boson mass M by

$$\varepsilon_i=\frac{\hbar^2 k^2}{2M}. \tag{2.2}$$

Notice that $\varepsilon_0=0$. This apparatus is an open system, namely it interacts *weakly* with a large external environment. If there were no other interactions, the Bose gas would relax with time towards the Gibbs distribution:

$$\rho_{\mathrm{A}}=\frac{1}{Z}\exp(-\beta H_{\mathrm{A}}+\beta\mu\mathcal{N}), \quad Z=\mathrm{tr}\exp(-\beta H_{\mathrm{A}}+\beta\mu\mathcal{N}), \tag{2.3}$$

where $T=1/\beta$ is the temperature and μ is the chemical potential. Both of these quantities are imposed by the environment. Finally $\mathcal{N}$ is the number operator

$$\mathcal{N}=\sum_i a_i^\dagger a_i. \tag{2.4}$$

In a realistic apparatus what we call the ''external'' environment is actually constituted by a large number of degrees of freedom which are part of the apparatus itself. Here we treat it as a separate thermal bath, which can exchange energy and particles with the Bose gas. We can imagine this bath itself as a Bose gas which is much larger than the apparatus, so that its intensive variables remain fixed once and forever. The bath is characterized by its temperature T, its chemical potential μ, and by a quantum coupling which induces a relaxation time γ^{-1} to the Bose gas. This situation corresponds to the grand canonical ensemble for the apparatus although the overall apparatus-bath system is isolated. Whereas the value of T is fixed by the overall energy, the value of μ is determined by the overall boson number. We shall focus below on a condensed gas, with $|\mu|/T$ small as $1/\sqrt{N}$. The tuning of μ is then achieved through a control of the overall particle number.

As a system examined by means of the apparatus, we take a particle living in one-dimensional space, with mass m, an external potential $V(x)$ and Hamiltonian

$$H_{\mathrm{S}}=\frac{p^2}{2m}+V(x). \tag{2.5}$$

The measured quantity is the position x, which is coupled to the Bose gas through the interaction

$$H_{\mathrm{I}}=-gxX, \quad X=\sqrt{\frac{\hbar}{2}}(a_0^\dagger+a_0), \tag{2.6}$$

where g is the coupling constant. The quantity X attached to the apparatus will be our pointer variable. It is macroscopically accessible through observation of the density of the Bose gas provided Bose condensation takes place, which requires low temperature and sufficiently small $|\mu|$. In this case we expect the signal x to be amplified, because X^2 will be extensive. On the other hand, a tested system interacting with all levels of the apparatus would certainly be less interesting, since the threshold of the influence on the apparatus would be diminished. We have chosen the simple interaction (2.6) for theoretical purposes, although it is not realistic since it can change the number of bosons.

Altogether the Hamiltonian of the apparatus and the system during the measurement reads

$$H=H_{\mathrm{A}}-\mu\mathcal{N}+H_{\mathrm{S}}+H_{\mathrm{I}}, \tag{2.7}$$

to which we should add the interaction of the Bose gas with the bath, see Appendix D. Since the number of bosons may change through exchanges with the bath and the observed system, we have included for convenience in Eq. (2.7) the contribution from the chemical potential. Later on we will show how to generalize this situation to an arbitrary tested system and measured observable, keeping the same measurement apparatus.

B. Bose condensation

As is well known, the three-dimensional ideal Bose gas in equilibrium undergoes a condensational phase transition at sufficiently large density or small chemical potential. Let us briefly recall this phenomenon, since in our setup this is a crucial property of the apparatus as an information-storing device. We consider here the Bose gas submitted to an external constant field source J, which later on will be identified with the term $\sqrt{\hbar/2}gx$ in H_{I}:

$$H_{\mathrm{B}}=\sum_i \varepsilon_i a_i^\dagger a_i-J(a_0+a_0^\dagger). \tag{2.8}$$

We work in the grand canonical ensemble. When there is no source term, it is known that despite different magnitudes of fluctuations in the condensed phase, the grand canonical and canonical ensembles are equivalent for the noninteracting Bose gas [23]. However, the source term in Eq. (2.8) can have a macroscopic effect only in the grand canonical ensemble. It controls the density of the condensate, which can vary owing to possible exchange with the bath. In the canonical ensemble where the overall density is given, the density of the condensate would be practically insensitive to the presence of the source. In our present situation the *density* $\langle a_0^\dagger a_0\rangle$, and not only the *phase* of $\langle a_0\rangle$, appears as an order parameter controlled by the field J. This property is specific for our model of noninteracting bosons, and would be invalid for a realistic Bose condensate with interaction.

At the equilibrium state with temperature T and (negative) chemical potential μ one diagonalizes the Gibbsian density matrix by shifting the lowest energy operators as

$$a_0=\tilde{a}_0-\frac{J}{\mu},\quad a_0^\dagger=\tilde{a}_0^\dagger-\frac{J}{\mu}. \tag{2.9}$$

This leads to

$$\langle a_0\rangle=-\frac{J}{\mu}, \tag{2.10}$$

$$\langle a_0^\dagger a_0\rangle=\frac{1}{e^{-\beta\mu}-1}+\frac{J^2}{\mu^2}. \tag{2.11}$$

The averages concerning excited states will not change, since the field acts only on the lowest mode. For the total density of particles one gets

$$\frac{N}{V}=\frac{1}{V}\sum_i\langle a_i^\dagger a_i\rangle=\frac{(2M)^{3/2}}{4\pi^2\hbar^3}\int_0^\infty\frac{d\varepsilon\sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)}-1}+\frac{1}{V}\frac{1}{e^{-\beta\mu}-1}+\frac{J^2}{V\mu^2}. \tag{2.12}$$

Here we went to the thermodynamical limit, which, if x and thus J are well defined, means to make the following change:

$$\mathrm{tr}(\cdots)\mapsto\int\frac{d^3x\,d^3k}{(2\pi)^3}(\cdots)=V\int\frac{d^3k}{(2\pi)^3}(\cdots)=V\frac{(2M)^{3/2}}{4\pi^2\hbar^3}\int_0^\infty d\varepsilon\sqrt{\varepsilon}(\cdots). \tag{2.13}$$

We also separated out the contribution coming from the lowest state $\varepsilon=0$.

We wish to measure x, which, if x is well defined, means that we wish to find the value of $J=\sqrt{\hbar/2}gx$ through a macroscopic observation of the Bose gas. This is feasible by observing the total density (2.12), provided that the last term in this equation is finite in the thermodynamical limit, which requires: $|\mu|\to 0$ when $N\to\infty$. In this case the particle density

$$\frac{N}{V}=\frac{N_\mathrm{n}}{V}+\frac{N_\mathrm{c}}{V} \tag{2.14}$$

splits into a noncondensed part,

$$\frac{N_\mathrm{n}}{V}=\frac{(2M)^{3/2}}{4\pi^2\hbar^3}\int_0^\infty\frac{d\varepsilon\sqrt{\varepsilon}}{e^{\beta\varepsilon}-1}=\frac{0.165869M^{3/2}}{\hbar^3}T^{3/2}, \tag{2.15}$$

and a condensed part

$$\frac{N_\mathrm{c}}{V}=\frac{T}{V|\mu|}+\frac{J^2}{V\mu^2}, \tag{2.16}$$

which is our order parameter. Since N_n/V is a known function of temperature, we can deduce N_c/V from the total density, provided that this total density is significantly larger than the critical value (2.15):

$$\frac{N}{V}>\frac{N_\mathrm{c}}{V}. \tag{2.17}$$

In order to use the Bose gas as a measurement device for J, we shall require Eq. (2.16) to be dominated by its last term, which means that

$$1\gg\frac{|\mu|}{T}\gg\frac{1}{N}, \tag{2.18}$$

and that

$$\frac{|\mu|}{J}=O\left(\frac{1}{\sqrt{N}}\right). \tag{2.19}$$

Under such conditions, the scalar source term x, which will be replaced later on by the coordinate of the tested particle, can be deduced from the density of the Bose gas and the characteristics of its bath:

$$x^2=\frac{2\mu^2V}{\hbar g^2}\left(\frac{N}{V}-\frac{N_\mathrm{n}}{V}\right). \tag{2.20}$$

The sign of x results through Eq. (2.10) from the phase of the condensate. Low temperatures will improve the efficiency of the measurement, since they provide small values for Eq. (2.15) and thus increase the ratio N/N_n. The chemical potential of the bath should be fixed at some small value when $N\to\infty$, so as to ensure Eqs. (2.18) and (2.19), and g should be such that as $g^2x^2\hbar/\mu^2$ is of order N.

The excited states of the Bose gas will not play a direct role in our model of measurement, since they are not coupled to the tested particle. Nevertheless, they contribute, together with the bath, to exchange bosons with the condensate, the density of which can thus be controlled by the chemical potential as well as by the source J.

Notice also that, although the various macroscopic states characterized by different values of the order parameter N_c/V can be distinguished, they appear on the same footing, because both their entropy and their energy are the same in the thermodynamical limit. Indeed the contribution of the condensate to the entropy is $\ln N_\mathrm{c}$, its contribution to $\langle H_\mathrm{B}\rangle$ is $2\mu N_\mathrm{c}$ where $|\mu|\ll T$, so that both become negligible as $N\to\infty$. This was required to prevent the apparatus from having an intrinsic bias.

C. Equations of motion

1. Dynamics of the apparatus in its bath

Before we examine the equations of motion of the overall system including the tested particle, the apparatus and the bath, we will investigate in this subsection the situation without the tested particle. At some remote initial time $t=t_0\ll 0$ the apparatus was in an arbitrary *nonequilibrium* state. At

that time it starts to interact with the bath, and the arising dynamics for the apparatus will be described by means of a weak-coupling quantum Langevin equation [24] (see Appendix D for the derivation of this equation from the Heisenberg equation associated with the apparatus-bath Hamiltonian):

$$\dot{a}_i=\frac{i}{\hbar}[H_{\rm A}-\mu\mathcal{N},a_i(t)]-\gamma a_i+\sqrt{2\gamma}b_i(t)$$
$$=-i(\omega_i+\alpha)a_i-\gamma a_i+\sqrt{2\gamma}b_i(t), \qquad (2.21)$$

where

$$\hbar\omega_i\equiv\varepsilon_i, \quad \hbar\alpha\equiv-\mu. \qquad (2.22)$$

Notice that the chemical potential μ is negative, whereas the parameter α is positive. The Langevin equations are written for the Heisenberg operators of the apparatus only, but the presence of the bath is reflected through a friction term $-\gamma a_i$, and a random Gaussian force operator $b_i(t)$, which satisfies

$$[b_i(t),b_k^\dagger(t')]=\delta_{ik}\delta(t-t'),$$
$$[b_i(t),b_k(t')]=[b_i^\dagger(t),b_k^\dagger(t')]=0, \qquad (2.23)$$
$$\langle b_i^\dagger(t)b_k(t')\rangle=\delta_{ik}\delta(t-t')n_i^{\rm eq},$$
$$\langle b_i(t)b_k(t')\rangle=\langle b_i^\dagger(t)b_k^\dagger(t')\rangle=0, \qquad (2.24)$$
$$n_i^{\rm eq}=\frac{1}{e^{\beta(\varepsilon_i-\mu)}-1}. \qquad (2.25)$$

The most important consequence of Eqs. (2.21), (2.23), (2.24), and (2.25) is that they ensure relaxation with the characteristic time $1/\gamma$ of the apparatus towards the Gibbs distribution (2.3) with the temperature T and chemical potential μ imposed by the bath. This can be seen from the following exact solution of Eq. (2.21):

$$a_i(t)=e^{-\gamma(t-t_0)-i(\omega_i+\alpha)(t-t_0)}a_i(t_0)$$
$$+\sqrt{2\gamma}\int_0^{t-t_0}ds\,e^{-\gamma s-i(\omega_i+\alpha)s}b_i(t-s). \qquad (2.26)$$

In particular, all possible moments $\langle a^{\dagger n}(t)a^m(t)\rangle$ calculated with Eq. (2.26) for $t-t_0\gg 1/\gamma$ are identical to those obtained through the Gibbs distribution (2.3). For example the average number of particles $\langle a_i^\dagger(t)a_i(t)\rangle$ in the level i evolves in time according to

$$\langle a_i^\dagger(t)a_i(t)\rangle\equiv n_i(t)=e^{-2\gamma(t-t_0)}n_i(t_0)+(1-e^{-2\gamma(t-t_0)})n_i^{\rm eq}, \qquad (2.27)$$

which shows that $n_i(t)$ relaxes to its Gibbsian stationary value $n_i^{\rm eq}$ at the characteristic time $1/(2\gamma)$.

Since, as shown by Eqs. (2.6) and (2.7), the tested particle interacts only with the lowest level of the apparatus, the equations of motion for the excited levels with $i\geqslant 1$ will be always given by Eq. (2.26). Therefore, in the further discussion we will leave these excited levels aside.

2. Equations of motion including the tested particle

Let us now consider the situation with the tested system. The interaction between the apparatus and the tested system is switched on at the initial time $t=0$. For $t\leqslant 0$ the overall initial state factorizes as in Eq. (1.1), where $\rho(0)$ is an arbitrary state of the tested particle and $R(0)$ is the Gibbs distribution of the apparatus given by Eq. (2.3). Indeed due to the assumed condition $t-t_0\gg 1/\gamma$ the apparatus had enough time to relax starting from any initial state at $t=t_0$. From now on we shall drop the index 0 in $a_0=a$, $b_0=b$. The Heisenberg-Langevin equation of motion for the lowest level of the apparatus reads for $t>0$

$$\dot{a}=\frac{i}{\hbar}[H,a(t)]-\gamma a+\sqrt{2\gamma}b(t)$$
$$=-i\alpha a+\frac{i}{\sqrt{2\hbar}}gx(t)-\gamma a+\sqrt{2\gamma}b(t). \qquad (2.28)$$

This equation is solved exactly as

$$a(t)=e^{-\gamma t-i\alpha t}a(0)+\frac{ig}{\sqrt{2\hbar}}\int_0^t ds\,e^{-\gamma s-i\alpha s}x(t-s)$$
$$+\sqrt{2\gamma}\int_0^t ds\,e^{-\gamma s-i\alpha s}b(t-s). \qquad (2.29)$$

For $\gamma t\gg 1$, and when x is constant, Eq. (2.29) expresses that the Bose gas relaxes towards an equilibrium state, where the particle number in the condensate is given by Eq. (2.16) with $J=\sqrt{\hbar/2}gx$, as ensured by the second term in the right-hand side (RHS) of Eq. (2.29). The average number of particles $n(t)=\langle a^\dagger(t)a(t)\rangle$ in the lowest state is evolving as

$$n(t)=e^{-2\gamma t}n(0)+\frac{g^2x^2}{2\hbar(\gamma^2+\alpha^2)}(1+e^{-2\gamma t}-2e^{-\gamma t}\cos\alpha t)$$
$$+(1-e^{-2\gamma t})n^{\rm eq}. \qquad (2.30)$$

The second term in the RHS of Eq. (2.30) is the contribution supplied by the source, which shifts the condensate density.

Although the evolution of the apparatus when there is no source leads to a well-defined equilibrium state where $n(0)=T/|\mu|$ is large but not extensive as $V\to\infty$, the small interaction with the tested particle is sufficient to change macroscopically n at times $t\gg 1/(2\gamma)$ if $|\mu|=\hbar\alpha$ is sufficiently small. This means that the apparatus together with its bath constitute a system which is *nearly nonergodic* when Bose condensation sets in.

The Heisenberg dynamics of the particle reads

$$\dot{F}=\frac{i}{\hbar}[H_{\rm S},F(t)]-\frac{ig}{\sqrt{2\hbar}}[x(t),F(t)](a(t)+a^\dagger(t)) \qquad (2.31)$$

for any operator F. We find in particular

$$\dot{x}=\frac{1}{m}p, \tag{2.32}$$

$$\dot{p}=-V'(x)+\int_0^t ds\,\chi(t-s)x(s)+\eta(t), \tag{2.33}$$

where

$$\eta(t)=\eta_0(t)+\eta_1(t), \tag{2.34}$$

$$\eta_0(t)=\sqrt{\hbar\gamma}g\int_0^t ds\,e^{-\gamma s}(b^\dagger(t-s)e^{i\alpha s}+b(t-s)e^{-i\alpha s}), \tag{2.35}$$

$$\eta_1(t)=\sqrt{\frac{\hbar}{2}}ge^{-\gamma t}(a^\dagger(0)e^{i\alpha t}+a(0)e^{-i\alpha t}), \tag{2.36}$$

$$\chi(t)=g^2e^{-\gamma t}\sin\alpha t. \tag{2.37}$$

The interaction of the particle with the Bose gas produces a force, which has a random part. The randomness of the noise $\eta(t)$ arises from two independent reasons: the statistical (uncertain) character of the initial state of the apparatus, which gives the contribution $\eta_1(t)$, and the random character of b, $b^\dagger$, which occurs through $\eta_0(t)$. Recall that at $t=0$ the apparatus was in equilibrium at temperature T and chemical potential μ,

$$\langle a(0)a^\dagger(0)\rangle=1+\frac{1}{e^{-\beta\mu}-1},\quad \langle a^\dagger(0)a(0)\rangle=\frac{1}{e^{-\beta\mu}-1}. \tag{2.38}$$

Since $b(t)$, $b^\dagger(t)$ are themselves Gaussian, $\eta(t)$ will be Gaussian as well, with the noise autocorrelation:

$$K(t,t')=\langle\eta(t);\eta(t')\rangle=K_0(t,t')+K_1(t,t'), \tag{2.39}$$

$$K_0(t,t')=\langle\eta_0(t);\eta_0(t')\rangle =\frac{g^2\hbar}{2}\cos[\alpha(t-t')] \times\coth\frac{\hbar\alpha}{2T}(e^{-\gamma|t-t'|}-e^{-\gamma(t+t')}), \tag{2.40}$$

$$K_1(t,t')=\langle\eta_1(t);\eta_1(t')\rangle =\frac{g^2\hbar}{2}\cos[\alpha(t-t')]\coth\frac{\hbar\alpha}{2T}e^{-\gamma(t+t')}, \tag{2.41}$$

where we define for any operators A, B:

$$\langle A;B\rangle\equiv\frac{1}{2}\langle AB+BA\rangle. \tag{2.42}$$

Notice that $K(t,t')$ is time-translation invariant, although its separate parts are not. It is seen that for $t+t'\gg 1/\gamma$, which corresponds to the stationary apparatus, only $K_0(t-t')$ persists. The most important effect of the bath on the dynamics of the tested particle is the appearance of a new characteristic correlation time $1/\gamma$ in addition to the time scale set up by the maximal frequency of the apparatus and to the universal quantum correlation time $\hbar/T$.

3. Brownian motion of the tested particle

In order to compare Eqs. (2.32)–(2.37) with the standard quantum Brownian motion approach, one integrates Eq. (2.33) by parts. This yields an equation [24],

$$\dot{p}=-V'(x)-\frac{1}{m}\int_0^t ds\,\tilde{\chi}(t-s)p(s)+\eta(t)-\tilde{\chi}(t)x(0) +\tilde{\chi}(0)x(t), \tag{2.43}$$

which has the usual form in terms of a friction kernel

$$\tilde{\chi}(t)=e^{-\gamma t}\frac{\alpha\cos\alpha t+\gamma\sin\alpha t}{\alpha^2+\gamma^2}, \tag{2.44}$$

and of a noise $\eta(t)$. Notice, however, that the friction kernel does not have a definite sign with t. The last term in Eq. (2.43) renormalizes the potential. As far as one is interested in the state of the tested particle itself, the full noise acting on it is $\eta(t)$. However, for the global state of the particle and the apparatus $\eta_1(t)$ is a deterministic object, and only η_0 remains as noise.

4. Validity of weak-coupling quantum Langevin equations

When substituting the above white-noise quantum Langevin equations for the actual interaction between the apparatus and the bath, a crucial fact was that their coupling is weak, so that the damping time $1/\gamma$ is much larger than both the (maximal) dynamical characteristic time $t_{\rm d}$ of the apparatus, and the characteristic correlation time of the bath $\hbar/T$. Under these conditions it is possible, as shown in Appendix D, to introduce an effective quantum noise $b(t)$ with white spectrum satisfying Eqs. (2.23)–(2.25), and get the Gibbs distribution as the result of relaxation. The time $t_{\rm d}$ is expressed as

$$t_{\rm d}\sim\frac{1}{\alpha}=\frac{\hbar}{|\mu|}, \tag{2.45}$$

as seen from the free part of Eq. (2.28). Altogether the parameters of the bath should satisfy

$$\alpha\gg\gamma\quad\text{or}\quad T\gg|\mu|\gg\hbar\gamma, \tag{2.46}$$

where we have taken into account the upper bound (2.18) on $|\mu|$. Notice that excited levels of the Bose gas have a lower dynamical time

$$t_{\rm d}^{(i)}\sim\frac{1}{\alpha+\omega_i}. \tag{2.47}$$

It is expected that in a nonideal Bose gas these characteristic times will influence also the lowest mode, making its characteristic dynamical time lower. The weak-coupling condition (2.46) is discussed with more details in Appendix D.

III. DYNAMICS OF THE MEASUREMENT PROCESS

A. Approximate conservation of the measured quantity

The above equations describe the joint evolution of the tested particle and the Bose gas in the bath which ensures relaxation. This evolution will describe a measurement of x, if the Bose condensate registers the statistical distribution of x in the initial state $\rho(0)$ under the conditions specified in the Introduction. In the present section we will take into account the following condition for ideality of the measurement.

If $x(t)$ did change with time during the process, the final state of the apparatus would be determined not only by the statistical distribution of $x(0)$ but rather by the whole history of its change. So to be sure that we indeed measure the quantity $x(0)$, we will require that the characteristic time for $x(t)$ to change appreciably from its initial value is much larger than the relaxation time $1/\gamma$ of the apparatus. This time is itself shorter than the duration θ of the measurement, since for $t=\theta$ the apparatus should display definite results through well-defined stationary states. Notice the conservation of $x(t)$ does not mean that the state of the particle is not changing. One only needs approximate conservation of the quantity x. For the considered model this is realized if over the time scale θ one can neglect the contribution of the kinetic energy $p^2/(2m)$ to the change of the coordinate x. Obviously, this is realized for sufficiently large m. The more precise condition $m \gg g^2\theta^3$ will be found in Appendix C, where we discuss the deviations from nonideality which arise from the motion of the particle during the measurement. For other systems this conservation of the measured quantity may result from a commutation relation: $[H_S,x]=0$, as stressed by Wigner [2] (see also [26] in this context). However, the actual mechanism is basically irrelevant provided that the time scale for the variation of x is much larger than the duration of the measurement.

Notice that taking the limit $m\to\infty$, as we do in the bulk of this paper, does not mean that we are dealing with a classical particle. In fact, the initial density matrix $\langle x'|\rho(0)|x''\rangle$ of the particle is arbitrary, which expresses that the quantity x is subject to a fully quantum probability distribution at the initial time owing to the presence of off-diagonal elements. The large mass enters the equation of motion (2.32), and it only implies that the diagonal elements $\langle x|\rho(t)|x\rangle$ remain unchanged during the measurement.

B. Wigner function and intermediate Wigner function

1. Wigner function

For solving the dynamics in the Schrödinger picture, we find it useful to employ the language of the Wigner function. The description through Wigner functions is, of course, completely equivalent to that through density matrices, but frequently, and in particular for quantum systems with linear dynamical evolution equations, this allows to obtain and understand results in a more easy way. This is because Wigner functions allow one to use the classical intuition as much as possible, which appears to be insightful for such systems.

We shall focus on the degrees of freedom x, p of the tested particle and a, $a^\dagger$ of the condensate, leaving aside the excited states of the Bose gas which relax independently. To introduce a Wigner function not only for the particle but also for the apparatus degrees of freedom, we represent the annihilation and creation operators in the canonical way:

$$a=\frac{1}{\sqrt{2\hbar}}(X+iP), \quad a^\dagger=\frac{1}{\sqrt{2\hbar}}(X-iP), \tag{3.1}$$

$$X=\sqrt{\frac{\hbar}{2}}(a+a^\dagger), \quad P=i\sqrt{\frac{\hbar}{2}}(a^\dagger-a), \tag{3.2}$$

$$[X,P]=i\hbar. \tag{3.3}$$

Recall that the connection between a density matrix $\langle x'|\rho|x''\rangle$ and the corresponding Wigner function $w(x,p)$ is given for each degree of freedom as

$$\left\langle x+\frac{1}{2}\xi\middle|\rho\middle|x-\frac{1}{2}\xi\right\rangle=\int\frac{dp}{2\pi\hbar}e^{i\xi p/\hbar}w(x,p), \tag{3.4}$$

$$w(x,p)=\int d\xi\, e^{-i\xi p/\hbar}\left\langle x+\frac{1}{2}\xi\middle|\rho\middle|x-\frac{1}{2}\xi\right\rangle. \tag{3.5}$$

Notice that in the present paper the normalization of the Wigner function is chosen as

$$\int\frac{dp\,dx}{2\pi\hbar}w(x,p)=1, \tag{3.6}$$

since the integration with this weight corresponds to a trace. The Wigner function of the particle and the condensate together will be denoted by $\mathcal{W}(X,P,x,p)$; those of the particle and condensate separately will be denoted as $w(x,p)$ and $W(X,P)$, respectively. Obviously, one has

$$\int\frac{dX\,dP}{2\pi\hbar}\mathcal{W}(X,P,x,p)=w(x,p),$$

$$\int\frac{dx\,dp}{2\pi\hbar}\mathcal{W}(X,P,x,p)=W(X,P). \tag{3.7}$$

The Wigner function at time t in the Schrödinger picture can readily be represented in terms of the Heisenberg operators. For example, for the tested particle the corresponding formula reads

$$w(x,p;t)=\langle \mathrm{tr}\rho(0)\hat{w}(x,p;t)\rangle,$$

$$\hat{w}(x,p;t)=\int\frac{da\,db}{4\pi^2}\exp[-iax-ibp+iax(t)+ibp(t)], \tag{3.8}$$

where $x(t)$, $p(t)$ are the Heisenberg operators of the particle, $\rho(0)$ is its initial state, and the average is taken with respect to the full noise $\eta=\eta_0+\eta_1$.

2. Intermediate Wigner function

We shall also find it convenient to use yet another formulation, where the degrees of freedom of the tested particle are left in the matrix representation, and the Wigner transformation (3.5) is taken only for the apparatus degrees of freedom. We will call this object the intermediate Wigner function, and denote it as $\mathcal{V}(X,P,x',x'')$, where x', x'' denote the corresponding matrix elements in the x representation,

$$\mathcal{V}(X,P,x',x'')=\int d\xi\, e^{-i\xi P/\hbar}\left\langle X+\frac{1}{2}\xi,x'\left|\mathcal{R}\right|X-\frac{1}{2}\xi,x''\right\rangle = \int dp\, e^{ip(x'-x'')/\hbar}\mathcal{W}\left(X,P,\frac{x'+x''}{2},p\right). \tag{3.9}$$

Here again the excited states of the Bose gas are left aside.

C. Exact solution of the equations of motion in the Heisenberg picture

We consider a heavy ($m\to\infty$), free ($V(x)=0$) particle. The equations of motion for the apparatus have already been solved as Eq. (2.26) for the excited states and Eq. (2.29) for the lowest state. The latter equation, in which $x(t-s)$ can be replaced by $x(0)$, is written in terms of X, P and $b=b_0$ as

$$X(t)=X(0)e^{-\gamma t}\cos\alpha t+P(0)e^{-\gamma t}\sin\alpha t + \frac{x(0)g}{\gamma^2+\alpha^2}[\alpha-(\gamma\sin\alpha t+\alpha\cos\alpha t)e^{-\gamma t}] + \sqrt{\hbar\gamma}\int_0^t ds\, e^{-\gamma s}[e^{-i\alpha s}b(t-s)+e^{i\alpha s}b^\dagger(t-s)], \tag{3.10}$$

$$P(t)=P(0)e^{-\gamma t}\cos\alpha t-X(0)e^{-\gamma t}\sin\alpha t + \frac{x(0)g}{\gamma^2+\alpha^2}[\gamma+(\alpha\sin\alpha t-\gamma\cos\alpha t)e^{-\gamma t}] + i\sqrt{\hbar\gamma}\int_0^t ds\, e^{-\gamma s}[e^{i\alpha s}b^\dagger(t-s)-e^{-i\alpha s}b(t-s)]. \tag{3.11}$$

For the tested particle we can solve Eqs. (2.32) and (2.33) when the operator $x(t)$ does not significantly change, as

$$x(t)=x(0), \tag{3.12}$$

$$p(t)=p(0)+x(0)\xi(t)+\int_0^t du\ \eta(u), \tag{3.13}$$

where

$$\xi(t)=\int_0^t du\int_0^u ds\,\chi(s)=g^2\frac{(\gamma^2+\alpha^2)\alpha t+(\gamma^2-\alpha^2)e^{-\gamma t}\sin\alpha t-2\alpha\gamma(1-e^{-\gamma t}\cos\alpha t)}{(\gamma^2+\alpha^2)^2}, \tag{3.14}$$

and $\chi(t)$, $\eta(t)$ are defined by Eqs. (2.34)–(2.37). Hereafter the following formulas will be used:

$$\int_0^t ds e^{-\gamma s}\cos\omega s=\frac{\gamma[1-e^{-\gamma t}\cos\omega t]+\omega e^{-\gamma t}\sin\omega t}{\gamma^2+\omega^2}, \quad \int_0^t ds\, e^{-\gamma s}\sin\omega s=\frac{\omega[1-e^{-\gamma t}\cos\omega t]-\gamma e^{-\gamma t}\sin\omega t}{\gamma^2+\omega^2}. \tag{3.15}$$

D. Exact solution of the equations of motion in the Schrödinger picture

Since the above dynamical equations are linear, there is a direct connection between the Heisenberg picture and the Schrödinger dynamics in terms of the overall Wigner function for the tested particle and the apparatus. The equation for the common Wigner function of the particle and the lowest mode has the form

$$\mathcal{W}(X,P,x,p;t)=\int dX_0 dP_0 dp_0\,\Phi(X,P,x,p;t|X_0,P_0,x,p_0;0)W(X_0,P_0;0)w(x,p_0;0). \tag{3.16}$$

Here we denote by X_0,P_0,p_0 the variables of the Wigner function at the initial time. The variable x of the Wigner function remains unchanged from the initial to the final state, since the Heisenberg operator $x(t)$ is conserved. In Eq. (3.16) we used the fact that the initial Wigner function $\mathcal{W}$ is factorized into the partial Wigner function of the lowest level $W(0)$ and that of the tested particle $w(0)$,

$$\mathcal{W}(X,P,x,p;0)=W(X,P;0)w(x,p;0)=w(x,p;0)\frac{\hbar}{\lambda}\exp\left[-\frac{1}{2\lambda}X^2-\frac{1}{2\lambda}P^2\right], \tag{3.17}$$

where we took into account the fact that at the initial time $t=0$ the apparatus has already relaxed to the equilibrium Gibbs distribution under the influence of the bath.

To find the transition kernel Φ, we notice that the Heisenberg operators $X(t)$, $P(t)$ and $p(t)$ of the Bose gas and the tested particle given by Eqs. (3.10)–(3.13) are *linear* and involve only the *Gaussian* noise $b,b^{\dagger}$. We can then use the method of Eq. (3.8) for the whole system. Recall that the moments of the Wigner function coincide with the corresponding *symmetrized* operator moments. Hence, Φ has the Gaussian form

$$\Phi(X,P,x,p;t|X_0,P_0,x,p_0;0)=\frac{\sqrt{\det B}}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}\sum_{i,k=1}^{3}B_{ik}L_iL_k\right), \tag{3.18}$$

where (L_1,L_2,L_3) is the following three-dimensional vector:

$$L_1=X-\langle X\rangle=X-X_0e^{-\gamma t}\cos\alpha t-P_0e^{-\gamma t}\sin\alpha t-\frac{xg}{\gamma^2+\alpha^2}[\alpha-(\gamma\sin\alpha t+\alpha\cos\alpha t)e^{-\gamma t}], \tag{3.19}$$

$$L_2=P-\langle P\rangle=P-P_0e^{-\gamma t}\cos\alpha t+X_0e^{-\gamma t}\sin\alpha t-\frac{xg}{\gamma^2+\alpha^2}[\gamma+(\alpha\sin\alpha t-\gamma\cos\alpha t)e^{-\gamma t}], \tag{3.20}$$

$$L_3=p-\langle p\rangle=p-p_0-x\xi(t)-\frac{gX_0}{\gamma^2+\alpha^2}[\gamma+(\alpha\sin\alpha t-\gamma\cos\alpha t)\,e^{-\gamma t}]-\frac{gP_0}{\gamma^2+\alpha^2}[\alpha-(\gamma\sin\alpha t+\alpha\cos\alpha t)\,e^{-\gamma t}]. \tag{3.21}$$

Here we have used Eqs. (3.10), (3.11), and (3.13), and the average $\langle\cdots\rangle$ is taken over the noise operators b, $b^{\dagger}$ directly and through η_0. The quantity $\xi(t)$ was defined in Eq. (3.14). Notice that since we are interested here in the common state of the particle and the apparatus, the term connected with η_1 which gives rise to the last two terms in Eq. (3.21) appears as deterministic. The Gaussian quantum noise enters through B, which is a 3×3 symmetric matrix with the following elements:

$$[B^{-1}]_{11}=\langle(X-\langle X\rangle)^2\rangle=[B^{-1}]_{22}=\langle(P-\langle P\rangle)^2\rangle=\frac{\hbar}{2}[1-e^{-2\gamma t}]\coth\frac{\hbar\alpha}{2T}, \tag{3.22}$$

$$[B^{-1}]_{12}=\langle X-\langle X\rangle;P-\langle P\rangle\rangle=0, \tag{3.23}$$

$$[B^{-1}]_{13}=\langle X-\langle X\rangle;p-\langle p\rangle\rangle=\left\langle X-\langle X\rangle;\int_0^t ds\,\eta_0(s)\right\rangle=\frac{g\hbar}{2}\left(\frac{\gamma-2\gamma e^{-\gamma t}\cos\alpha t+\gamma e^{-2\gamma t}}{\gamma^2+\alpha^2}\right)\coth\frac{\hbar\alpha}{2T}, \tag{3.24}$$

$$[B^{-1}]_{23}=\langle P-\langle P\rangle;p-\langle p\rangle\rangle=\left\langle P-\langle P\rangle;\int_0^t ds\,\eta_0(s)\right\rangle=-\frac{g\hbar}{2}\left(\frac{\alpha-2\gamma e^{-\gamma t}\sin\alpha t-\alpha e^{-2\gamma t}}{\gamma^2+\alpha^2}\right)\coth\frac{\hbar\alpha}{2T}, \tag{3.25}$$

$$[B^{-1}]_{33}=\langle(p-\langle p\rangle)^2\rangle=\int_0^t\int_0^t ds_1ds_2K_0(s_1,s_2)=\frac{g^2\hbar}{2}\coth\frac{\hbar\alpha}{2T}\left(\frac{2\gamma t+1-e^{-2\gamma t}}{\gamma^2+\alpha^2}+\frac{4\gamma(\gamma e^{-\gamma t}\cos\alpha t-\alpha e^{-\gamma t}\sin\alpha t-\gamma)}{(\gamma^2+\alpha^2)^2}\right). \tag{3.26}$$

If we adopt the following notations

$$\lambda(t)=[B^{-1}]_{11},\quad \zeta(t)=\frac{[B^{-1}]_{13}}{\lambda(t)},\quad \sigma(t)=\frac{[B^{-1}]_{23}}{\lambda(t)}, \tag{3.27}$$

$$\Delta(t)=\frac{[B^{-1}]_{33}}{\lambda(t)}-\sigma(t)^2-\zeta(t)^2, \tag{3.28}$$

then we can write the matrix B^{-1} as

$$B^{-1}=\lambda\begin{pmatrix}1 & 0 & \zeta\\ 0 & 1 & \sigma\\ \zeta & \sigma & \Delta+\zeta^2+\sigma^2\end{pmatrix}, \tag{3.29}$$

and hence the matrix B will read

$$B=\frac{1}{\lambda\Delta}\begin{pmatrix}\Delta+\zeta^2 & \zeta\sigma & -\zeta\\ \zeta\sigma & \Delta+\sigma^2 & -\sigma\\ -\zeta & -\sigma & 1\end{pmatrix}=\frac{1}{\lambda}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\frac{1}{\lambda\Delta}\begin{pmatrix}-\zeta\\-\sigma\\1\end{pmatrix}\otimes(-\zeta\quad-\sigma\quad 1). \tag{3.30}$$

Equation (3.18) for Φ can altogether be written in the explicit form:

$$\Phi(X,P,x,p;t|X_0,P_0,x,p_0;0)=\frac{1}{\sqrt{(2\pi)^3\lambda^3\Delta}}\exp\left(-\frac{1}{2\lambda}(L_1^2+L_2^2)-\frac{1}{2\lambda\Delta}(L_3-\zeta L_1-\sigma L_2)^2\right), \tag{3.31}$$

where λ, ζ, σ, Δ, L_1, L_2 and L_3 are defined by Eqs. (3.19)–(3.28) as functions of time.

E. Measurement of a variable with a discrete spectrum

The results reported so far were obtained for the measurement of the coordinate x which has a continuous spectrum. It is of clear interest to indicate how the obtained results can be generalized for other specific situations, e.g., for measurement of spin. Here we provide a simple remark, which will set our results in a more general context.

Let us assume that the interaction between the tested system and the apparatus is still given by Eq. (2.6) but now the tested system is completely arbitrary, and x in this equation refers to one of its observables. In particular, it can have a discrete spectrum. For simplicity we still neglect the self-Hamiltonian of the tested system. In this general case, the complete Wigner function for the system and the lowest level is no longer defined. Nevertheless, the intermediate Wigner function of Eq. (3.9) is still perfectly defined. Recall that this function employs (X,P) variables for the lowest mode, but uses the matrix elements (x',x'') in the eigenrepresentation of the measured quantity x. As we show in Appendix A the intermediate Wigner function corresponding to Eq. (3.16) adequately describes the general situation that we consider. Though the complete density matrix of the measured system and the lowest mode might also be used, the intermediate Wigner function $\mathcal{V}$ is a more convenient object to deal with.

An illustrative example for the measurement of an observable with a discrete spectrum is the spin-boson Hamiltonian [25], of which we only need to specify the interaction part

$$H_I=\frac{1}{2}g\sigma_z X, \tag{3.32}$$

where the measured observable $\frac{1}{2}\sigma_z$ is the z-component of spin for the tested system.

The situation of a discrete spectrum measurement will also be encountered below for our original model in spite of the continuity of the coordinate x. Actually, we shall consider an initial density operator $\rho(0)$ involving two distinct values of x only, a situation which does not differ much from a genuine discrete spectrum.

IV. IDEAL MEASUREMENT: POSTMEASUREMENT STATES

In this section we will be interested in the postmeasurement situation. Let us first resume the *conditions of ideality* of the measurement that we have encountered above. The relaxation time of the apparatus should be small compared to the duration of the measurement:

$$\gamma\theta\gg 1. \tag{4.1}$$

The coupling with the bath should however be small on the scale of the dynamical time (2.45), as shown in Appendix D:

$$\gamma\ll\alpha=\frac{|\mu|}{\hbar}. \tag{4.2}$$

On the other hand, if we denote by $\tilde{x}$ a typical value of the coordinate to be measured,

$$\tilde{x}^2=\mathrm{tr}(\rho(0)x^2), \tag{4.3}$$

its coupling with the apparatus should produce a finite condensate density, which according to Eq. (2.20) is expressed as

$$\frac{\hbar g^2\tilde{x}^2}{\mu^2}=O(N). \tag{4.4}$$

The fact that the bath ensures Bose condensation, but that the condensate density remains dominated by the coupling with the particle, imposes the condition (2.18), that is

$$1\gg\frac{|\mu|}{T}\gg\frac{1}{N}. \tag{4.5}$$

To fix ideas we shall assume in the following that

$$\frac{|\mu|}{T}=O\left(\frac{1}{\sqrt{N}}\right), \tag{4.6}$$

with T finite in the thermodynamical limit. This will imply, from Eq. (4.4), that the coupling constant g is finite. Finally, if $\tilde{p}$ is the characteristic value of the particle momentum, the approximate conservation of x during the measurement means that

$$\frac{\tilde{p}}{m}\theta \ll \tilde{x} \tag{4.7}$$

or $m \gg g^2\theta^3$ as shown in Appendix C. We expect that under these conditions both the tested system and the Bose gas have reached at time θ a quasistationary state that we wish to study.

By using the condition (4.1) we first recall that the excited states become decoupled and thermalized with the bath, since their Wigner function at time θ deduced from Eq. (2.26) reads

$$W_k(X_k,P_k)=\frac{2}{\coth\dfrac{\varepsilon_k+\hbar\alpha}{2T}}\exp\left[-\frac{1}{\hbar\coth\dfrac{\varepsilon_k+\hbar\alpha}{2T}}\times(X_k^2+P_k^2)\right]. \tag{4.8}$$

The transition kernel Φ between times 0 and θ can be obtained from Eqs. (3.19)–(3.28), and (3.31). Under conditions (4.1), (4.2), (4.5), and (4.7) its various ingredients reduce to

$$\lambda(\theta)=\frac{\hbar}{2}\coth\frac{\hbar\alpha}{2T}\simeq\frac{T}{\alpha},\quad \zeta(\theta)=0,\quad \sigma(\theta)=-\frac{g}{\alpha}, \tag{4.9}$$

$$\Delta(\theta)=\frac{2g^2\gamma\theta}{\alpha^2}, \tag{4.10}$$

$$L_1(\theta)=X-\frac{gx}{\alpha},\quad L_2(\theta)=P,$$

$$L_3(\theta)=p-p_0-\frac{xg^2\theta}{\alpha}-\frac{g}{\alpha}P_0, \tag{4.11}$$

$$L_3-\zeta L_1-\sigma L_2=p-p_0+\frac{g}{\alpha}(P-P_0)-\frac{xg^2\theta}{\alpha}, \tag{4.12}$$

where we have used $\xi(\theta)=g^2\theta/\alpha$ as follows from Eq. (3.14). We note that Eqs. (4.5) and (4.6) imply

$$N\hbar \gg \lambda(\theta) \gg \hbar,\quad \lambda=\hbar O(\sqrt{N}), \tag{4.13}$$

respectively, and that Eqs. (4.1) and (4.4) imply

$$\Delta(\theta)\gg\frac{N\hbar}{\tilde{x}^2}, \tag{4.14}$$

while $g\tilde{x}/\alpha$ is of order $\sqrt{N\hbar}$.

The postmeasurement state of the tested particle and the apparatus will be investigated below in three steps. First we will discuss the partial state of the apparatus and that of the particle. Later on we shall turn to the global state.

A. Apparatus

As seen, the apparatus itself behaves as a Bose gas subject to a source field proportional to $gx(0)$. Owing to Eqs. (4.13) and (4.14) the variance $\lambda\Delta$ of L_3 in Eq. (3.31) is much larger than $(N\hbar/\tilde{x})^2$. We can therefore readily trace out the tested particle from Eqs. (3.16) and (3.31) by integrating over both p and p_0. The factor $w(x,p_0;0)$ of Eq. (3.16) thus generates the probability density

$$\int\frac{dp_0}{2\pi\hbar}w(x,p_0;0)=\langle x|\rho(0)|x\rangle \tag{4.15}$$

for the coordinate x in the initial state of the tested particle. The resulting expression of the Wigner function of the apparatus at time θ has the expected form

$$W(X,P;\theta)=\int dx\langle x|\rho(0)|x\rangle W_x(X,P), \tag{4.16}$$

$$W_x(X,P)=\frac{\hbar}{\lambda}\exp\left[-\frac{1}{2\lambda}\left(X-\frac{gx}{\alpha}\right)^2-\frac{1}{2\lambda}P^2\right]. \tag{4.17}$$

The Wigner function $W_x(X,P)$, where $\lambda=T/\alpha$, describes the quantum Gibbs distribution of the apparatus at temperature T and chemical potential $\mu=-\hbar\alpha$, with a classical source $J=\sqrt{\hbar/2}\,gx$.

For each possible value of the coordinate x of the particle the apparatus is thus in an equilibrium state, with an order parameter proportional to x. In spite of the quantum nature of the variable $x(0)$ which is governed by the initial density operator $\rho(0)$, it acts on the apparatus as a *classical random* object. We can understand this classical feature by noting that because $x(0)=x(t)$ during the process, $a(t)$ and $a^\dagger(t)$ commute with it: $[a(t),x(0)]=0$, $[a^\dagger(t),x(0)]=0$. Therefore, the situation is very similar to that considered in Sec. II B, with $J=gx(0)\sqrt{\hbar/2}$. However, there is a subtle point, since the field J is now random. Its quantum randomness arises from the initial state of the tested particle, which in general is not an eigenstate of the coordinate operator. Since the off-diagonal part of $\rho(0)$ disappears owing to the large size of $\lambda(\theta)\Delta(\theta)$, to be inserted in Eq. (3.31), Eq. (4.16) shows that the quantum nature of the randomness is suppressed. It is seen as well that the resulting classical randomness is quenched, which means that all extensive quantities have to be calculated for a fixed field and then averaged at the last step.

Notice that for the continuous spectrum a certain difficulty may arise if (for example) the initial state of the particle is an eigenstate of the coordinate: $\langle x'|\rho(0)|x''\rangle=\delta(x'-x_0)\delta(x''-x_0)$, in which case the quantity (4.15) diverges. There are several standard ways to overcome this difficulty [3]. The simplest one is to consider a Gaussian packet $\langle x|\psi\rangle=(2\pi\epsilon)^{-1/4}\exp[-(x-x_0)^2/(4\epsilon)]$ instead of a precise eigenstate of the coordinate. This state has a normalizable Wigner function,

$$w(x,p;0)=2\exp\left[-\frac{(x-x_0)^2}{2\epsilon}-\frac{2\epsilon p^2}{\hbar^2}\right], \tag{4.18}$$

and we can let $\epsilon \to 0$ in the last stage of calculations. Further on we will always assume that this procedure is implied when necessary. This justifies the replacement of Eq. (4.15) by $\delta(x-x_0)$.

Since the apparatus is a nonergodic system, Eq. (4.16) means that it will occupy with *a priori* probabilities $\langle x|\rho(0)|x\rangle$ a state which is determined by the initial value of the coordinate. If the initial state of the particle is an eigenstate of the coordinate operator the field is not random, and there is only one state to which the apparatus can relax. So in this case we have a definite prediction, as it should be.

In each state (4.17) of the apparatus, the momentum P associated with the lowest mode fluctuates around the value 0 exactly as in the Gibbsian state imposed by the bath, but its coordinate X is shifted by $xg\hbar/|\mu|$. This shift is of order $\sqrt{N\hbar}$ according to Eq. (4.4), and it produces a finite shift in the density

$$\frac{N_c}{V}=\frac{1}{2\hbar V}\langle X^2+P^2-\hbar\rangle \simeq \frac{\hbar g^2 x^2}{2\mu^2 V} \tag{4.19}$$

of the condensate. The statistical fluctuation of X, equal to $\sqrt{\lambda}$, is small compared to $\langle X\rangle \propto \sqrt{N\hbar}$ owing to the first condition in Eq. (4.13), namely $\lambda \ll N\hbar$.

The second condition in Eq. (4.13), namely $\lambda \gg \hbar$, entails that the variables X and P behaves as *classical* random variables, which is a natural requirement for the pointer variable of an apparatus. Compared to the shift $\langle X\rangle$ their fluctuations are of relative order $N^{-1/4}$ if we choose $\lambda=\hbar O(\sqrt{N})$.

1. Amplification and registration

We have just seen that the interaction of the tested particle and the apparatus results in a *macroscopic* change in the condensate density: It fixes the expectation value $\langle X\rangle$ to $xg\hbar/|\mu|$ within fluctuations which are small in relative value. This large effect is a consequence of the condition (4.4). The coupling g is sufficiently large to produce a shift in the number N_c, which is of the same order as the total particle number. However, it is sufficiently small so that the contribution of the interaction Hamiltonian H_I to the energy of the apparatus is negligible.

The amplification of the influence of the tested system on the apparatus is due here to the smallness of $|\mu|$. The bath, which imposes on the apparatus the condition (4.5), prepares it before the measurement in a state where the condensate density is not yet finite, but where the smallness of $|\mu|$ makes the apparatus very sensitive to a source coupled to X. By making successive macroscopic observations of the value of X, one can then find the statistics of $x(0)$ through $\langle x|\rho(0)|x\rangle$. There is a one-to-one correspondence between $\langle X\rangle$ and $x(0)$, without bias because the various values of the order parameter $\langle X\rangle$ yield identical values for the energy as well as for the entropy.

A peculiarity of the model comes from the fact that the amplification factor $g/|\mu|$ depends on the coupling constant g and on the chemical potential of the bath. These quantities need to be known to let us determine $x(0)$ through $\langle X\rangle$.

Moreover, if we wish to *register* the result of a measurement, which is the value reached by X at the time θ when the interaction g is switched off, we need to imagine that the exchange of bosons between the bath and the apparatus is also switched off at the same time θ. The overall density of bosons as well the condensate density thereafter remain fixed in the apparatus, which is in canonical equilibrium after the time θ. Another theoretical procedure to freeze the condensate density at the value (4.19) would consist in switching off the coupling γ before the end of the measurement. The fate of the tested particle after the time θ is considered in Appendix B.

2. Robustness

We have explained how our apparatus realizes amplification of weak signals. This is only half of the way towards a good information storing device, because we yet should see whether another important property which is *robustness* is satisfied. In other words, if under influence of a weak field the apparatus has relaxed to a definite state, then what is the probability that it will leave this state spontaneously? If this transition probability is small, and can be made as small as it is desired, then the property of robustness is present.

Let us assume that the apparatus has been brought into a state with

$$\langle X\rangle=\frac{gx}{\alpha}. \tag{4.20}$$

In this state the apparatus has Wigner function W_x and density matrix R. We wish to calculate the transition probability to another state R' associated with x' under the effect of some perturbation. If these states were pure, the transition probability would read as usual:

$$\Pr(x\to x')=\mathrm{tr}(RR'). \tag{4.21}$$

For mixed states we use the same formula in terms of the overlap:

$$\Pr(x\to x')\propto \mathrm{tr}(RR')=\int \frac{dX\,dP}{2\pi\hbar} W_x(X,P)W_{x'}(X,P). \tag{4.22}$$

To be normalized this expression should be divided by $\Pr(x\to x)$. Using Eq. (4.17) one gets

$$\Pr(x\to x')=\exp\left[-\frac{g^2(x-x')^2}{4\lambda\alpha^2}\right]. \tag{4.23}$$

It is clear that above the phase transition point, when both $\alpha=-\mu\hbar$ and λ [defined by Eqs. (3.22) and (3.27)] are finite, this transition probability is of order one, so that no robustness is present as was to be expected.

Let us consider the situation below phase transition where $\lambda \sim T/\alpha$. We then have

$$\Pr(x\to x')=\exp\left[-\frac{\hbar g^2(x-x')^2}{4T|\mu|}\right]. \tag{4.24}$$

According to Eq. (4.5) the exponent is of order $-N|\mu|/T$. With Eq. (4.6) this exponent behaves as $-\sqrt{N}$ in the thermodynamical limit, provided x and x' differ by a quantity which remains finite as $N\to\infty$. The probability therefore vanishes, as it should.

The fact that the overlap between states of the apparatus associated with different values of x is negligible also expresses that different positions of the pointer variable constitute *exclusive* events.

3. Accuracy of measurement

The robustness reflects stability of the apparatus with respect to external perturbations. Another quantity, the *accuracy*, characterizes the strength of the noise due to the initial uncertainty of the tested particle, and due to spontaneous thermal fluctuations induced by the bath. One can estimate the accuracy of the measurement by evaluating the following quantity

$$\Sigma=\frac{\{\langle X^2\rangle\}_{\rm av}-\{\langle X\rangle^2\}_{\rm av}}{\{\langle X\rangle^2\}_{\rm av}}, \tag{4.25}$$

where the average $\langle\cdots\rangle$ is taken with respect to the state of the apparatus, and where $\{\cdots\}_{\rm av}$ denotes the average over the initial distribution of the particle. This is the signal-to-noise ratio, and $\Sigma\ll 1$ corresponds to a good measurement of X. Having used Eqs. (2.10), (2.11), and (4.5), one finally gets

$$\Sigma=\frac{T|\mu|}{\hbar g^2\{x^2\}_{\rm av}}. \tag{4.26}$$

In the region where the condensational phase transition exists this quantity is small as $T/(|\mu|N)=O(N^{-1/2})$, provided that the thermodynamical limit is taken and that $\{x^2\}_{\rm av}$ is finite. The accuracy is thus governed by thermal noise.

Note also that apart from the above uncertainty the derivation of the measured quantity x from the pointer variable X by means of Eq. (4.19) involves the ratio g/μ. The accuracy of the measurement is, of course, spoiled if the coupling constant g and the chemical potential of the bath are not controlled with precision.

B. Tested particle

Let us now consider the partial state of the tested particle. At time θ we can find this state by tracing out the apparatus from Eq. (3.16) using the approximations (4.9)–(4.12). In this calculation we first note that the memory about the initial value X_0 is lost. The variable P_0 enters through the last term of the exponent of Φ, which at the time θ reads

$$-\frac{1}{2\lambda\Delta}(L_3-\zeta L_1-\sigma L_2)^2 =-\frac{1}{2\lambda\Delta}\left(p-p_0+\frac{g}{\alpha}(P-P_0)-\frac{xg^2\theta}{\alpha}\right)^2. \tag{4.27}$$

Since the apparatus is nearly in equilibrium at both times 0 and θ, P and P_0 are of order $\sqrt{\lambda}=\sqrt{T/\alpha}$. We can thus neglect the term depending on the apparatus in the bracket of Eq. (4.27), because

$$\frac{1}{2\lambda\Delta}\frac{g^2}{\alpha^2}(P-P_0)^2=\frac{\alpha^3}{4g^2T\gamma\theta}\frac{g^2}{\alpha^2}\frac{T}{\alpha}=\frac{1}{4\gamma\theta} \tag{4.28}$$

is small. This means that the overall system forgets about the initial state of the apparatus for $\gamma\theta\gg 1$. We can therefore readily integrate over the initial state $W(X_0,P_0;0)$ of the apparatus, then over the variables X and P, which yields

$$w(x,p;\theta)=\int dp_0 w(x,p_0;0)\frac{1}{\sqrt{2\pi\lambda\Delta}} \times\exp\left[-\frac{1}{2\lambda\Delta}\left(p-p_0-x\frac{g^2\theta}{\alpha}\right)^2\right]. \tag{4.29}$$

Due to the large value of $\lambda\Delta$ the exponential factor in Eq. (4.29) is nearly constant. Indeed, we have

$$\frac{\lambda\Delta\tilde{x}^2}{\hbar^2}=\frac{2\hbar g^2\tilde{x}^2}{\mu^2}\frac{T}{|\mu|}\gamma\theta=\gamma\theta O(N^{3/2}), \tag{4.30}$$

where $\tilde{x}$ is defined by Eq. (4.3), so that the only effect of this exponential is to produce a cutoff which ensures the normalization of $w(x,p;\theta)$. Otherwise, $w(x,p;\theta)$ is practically independent of p, and its dependence on x is the same as that of the probability density $\langle x|\rho(0)|x\rangle$ as expected.

The density matrix in the x basis associated with the Wigner function (4.29) is given by

$$\langle x'|\rho(\theta)|x''\rangle=\langle x'|\rho(0)|x''\rangle\exp\left[-\frac{\lambda\Delta}{2\hbar^2}(x'-x'')^2 +\frac{ig^2\theta}{2|\mu|}(x'^2-x''^2)\right]. \tag{4.31}$$

The large value of $\lambda\Delta$ ensures that it practically reduces to the diagonal part of $\rho(0)$ in the basis x.

1. Decoherence time

The above expressions for $w(\theta)$ or $\rho(\theta)$ show that the decoherence of the state of the tested particle with respect to x has been achieved at the time θ.

In order to understand *how and when* this *decoherence* takes place during the interaction process between times 0 and θ, we return to the equations of motion of the tested particle which are given by Eqs. (3.8), (3.12), and (3.13). Both terms η_0 and η_1 in Eq. (3.13) should here be treated as noise, since we eliminate the apparatus. The fluctuations of this noise are given by Eqs. (2.40) and (2.41). Altogether we get for the Wigner function $w(x,p;t)$ of the tested particle:

$$w(x,p;t)=\int dp_0 w(x,p_0;0)\frac{1}{\sqrt{2\pi d(t)}} \times\exp\left[-\frac{1}{2d(t)}(p-p_0-x\xi(t))^2\right], \quad (4.32)$$

where we define $d(t)$ by

$$d(t)=\int_0^t\int_0^t dt_1dt_2K(t_1,t_2)=\int_0^t\int_0^t dt_1dt_2K_0(t_1,t_2) +\frac{g^2\hbar}{2}\coth\frac{\hbar\alpha}{2T}\left(\frac{1-2e^{-\gamma t}\cos\alpha t+e^{-2\gamma t}}{\gamma^2+\alpha^2}\right), \quad (4.33)$$

and $\int_0^t\int_0^t dt_1dt_2K_0(t_1,t_2)$ is given by Eq. (3.26). By using Eqs. (3.24) and (3.26) we find

$$d(t)=[B^{-1}]_{33}+\frac{g}{\gamma}[B^{-1}]_{13}. \quad (4.34)$$

With the help of formulas (3.4) and (3.5) one gets the evolution of the density matrix

$$\langle x'|\rho(t)|x''\rangle=\langle x'|\rho(0)|x''\rangle\exp\left[-\frac{d(t)}{2\hbar^2}(x'-x'')^2 +\frac{i\xi(t)}{2\hbar}(x'^2-x''^2)\right]. \quad (4.35)$$

It is seen that the diagonal elements of the density matrix (those for which $x'=x''$) are not changed at all, whereas the off-diagonal ones are damped with rate $d(t)/\hbar^2$. In other words, the density matrix of the tested particle tends to the mixture formed by eigenstates of the coordinate operator. Indeed, the decoherence factor $d(t)$, which measures how the density matrix is squeezed in terms of $x'-x''$, increases from 0 to ∞ as the time t goes from 0 to ∞. Its asymptotic forms at short and long times are derived from Eqs. (3.24) and (3.26) as

$$d(t)=\frac{g^2T}{\alpha}t^2, \quad \gamma t\ll\alpha t\ll 1, \quad (4.36)$$

$$d(\theta)=\frac{2g^2T}{\alpha^3}\gamma\theta, \quad \gamma\theta\gg 1. \quad (4.37)$$

For long times $\gamma\theta\gg 1$ the exponent of Eq. (4.35) behaves as

$$\frac{d(\theta)}{2\hbar^2}(x'-x'')^2\sim\frac{\hbar g^2(x'-x'')^2}{\mu^2}\frac{T}{|\mu|}\gamma\theta=\gamma\theta O(N^{3/2}), \quad (4.38)$$

and we recover as in Eq. (4.31) the strong damping of the off-diagonal terms, which allows us to obtain the proper state of the tested particle after the measurement.

An analogous investigation for short times $\gamma t\ll\alpha t\ll 1$ can be carried out using Eq. (4.36):

$$\frac{d(t)}{2\hbar^2}(x'-x'')^2\sim\frac{\hbar g^2(x'-x'')^2}{\mu^2}\frac{T}{|\mu|}\left(\frac{Tt}{\hbar}\right)^2=\left(\frac{Tt}{\hbar}\right)^2O(N^{1/2}). \quad (4.39)$$

This quantity can be much larger than 1 even in the region of short times t for which $\alpha t=|\mu|t/\hbar\ll 1$, provided

$$N^{-1/4}\ll\frac{Tt}{\hbar}\ll\frac{T}{|\mu|}=O(N^{1/2}). \quad (4.40)$$

The off-diagonal terms of $\rho(t)$ thus disappear at the very beginning of the interaction of the particle with the apparatus, after a delay of order $N^{-1/4}$. The characteristic time

$$\tau\sim\frac{\hbar}{TN^{1/4}} \quad (4.41)$$

over which the reduction of the state of the tested particle takes place is thus *much shorter* than the *duration* θ of the measurement, since

$$\theta\gg\frac{1}{\gamma}\gg\frac{\hbar}{|\mu|}\gg\tau. \quad (4.42)$$

The first inequality in Eq. (4.42) ensures the relaxation of the apparatus. The second inequality ensures that the stationary state will be Gibbsian. The third inequality indicates that the decoherence of the tested particle takes place on a time scale much smaller that the dynamical time $\hbar/|\mu|$ given by Eq. (2.45). Note also that $\hbar/T$ itself is an important characteristic time scale in quantum statistical physics, which characterizes the relevance of quantum versus thermal effects. As shown in Eq. (4.41) the macroscopic size of the apparatus reduces these quantum effects by a factor $N^{1/4}$.

The *collapse time* τ, which in textbook discussions is either taken to be zero or identified with the *measurement time* θ itself, is definitely different from the latter in our model. Another interesting aspect of this difference is that, at those short time scales given by Eq. (4.41), the change in the apparatus is still negligible. Indeed, the variables of the bath have not yet changed at those scales, since the characteristic time over which the equilibrium between the bath and the apparatus sets in is $1/\gamma$. On the other hand, the energy associated with the particle has as well not changed, because the influence of the kinetic energy can be neglected, and the coordinate $x(t)$ remains constant. In fact we shall study in the next subsection the change in momentum associated with the second, imaginary term in the exponent of Eq. (4.35). According to Eqs. (4.36) and (4.45) below, this term is negligible compared to the first one for $\alpha t\ll 1$ since their ratio for such short times is of order $\hbar\xi(t)/d(t)\sim\alpha t|\mu|/6T=\alpha tO(N^{-1/2})$. Altogether, this means that Eqs. (4.39), (4.41), and (4.42) provide an example of a situation where the reduction of the state of the tested particle occurs long before achievement of the measurement process, and without

any energy cost, although this reduction is a consequence of the interaction with the apparatus.

2. Back reaction of the apparatus

The post-measurement Wigner function (4.29) of the tested particle involves a shift $xg^2\theta/\alpha$ of the momentum, that we have not yet discussed. The density matrix (4.31) accordingly exhibits oscillations within the small region in $x'-x''$ where $\langle x'|\rho(t)|x''\rangle$ is significant. Let us evaluate the order of magnitude of the corresponding average momentum:

$$\langle p\rangle=\frac{\hbar x g^2\theta}{\mu}=\frac{\hbar g^2\tilde{x}^2\theta}{\mu^2}\frac{\alpha}{\gamma}\frac{x\hbar}{\tilde{x}^2}\gamma\theta=\frac{\hbar}{\tilde{x}}\frac{\alpha}{\gamma}\gamma\theta O(N)\gg\frac{\hbar}{\tilde{x}}O(N). \tag{4.43}$$

The fluctuations of p, of order $\sqrt{\lambda\Delta}=\sqrt{d(\theta)}$, are also large, because the final state is localized in the x space. They are expressed by Eq. (4.30). Although large, the value of $\langle p\rangle$ would be ineffective if it were smaller than these fluctuations. However, the ratio of the shift to the fluctuations is given at the time θ by

$$\frac{\langle p\rangle^2}{\langle p^2\rangle-\langle p\rangle^2}=\frac{\hbar g^2x^2}{2\mu^2}\frac{T}{|\mu|}\left(\frac{\alpha}{\gamma}\right)^2\gamma\theta\gg\left(\frac{\alpha}{\gamma}\right)^2\gamma\theta O(N^{1/2}), \tag{4.44}$$

a large number in the thermodynamical limit.

The shift of $\langle p\rangle$ is therefore an important effect of the measurement. Let us look how this shift increases at short times. By using Eqs. (4.32) and (3.14) we find

$$x\xi(t)\sim\frac{g^2\alpha x t^3}{6},\quad \gamma t\ll\alpha t\ll 1. \tag{4.45}$$

At the time τ when decoherence is being achieved, the shift (4.45) is of order

$$x\xi(\tau)\sim\frac{\hbar g^2x^2}{\mu^2}\left(\frac{\mu}{T}\right)^3N^{-3/4}\frac{\hbar}{x}=\frac{\hbar}{x}O(N^{-5/4}). \tag{4.46}$$

We see that the change in $\langle p\rangle$ begins to be significant long after decoherence has taken place. This is consistent with the fact anticipated at the end of Sec. IV B 1, that very little is yet changed in the apparatus at the time τ.

The ratio (4.44) is of order 1 at a time τ_1 such that $[x\xi(\tau_1)]^2=d(\tau_1)$, where the shift becomes comparable with the fluctuation. Using Eqs. (4.36) and (4.45) one finds

$$\tau_1=\frac{\hbar}{T}N^{3/8},\quad \tau\ll\tau_1\ll\frac{1}{\alpha}. \tag{4.47}$$

Here as in τ, the time-scale is given by $\hbar/T$, but now the thermodynamical limit produces an enhancement.

The large shift of $\langle p\rangle$ can be attributed to the interaction process which takes place between the apparatus and the particle in the time interval τ_1, θ. During this period, the particle acts upon the lowest mode so as to drive it towards a state with a finite density of the condensate. In response this large effect produces a boost in the average momentum $\langle p\rangle$. The phenomenon can be traced back to the second term in the RHS of Eq. (2.33) for p. The factor $\chi(t)$ which enters this term describes a deterministic effect produced on the particle by the apparatus in contact with the bath. The increase of $|\langle p\rangle|$ is thus a cumulative effect of friction, which accompanies the rise of $|\langle X\rangle|$.

Altogether the interaction of the tested particle with the apparatus produces on this particle two effects. It first reduces the state, suppressing the off-diagonal terms in x during the time τ. Later on, between the times τ_1 and θ, it yields a large value to the average momentum without spreading the distribution in x, as seen in Eqs. (4.29) and (4.32). This second effect is probably connected with specific features of our model, namely, the choice of the apparatus and of its order parameter, and the form of the interaction Hamiltonian between the apparatus and the tested particle. It is also related to the existence of the continuous spectrum for x, which allows the rapid oscillations exhibited by Eqs. (4.31) and (4.35).

One may wonder whether the large value of $\langle p\rangle$ reached at the time θ is compatible with our hypothesis that the Heisenberg operator $x(t)$ has remained practically constant over the time interval $(0,\theta)$. We show in Appendix C that, contrary to $p(t)$, the equation of motion for $x(t)$ contains no systematic drift term arising from the coupling with the apparatus and hence with the bath; it involves only a noise term which does not affect much $x(t)$. We were thus entitled to neglect the variations of x between the times 0 and θ.

3. Einstein-Podolsky-Rosen experiment and speed of quantum signals

The above analysis allows us to discuss an experiment of the Einstein-Podolsky-Rosen (EPR) type. Let us suppose that the tested system consists of two particles denoted by 1 and 2. They do not interact for $t>0$, but they did interact in the past, which is reflected in an entangled wave function of the tested system at the initial time $t=0$:

$$\rho(0)=|\psi\rangle\langle\psi|,\quad |\psi\rangle=\sum_k\alpha_k|x_k\rangle|y_k\rangle, \tag{4.48}$$

where $|x_k\rangle$ are eigenfunctions of the operator x for the first particle, and $|y_k\rangle$ are arbitrary normalized, not necessarily orthogonal functions in the Hilbert space of the second particle. As indicated by Eq. (4.18) a small dispersion should be allowed for x so as to normalize $|x_k\rangle$. The measurement of the observable x is realized as above, namely the first particle couples through its operator x with the apparatus as expressed by Eq. (2.6). However, the second particle *does not* interact with the apparatus. Equations (4.35) and (4.39) take place as above with the slight difference that $\langle x'|\rho(t)|x''\rangle$, $\langle x'|\rho(0)|x''\rangle$ are matrices in the Hilbert space of the second particle. In particular, the reduction (collapse) of the initial state occurs on the time scale predicted by Eqs. (4.39) and (4.41), and it now provides

$$\rho(\tau)\simeq\sum_k |\alpha_k|^2|x_k\rangle|y_k\rangle\langle y_k|\langle x_k|. \tag{4.49}$$

The remarkable feature of quantum mechanics is that although only one subsystem is involved in the measurement, the total state of the tested system is reduced.

This general analysis can be illustrated by the standard example of two spins produced in the singlet state by the decay of an object with angular momentum 0. The initial density operator (4.48) is

$$\langle s_1 s_2|\rho(0)|s_1' s_2'\rangle=\frac{1}{2}\delta_{s_1+s_2,0}(\delta_{s_1,s_1'}\delta_{s_2,s_2'}-\delta_{s_1,s_2'}\delta_{s_2,s_1'}). \tag{4.50}$$

As above we perform measurement on the z component of the first spin only. For large N Eq. (4.39) leads to reduction $s_1=s_1'$ after a delay τ. This automatically implies $s_2=s_2'$:

$$\langle s_1 s_2|\rho(\tau)|s_1' s_2'\rangle\simeq\frac{1}{2}\delta_{s_1+s_2,0}\delta_{s_1,s_1'}\delta_{s_2,s_2'}. \tag{4.51}$$

Let us give some quantitative estimate of the characteristic reduction time scale τ. For a temperature $T\simeq 1\mathrm{K}$ we obtain

$$\tau=10^{-11}N^{-1/4}\ \text{s}. \tag{4.52}$$

Estimating $N\sim 10^{24}$ for a macroscopic system, we get $\tau\sim 10^{-17}$ s. For a distance of 1 m between spins this would lead to a speed of order 10^{17} m/s. Of course, this does not mean that there is an information transfer at this speed, but only a change in our knowledge through ρ of the quantum correlations of the two spins. Indeed, as we noticed at the end of Sec. IV B 1 the energy of the system is constant for times of order τ.

C. The common state of the tested particle and apparatus

In the evaluation of the common Wigner function (3.16) of the particle and the apparatus, we use for the transition kernel Φ the approximate expressions (4.9)–(4.12) as above. We recall that in the limit $\gamma\theta\gg 1$ the variance of p,

$$d(\theta)=\lambda(\theta)\Delta(\theta)=\frac{2g^2T}{\alpha^3}\gamma\theta=\frac{\hbar^2}{\tilde{x}^2}\gamma\theta O(N^{3/2}), \tag{4.53}$$

is large as well as $\lambda(\theta)=T/\alpha$. We saw at the beginning of Sec. IV B that this implies a loss of memory about the initial state $W(X_0,P_0;0)$ of the apparatus. In Eq. (3.16) we thus integrate over the initial Wigner function (3.17) and find

$$\mathcal{W}(X,P,x,p;\theta)=\int dp_0\Psi(X,P,x,p;\theta|x,p_0)w(x,p_0;0), \tag{4.54}$$

where

$$\Psi(X,P,x,p;\theta|x,p_0)=\frac{\hbar}{\lambda(\theta)\sqrt{2\pi d(\theta)}}\exp\left[-\frac{1}{2\lambda}\left(X-x\frac{g}{\alpha}\right)^2-\frac{1}{2\lambda}P^2-\frac{1}{2d(\theta)}\left(p-p_0-\xi(\theta)x+\frac{g}{\alpha}P\right)^2\right]. \tag{4.55}$$

For $d(\theta)\to\infty$ the expression (4.54) reduces to the product of the Wigner function $W_x(X,P)$ of the apparatus, defined by Eq. (4.17) for each possible value of x, and that $w(x,p;\theta)$ of the particle given by Eq. (4.29). This factorization merely expresses both the reduction of the initial state $w(x,p;0)$ into $w(x,p;\theta)$ and the registration by the apparatus of the classical random variable x.

The finite size of $d(\theta)$ in Eq. (4.55) entails a nonideality of the measurement that we now consider. We shall find it convenient to rewrite Eqs. (4.54) and (4.55) in two alternative forms. First we may use as indicated in Sec. III B 2 the density matrix representation only for the tested particle, while keeping the Wigner representation for the apparatus (intermediate Wigner function)

$$\mathcal{V}(X,P,x',x'';\theta)=\langle x'|\rho(\theta)|x''\rangle\frac{\hbar}{\lambda}\exp\left[-\frac{1}{2\lambda}\left(X-\frac{g}{\alpha}\frac{x'+x''}{2}\right)^2-\frac{1}{2\lambda}P^2-\frac{g}{\alpha}\frac{i}{\hbar}(x'-x'')P\right], \tag{4.56}$$

where the density matrix $\rho(\theta)$ reduces to a nearly diagonal form in x as expressed by Eqs. (4.31) and (4.35). And finally the same expression can be presented in the complete density matrix representation:

$$\langle X',x'|\mathcal{R}(\theta)|X'',x''\rangle=\langle x'|\rho(\theta)|x''\rangle\frac{1}{\sqrt{2\pi\lambda}}\times\exp\left[-\frac{1}{2\lambda}\left(\frac{X'+X''}{2}-\frac{g}{\alpha}\frac{x'+x''}{2}\right)^2-\frac{\lambda}{2\hbar^2}\left(X'-X''-\frac{g}{\alpha}(x'-x'')\right)^2\right]. \tag{4.57}$$

It is seen that the last small term $g(x'-x'')/\alpha$ in the RHS of Eqs. (4.56) or (4.57) quantifies the entanglement, that is, the degree of quantum correlations between the apparatus and the particle. Indeed, if in Eq. (4.57) we neglect this factor, we again find that the overall density matrix simply *factorizes* into the contributions studied above separately for the apparatus and the particle

$$\langle X',x'|\mathcal{R}(\theta)|X'',x''\rangle\simeq\langle X'|R_x|X''\rangle\langle x'|\rho(\theta)|x''\rangle, \tag{4.58}$$

where we check, using $\lambda=-\hbar T/\mu$ and $\mu=-\hbar\alpha$, that

$$R_x=\frac{1}{\sqrt{2\pi\lambda}}\int dX'dX''|X'\rangle\langle X''|\exp\left[-\frac{1}{2\lambda}\left(\frac{X'+X''}{2}-\frac{gx}{\alpha}\right)^2-\frac{\lambda}{2\hbar^2}(X'-X'')^2\right] \tag{4.59}$$

is the Gibbsian density operator of the apparatus for the pointer variable $x=(x'+x'')/2$. This factorization, together with the fact that $\rho(\theta)$ describes the reduced state of the tested particle, shows that the program set up in the introduction by Eqs. (1.1)–(1.8) is achieved, and that our model describes an ideal measurement provided the various conditions (4.9)–(4.12) are satisfied.

Let us turn to a more detailed discussion of the off-diagonal terms in the overall density matrix (4.57). As we already discussed in Sec. III E, Eqs. (4.56) and (4.57) are valid for the measurement of any Hermitian operator S of the tested particle, in particular those with a discrete spectrum. When dealing with this case, $\{|x\rangle\}$ should be directly substituted by the eigenbase of S. Then the analogue of Eq. (4.57) reads

$$\mathcal{R}(\theta)=\sum_i p_i|s_i\rangle\langle s_i|\otimes R_i(\theta)+\sum_{i\neq k}\exp\left[\frac{i\xi(\theta)}{2\hbar}(s_i^2-s_k^2)-\frac{d(\theta)}{2\hbar^2}(s_i-s_k)^2\right]\langle s_i|\rho(0)|s_k\rangle|s_i\rangle\langle s_k|\otimes R_{ik}, \tag{4.60}$$

where $p_i=\langle s_i|\rho(0)|s_i\rangle$ is the initial distribution of the measured quantity S of the tested particle, and where

$$R_{ik}=\frac{1}{\sqrt{2\pi\lambda}}\int dX'dX''|X'\rangle\langle X''|\exp\left[-\frac{1}{2\lambda}\left(\frac{X'+X''}{2}-\frac{g}{\alpha}\frac{s_i+s_k}{2}\right)^2-\frac{\lambda}{2\hbar^2}\left(X'-X''-\frac{g}{\alpha}(s_i-s_k)\right)^2\right] \tag{4.61}$$

satisfies

$$\mathrm{tr}_A R_{ik}=\exp\left[-\frac{\lambda g^2}{\alpha^2}\frac{1}{2\hbar^2}(s_i-s_k)^2\right],\quad \mathrm{tr}_A R_{ik}^2=\frac{\hbar}{2\lambda}=\frac{|\mu|}{2T}. \tag{4.62}$$

For $i\neq k$, R_{ik} has almost the same form as $R_{ii}\equiv R_i$, given for the continuous case by Eq. (4.59), but with slightly shifted off-diagonal matrix elements. Notice that this shift is due to entanglement between the apparatus and the particle.

It is seen from Eq. (4.60) that the off-diagonal terms of $\mathcal{R}$ are strongly suppressed with the exponential factor

$$\exp\left[-\frac{d(\theta)}{2\hbar^2}(s_i-s_k)^2\right]. \tag{4.63}$$

We again find the expected features for an ideal measurement, without the difficulties of the continuous spectrum discussed in Eq. (4.18) and Sec. IV B 2.

In order to understand the above suppression in terms of observables, let us imagine that one calculates the average in the state (4.60) of some observable $\mathcal{F}$ with matrix elements $\langle X'|\mathcal{F}_{ik}|X''\rangle$ in the Hilbert space of the particle and the apparatus:

$$\langle\mathcal{F}\rangle=\sum_i p_i\mathrm{tr}_A(\mathcal{F}_{ii}R_i)+\sum_{i\neq k}\exp\left[\frac{i\xi(\theta)}{2\hbar}(s_i^2-s_k^2)-\frac{d(\theta)}{2\hbar^2}(s_i-s_k)^2\right]\langle s_i|\rho(0)|s_k\rangle\mathrm{tr}_A(\mathcal{F}_{ki}R_{ik}). \tag{4.64}$$

To ensure that the second sum in the RHS of Eq. (4.60) is non-negligible in the thermodynamic limit, one needs

$$|\mathrm{tr}_A(\mathcal{F}_{ki}R_{ik})|\propto\exp\left[\frac{d(\theta)}{2\hbar^2}(s_i-s_k)^2\right] \tag{4.65}$$

at least for one pair $(i\neq k)$. Using

$$|\mathrm{tr}_A(\mathcal{F}_{ki}R_{ik})|^2\leqslant\mathrm{tr}_A\mathcal{F}_{ik}\mathcal{F}_{ki}\mathrm{tr}_A R_{ik}^2 \tag{4.66}$$

and Eq. (4.62), one can write the condition (4.65) as

$$\mathrm{tr}_A\mathcal{F}_{ik}\mathcal{F}_{ki}\geqslant\frac{2T}{|\mu|}\exp\left[\frac{d(\theta)}{\hbar^2}(s_i-s_k)^2\right]. \tag{4.67}$$

No general principle prohibits the existence of such an observable $\mathcal{F}$ which will satisfy Eq. (4.67). However, it is needless to mention that in the considered large N limit it would be quite pathological. So, under reasonable conditions one only has the diagonal term in Eq. (4.64).

The same conclusions as we just drew from the large size of $d(\theta)$ do hold in the continuous case given by Eq. (4.57), except for divergences associated with the continuous spectrum, and we can rewrite Eq. (4.57) in the form

$$\mathcal{R}(\theta)=\int dx p(x)|x\rangle\langle x|\otimes R_x(\theta), \tag{4.68}$$

where $p(x)=\langle x|\rho(0)|x\rangle$, which exhibits the form required by the ideal measurement conditions.

Let us finally discuss for illustration two examples.

1. Transformation of an eigenstate

If the initial state $|x_1\rangle$ of the particle is an eigenstate of coordinate, one has for the initial density matrix and Wigner function,

$$\langle x'|\rho(0)|x''\rangle=\langle x'|x_1\rangle\langle x_1|x''\rangle,\quad w(x,p;0)=\delta(x-x_1). \tag{4.69}$$

More precise normalization according to Eq. (4.18) provides a small width ϵ to the δ function in w and multiplies it by

$$\sqrt{8\pi\epsilon}\exp\left(-\frac{2\epsilon p^2}{\hbar^2}\right). \tag{4.70}$$

The state of the apparatus and particle for $\gamma\theta \gg 1$ will be

$$\mathcal{W}(X,P,x,p;\theta) = W_{x_1}(X,P)\,\delta(x-x_1), \tag{4.71}$$

where $W_{x_1}(X,P)$, given by Eq. (4.17), is the final Wigner function of the apparatus. As expected the measurement does not change the state of the particle; it leaves the particle and the apparatus uncorrelated, apart from the registration of the value x_1 in the latter.

2. Decay of initial superpositions: May Schrödinger kittens survive?

Let the initial state of the particle be a superposition of two different eigenvectors $|x_1\rangle$ and $|x_2\rangle$ of the coordinate operator, which appear with amplitudes φ_1, φ_2. For simplicity we will take these amplitudes real. At the initial time one has

$$\langle x'|\rho(0)|x''\rangle = \sum_{i,k=1}^{2} \varphi_i\varphi_k\langle x'|x_i\rangle\langle x_k|x''\rangle. \tag{4.72}$$

The corresponding Wigner function reads, within the regularization of Eq. (4.18) which provides a normalization factor (4.70),

$$w(x,p;0) = \sum_{i=1}^{2} \varphi_i^2\,\delta(x-x_i) + 2\varphi_1\varphi_2\,\delta\left(x-\frac{x_1+x_2}{2}\right)\cos\frac{p(x_1-x_2)}{\hbar}. \tag{4.73}$$

The physical interpretation of this formula is obvious. The first two, incoherent terms refer to localized states at x_1 and x_2. The cross term, which describes coherence, is localized half-way between x_1 and x_2; through its oscillations it is associated with quantum *interference*, a fact which is clear when one notices that it makes the Wigner function alternatively positive and *negative* along the line $y=(x_1+x_2)/2$. In other words, the initial state is highly non-classical.

Using Eq. (4.55) one finds the common Wigner function of the apparatus and the particle as

$$\mathcal{W}(X,P,x,p;\theta) = \sum_{i=1}^{2} \varphi_i^2 W_{x_i}(X,P)\,\delta(x-x_i) + \mathcal{W}_{\rm if}(X,P,x,p;\theta), \tag{4.74}$$

where the contribution from the interference term of $w(x,p;0)$ after integration over p_0 is

$$\mathcal{W}_{\rm if}(X,P,x,p;\theta) = 2\varphi_1\varphi_2\exp\left[-\frac{d(\theta)}{2\hbar^2}(x_2-x_1)^2\right] \times W_y(X,P)\,\delta(x-\overline{x}) \times\cos\left[\frac{(x_1-x_2)}{\hbar}\left(p+\frac{g}{\alpha}P-x\frac{g^2\theta}{\alpha}\right)\right], \tag{4.75}$$

with $\overline{x}=(x_1+x_2)/2$. The Wigner function (4.74) is a sum of two contributions $W_{\rm m}$ and $W_{\rm if}$. The first one, $W_{\rm m}$, is the expression (4.68) describing an ideal measurement. It is a positive function and just consists as expected of the incoherent mixture of two measured values x_1 and x_2 with classical probabilities φ_1^2 and φ_2^2.

The interference term $\mathcal{W}_{\rm if}$ is strongly suppressed due to the factor $d(\theta)$, which according to Eq. (4.52) yields in Eq. (4.75) an exponent of order $\gamma\theta N^{3/2}$ provided $|x_2-x_1|$ is not very small. The disappearance of the contribution of the interference term expresses that the initially existing *Schrödinger cats* are automatically *suppressed*, so that a classical interpretation can be given to the final result of the measurement.

For the continuous spectrum there can be cases where $|x_2-x_1| \ll |x_1|$. Since the initial superposition is then small, this situation can be called *Schrödinger kitten*. As seen from Eq. (4.75), the decay of such a state becomes less efficient as $|x_2-x_1|$ decreases. We may thus wonder whether Schrödinger kittens could partly survive in a non-ideal measurement process. We note, however, that the values x_1 and x_2 can be separated in a measurement only if the transition probability of Eq. (4.24) is negligible, which requires

$$\frac{\hbar g^2}{4T|\mu|}(x_2-x_1)^2 \gg 1. \tag{4.76}$$

Since the exponent in the damping factor of Eq. (4.75) which characterizes the decoherence,

$$\gamma\theta\frac{T^2}{\mu^2}\,\frac{\hbar g^2}{T|\mu|}(x_2-x_1)^2, \tag{4.77}$$

is much larger by a factor of order $\gamma\theta N$ than the exponent (4.76) which characterizes the robustness of the measurement, even the weakest Schrödinger kittens disappear in any measurement process. Any kitten that can be detected by distinguishing from each other the two interfering values of x has the same fate as a cat: it does not survive.

V. SUMMARY AND CONCLUSIONS

In the present paper we have studied a simple model in order to get better insight on the question of quantum measurement. Our purpose was to describe in full detail the dynamical process which takes place during the measurement, due to the coupling between the tested object and the appa-

ratus. As usual in this problem, we rely on the fact that the apparatus is a macroscopic object, so as to ensure the rapid decoherence which is needed to explain the classical nature of the interpretation of a measurement. However, in our approach, this type of irreversible behavior is merged with the idea that the apparatus should be able to evolve indifferently towards different macroscopic states. The selection of the outcome should be controlled by a small interaction with the microscopic observed object. The evolution of the apparatus should therefore be nonergodic, which we realize by identifying the pointer variable with an order parameter in a phase transition. The interaction with the measured system behaves as a small source which drives the actual value of the order parameter.

We wanted our model to fulfill the requirements on ideal measurements listed by the end of the Introduction. We also wished to be able to produce, in a consistent framework and from the first principles, using standard methods of quantum statistical mechanics, a full solution for the equations of motion which describe the dynamical process of measurement. This led us to choose an extremely simple model. Our apparatus is a noninteracting Bose gas in contact with a particle and energy reservoir, which can undergo a Bose-Einstein condensation. The pointer variable is the condensate density, which is sensitive to a coupling of the lowest-energy level of the gas with the tested microscopic system. To fix ideas we have chosen for this system a one-dimensional particle, the position of which is to be measured, but generalization to other systems is straightforward. When the interaction between the system and the apparatus is switched on, the condensate density relaxes to a value in one-to-one correspondence with the possible values of the position of the tested particle. We find that the randomness of this position is directly reflected by the statistics of the possible outcomes. The off-diagonal elements of the initial density matrix of the particle are suppressed by the process, and only classical probabilities enter the description of the correlations between the initial position of the particle and the pointer variable.

The various parameters of the model can be tuned, so as to explore the validity of the approximations which ensure that the measurement is ideal. The following requirements, which were expressed mathematically at the beginning of Sec. IV, are needed for ideality.

(a) The apparatus should be macroscopic. This large size plays a double role. On the one hand it ensures through decoherence the appearance of definite results in the measurement process. This means that in its final state the overall system composed by the apparatus and the tested system may be found in different mutually exclusive states with probabilities given by the initial distribution of the measured quantity. On the other hand, the macroscopic number of condensed bosons ensures a robust and accurate registration of the measured observable.

(b) Before the interaction with the tested system starts to act, the apparatus should be prepared in a state which is extremely sensitive to this interaction with the tested microscopic system. When the coupling is switched on at some initial time, the initial state of the apparatus becomes thus unstable, and it relaxes to another state determined by the measured object. This was achieved by letting the initial number of condensed Bosons be already large (as $\sqrt{N}$) but not yet macroscopic (as N).

(c) The relaxation time of the apparatus should be larger than the dynamical time, so that the equilibrium properties of the apparatus are not affected by the bath.

(d) The coupling constant g should be finite, so that the source term produces a macroscopic effect on the condensate although the interaction term in the Hamiltonian is not extensive.

(e) The duration of the measurement should be larger than the relaxation time of the apparatus.

(f) The statistical distribution of the measured quantity, here the position of the tested particle, should remain constant during the whole measurement process.

The model explains the collapse of the state of the tested particle as an effect of its coupling to the lowest level of the Bose gas. The thermal bath acts only *indirectly*, through the apparatus. We can therefore understand why this decoherence process eliminates the off-diagonal elements of the density matrix in the x basis and not in another basis. Indeed, the quantum noise due to the environment affects directly the pointer variable X of the apparatus, and the tested system feels it only because it is coupled to X through the term $-gxX$ of the Hamiltonian. The association of the decoherence with the measured observable is thus a natural outcome of the model. Thus in our model the decoherence is determined by the interaction between the tested system and the apparatus. This interaction is a tunable property and can be controlled. Let us notice that in the standard decoherence approach [9,11] this process depends on the interaction between the apparatus and its environment which is a hardly controllable quantity due to the very definition of (unobservable) environment.

Remarkably, the collapse of the state of the measured system takes place at the very beginning of its interaction with the apparatus, over a time $\tau=\hbar/(TN^{1/4})$ which has the usual features of a decoherence time, proportional to $\hbar/T$ and small in the thermodynamic limit of the apparatus. This time scale should be contrasted to the much larger time scale $1/\gamma$ associated with the relaxation of the apparatus. Once the state of the tested particle is reduced, very little has yet been changed in the macroscopic apparatus. It still takes a long time, of order $1/\gamma$, for the apparatus to reach its new equilibrium position determined by the system.

The reduction takes place for both the states of the tested particle and the apparatus, which remain only coupled by classical correlations at the end of the measurement. We have estimated the order of magnitude of the corrections to this ideal situation, and seen that they become extremely small as N increases. In particular the broadening of the density matrix of the particle around its diagonal elements is of order less than $N^{-3/2}$. Related to this aspect is the suppression of Schrödinger cats (interference effects of states located at two different positions), and even of Schrödinger kittens (similar states at two nearby positions), which cannot survive a robust measurement.

If the tested system involves degrees of freedom other than the one which is measured, our model shows that they remain unaffected by the process. As expected for an ideal

measurement, the density matrix of the system is then simply projected according to Eq. (1.7) by the interaction with the apparatus. In particular, in Einstein-Podolsky-Rosen setups, our analysis confirms that the measurement of one particle implies the collapse of the full system, thus also of constituents that are spatially separated. This means that quantum information is transferred at a large speed, which depends on the size of the apparatus, and can thus be extremely large.

Although oversimplified, our model has many generic features which are expected to occur in realistic measurement processes. However, taking advantage of the lack of interactions in the Bose gas, we have used an order parameter which is the density of the condensate. This is a peculiar property, since in a real Bose gas only the phase of the condensate, not its amplitude is an order parameter. A drawback of this situation is the fact that the back reaction of the apparatus on the system is stronger here than expected for a truly ideal measurement: Indeed we have seen that, apart from the reduction of the state to a nearly diagonal density matrix, a large amount of momentum is transferred by the apparatus to the particle. Another drawback of our model is that it simulates nonergodicity, rather than completely involving it. Although very sensitive to perturbations, our initial state is stable. It would be desirable to work out a more elaborate model where the initial state is metastable, and can be displaced towards several possible truly equilibrium states characterized by an order parameter, this displacement being controlled by the interaction with the system.

Other difficulties have been encountered above due to the continuity of the spectrum of the measured quantity. For instance, the large back reaction on the particle momentum is related to this continuity. However, such difficulties are not an artifact of our model as continuous spectra are known to cause difficulties in many other circumstances.

ACKNOWLEDGMENTS

R.B. is grateful for discussion with M. Gaudin. Th.M.N. acknowledges hospitality during a visit to the CEA Saclay. A.E.A. acknowledges support by NATO and INTAS and hospitality at the University of Amsterdam.

APPENDIX A: SOLUTION OF EQUATIONS OF MOTION FOR AN ARBITRARY MEASURED QUANTITY

Here we discuss the solution of equations of motion (2.28) and (2.31) for an interaction Hamiltonian

$$\hat{H}_S = -g\hat{X}\hat{S}, \tag{A1}$$

where $\hat{S}$ is an arbitrary Hermitean operator, which can in particular have discrete spectrum. In the present appendix we will distinguish operators by a hat. Let us rewrite Eq. (2.28) in terms of $\hat{X}$ and $\hat{P}$:

$$\frac{d}{dt}\hat{X} = \alpha\hat{P} - \gamma\hat{X} + \hat{f}_X(t), \quad \hat{f}_X(t) = \sqrt{\hbar\gamma}(\hat{b}^\dagger(t) + \hat{b}(t)),$$

$$\frac{d}{dt}\hat{P} = -\alpha\hat{X} - \gamma\hat{P} + g\hat{S} + \hat{f}_P(t),$$

$$\hat{f}_P(t) = i\sqrt{\hbar\gamma}(\hat{b}^\dagger(t) - \hat{b}(t)). \tag{A2, A3}$$

For a general operator of $\hat{S}$ no Wigner function exists in its standard sense, but for us it will be enough to operate with the intermediate Wigner function $\mathcal{V}(X,P,s',s'')$ defined in Sec. III B 2. Recall that this object is a Wigner function with respect to the lowest level of the apparatus (here the corresponding Wigner function can be defined), but is a density matrix in the eigenrepresentation $\{|s\rangle\}$ of $\hat{S}$.

To write the equations of motion in terms of $\mathcal{V}$ we will use a method described in [27], which replaces the operator equations (A2) and (A3) by the stochastic equation:

$$\begin{aligned}\partial_t\tilde{\mathcal{V}}(X,P,s',s'') = &-\partial_X([\alpha P - \gamma X]\tilde{\mathcal{V}}) + \partial_P([\alpha X + \gamma P]\tilde{\mathcal{V}}) \\ &- \partial_X(f_X\tilde{\mathcal{V}}) - \partial_P(f_P\tilde{\mathcal{V}}) - \frac{s'+s''}{2}g\partial_P\tilde{\mathcal{V}} \\ &+ \frac{ig}{\hbar}X(s'-s'')\tilde{\mathcal{V}}.\end{aligned} \tag{A4}$$

Here $f_X(t)$ and $f_P(t)$ behave as classical noises, which have exactly the same average and autocorrelation as the corresponding quantum quantities $\hat{f}_X(t)$ and $\hat{f}_P(t)$ after the symmetrization of Eq. (2.42). The true intermediate Wigner function $\mathcal{V}$ is obtained from $\tilde{\mathcal{V}}$ by averaging with respect to these noises $f_X(t)$ and $f_P(t)$,

$$\mathcal{V}(X,P,s',s'') = \langle\tilde{\mathcal{V}}(X,P,s',s'')\rangle. \tag{A5}$$

The first line in the RHS of Eq. (A4) is the standard drift contribution of the Liouville-Wigner equation. The last line in this equation is as well explained rather simply: This is just the result of the Wigner transformation (3.4) which was applied for the apparatus to the corresponding term $(i/\hbar)\langle s'|[H,\rho]|s''\rangle$ in the density matrix representation.

Now it is easy to see by direct substitution that the solution of Eq. (A4) reads

$$\begin{aligned}\tilde{\mathcal{V}}(X,P,s',s'') = &\exp\left[\frac{ig}{\hbar}(s'-s'')\int_0^t dt' X(t')\right] \\ &\times\delta(X - X(t))\delta(P - P(t)),\end{aligned} \tag{A6}$$

where $X(t)$ and $P(t)$ are the solution of the following c-number equations:

$$\frac{d}{dt}X = \alpha P - \gamma X + f_X(t), \tag{A7}$$

$$\frac{d}{dt}P = -\alpha X - \gamma P + g\frac{s'+s''}{2} + f_P(t). \tag{A8}$$

This means that one has the following solution for $\mathcal{V}$:

$$\mathcal{V}(X,P,s',s'') = \int dX_0 dP_0 \left\langle \exp\left[\frac{ig}{\hbar}(s'-s'')\int_0^t dt' X(t')\right] \times \delta(X-X(t))\,\delta(P-P(t)) \right\rangle \times W(X_0,P_0)\langle s'|\rho(0)|s''\rangle, \tag{A9}$$

where $W(X_0,P_0)$ is the initial Wigner function of the lowest level, and $X_0=X(0)$, $P_0=P(0)$ are initial values of $X(t)$, $P(t)$ which are inherent in Eqs. (A7) and (A8). Since the exact solution of these equations is available in the form of Eqs. (3.10) and (3.11), a little patience is sufficient to verify that $\mathcal{V}(X,P,s',s'')$ in Eq. (A9) coincides with the intermediate Wigner function corresponding to Eqs. (3.16) and (3.18) provided that one makes the identification $s'\to x'$, $s''\to x''$.

APPENDIX B: THE STATE OF THE PARTICLE AFTER MEASUREMENT

Here we discuss the postmeasurement evolution of the state of the particle. We will assume that at time $t=\theta$ the interaction between the particle and the apparatus has been instantaneously switched off. The switching is needed to ensure that the measurement will remain ideal, whereas its instantaneous character is taken for simplicity. Indeed, if the tested particle is interacting with the apparatus long enough, its coordinate will start to change due to its own Hamiltonian H_S, as well as due to interaction with the apparatus. Since the apparatus is itself interacting with the bath, sufficiently long interaction of the particle and apparatus will finally lead to relaxation of the particle towards certain steady state, which is independent of its initial state. This will violate the condition of ideality.

Therefore, the time θ was assumed to be much larger than $1/\gamma$, so that the apparatus has enough time to relax to its stationary state and monitor the results of measurement. On the other hand, θ was assumed to be small enough so that effects connected with change of x are not yet relevant. For $t\geqslant\theta$ the tested particle thus follows its free evolution, which is described by free Heisenberg equations:

$$p(t)=p(\theta), \tag{B1}$$

$$x(t)=x(\theta)+\frac{t-\theta}{m}p(\theta). \tag{B2}$$

Due to the instantaneous character of the switching, $p(\theta)$, $x(\theta)$ are those operator values which the particle reached during the interaction with the apparatus.

The dynamics from $t=0$ to $t=\theta$ is described by Eq. (4.32), and for $t>\theta$ one has

$$w(x,p;t)=\int dx_0 dp_0 w(x_0,p_0,\theta)\,\delta(p-p_0) \times \delta\left(x-x_0-\frac{t-\theta}{m}p_0\right). \tag{B3}$$

Let us now investigate $w(x,p;t)$ at $t>\theta$ for the initial state at $t=0$ given by Eq. (4.73):

$$w(x,p;t) = \sum_{k=1}^{2}\varphi_i^2\,\delta\left(x-x_i-\frac{t-\theta}{m}p\right) + 2\varphi_1\varphi_2\,\delta\left(x-\frac{t-\theta}{m}p-\frac{x_1+x_2}{2}\right) \times \exp\left[-\frac{d(\theta)(x_1-x_2)^2}{2\hbar^2}\right]\cos\left[\frac{(x_1^2-x_2^2)}{2\hbar}(p-\xi(\theta)x)\right]. \tag{B4}$$

It is seen that the last contribution to this equation is due to incomplete reduction. If it is not suppressed totally in the course of the measurement (because N is not sufficiently large), it will persist during further evolution, and might in principle be observed.

APPENDIX C: MOTION OF THE PARTICLE DURING MEASUREMENT

The purpose of the present Appendix is to discuss what happens if the change with time of the measured quantity x is not negligible, and if it cannot be treated as a constant of motion. In practice the conservation of the measured quantity is ensured only over short times. However the duration θ of the interaction with the apparatus should be sufficient so as to ensure registration, and there will arise a source of nonideality. Our purpose here is not to develop a full account of this nonideality, but just to display on which time-scales its presence is not relevant.

We will investigate Eqs. (2.28), (2.32), and (2.33) on times where changes of the measured quantity x become noticeable. To keep the situation free of instabilities, we will make a natural assumption that the tested particle is subjected to a confining potential. For simplicity this potential will be taken to be harmonic: $V(x)=m\omega_0^2x^2/2$.

1. Dynamics

The general solution of Eqs. (2.32) and (2.33) is obtained with help of Laplace transformation. Recall the following standard relations between functions $A(t)$, $B(t)$ and their Laplace transforms $\hat{A}(s)=\int_0^\infty dt\,e^{-st}A(t)$ and $\hat{B}(s)$ denoted in this appendix with a hat:

$$\mathcal{L}\left\{\int_0^t dt' A(t-t')B(t')\right\}=\hat{A}(s)\hat{B}(s),$$

$$\mathcal{L}\{\dot{A}\}=-A(0)+s\hat{A}(s), \tag{C1}$$

where $\dot{A}=(d/dt)A$. Thus the solution of Eqs. (2.32) and (2.33) reads

$$\hat{x}(s)=\frac{1}{m}\hat{f}(s)\,\hat{\eta}(s)+(\dot{x}(0)+sx(0))\hat{f}(s), \tag{C2}$$

$$\hat{f}(s)=\frac{1}{s^2+\omega_0^2-\frac{1}{m}\hat{\chi}(s)}, \tag{C3}$$

where

$$\hat{\chi}(s)=g^2\frac{\alpha}{\alpha^2+(\gamma+s)^2} \tag{C4}$$

is the Laplace transform of $\chi(t)$ [see Eq. (2.37)]. Finally one has

$$x(t)=x(0)\dot{f}(t)+\frac{1}{m}p(0)f(t)+\frac{1}{m}\int_0^t dt' f(t-t')\,\eta(t'), \tag{C5}$$

$$p(t)=p(0)\dot{f}(t)+mx(0)\ddot{f}(t)+\int_0^t dt'\dot{f}(t-t')\,\eta(t'), \tag{C6}$$

where $f(t)$ is the inverse Laplace transform of $\hat{f}(s)$.

Substituting Eq. (C5) into Eq. (2.29), we obtain the solution of the Heisenberg equation for the pointer variable:

$$X(t)=gx(0)\int_0^t dt' e^{-\gamma t'}\sin\alpha t'\dot{f}(t-t')+\frac{gp(0)}{m}\times\int_0^t dt' e^{-\gamma t'}\sin\alpha t' f(t-t')+\frac{1}{g}\mathcal{L}^{-1}\left\{\left(\frac{\hat{\chi}}{m}\hat{f}+1\right)\hat{\eta}\right\}. \tag{C7}$$

Its last term, where η is given by Eqs. (2.34)–(2.36) describes the effect of the noise $b,b^\dagger$ and the remanence of the initial conditions $X(0),P(0)$ of the apparatus. We recover Eq. (3.10) in the large m limit, for which $f(t)=t$ if $\omega_0 t\ll 1$. However, instead of being controlled by $x(0)$ only, the order parameter $X(t)$ now depends on the initial state of the particle through both $x(0)$ and $p(0)$. Since due to the uncertainty relation $\langle p(0)^2\rangle\langle x(0)^2\rangle\geqslant\hbar^2/4$ these quantities cannot have definite values simultaneously, $\langle X(t)\rangle$ is always fluctuating with the initial state of the particle. This violates conditions (1.4), (1.5) according to which if the tested system starts its evolution from one of the eigenstates of the measured quantity, then the result displayed by the apparatus can be definite. Moreover, Eq. (C7) shows that the statistics of $X(t)$ is governed by the statistics of $x(0),p(0)$, or equivalently by the full density operator $\rho(0)$ of the particle at the beginning of the measurement, including off-diagonal elements. It is the disappearance of the term in $p(0)$ in Eq. (3.10) which allows $X(0)$ at the end of the measurement to depend only on the diagonal elements $\langle x|\rho(0)|x\rangle$. As we shall see below, this occurs for $m\gg g^2\theta^3$, that is, for a sufficiently short duration of the measurement. Once again this shows that the short-time limit is a necessary condition for (nearly) ideal measurement of the coordinate.

Remember that the Heisenberg equation (2.28) for $X(t)$ depended only on the position operator $x(t)$, not on the momentum operator $p(t)$. Its solution given by Eq. (2.29) expresses the action of the particle on the apparatus as a *memory effect* depending on $x(t)$ at earlier times. Using the equation of motion (C5) for $x(t)$, we have here re-expressed $X(t)$ in terms of $x(0)$ and $p(0)$. The momentum of the particle thus came out from the elimination of the history of $x(t)$.

2. Short-time expansion

To investigate the short-time behavior of the tested particle in more detail, we adopt the following large-mass approximation:

$$\hat{f}(s)=\frac{1}{s^2+\omega_0^2}+\frac{1}{(s^2+\omega_0^2)^2}\frac{\hat{\chi}(s)}{m}, \tag{C8}$$

$$f(t)=\frac{\sin\omega_0 t}{\omega_0}+\frac{\Omega^3}{2\omega_0^3}\int_0^t dt'[\sin(\omega_0 t')-t'\omega_0\cos\omega_0 t']\sin(\alpha[t-t'])e^{-\gamma(t-t')}, \tag{C9}$$

where

$$\Omega=\left(\frac{g^2}{m}\right)^{1/3} \tag{C10}$$

is the characteristic frequency connected with the mass of the particle and the interaction with the apparatus. This frequency will be assumed to be the smallest characteristic frequency in our problem. Equation (C9) is a short-time expansion, which is valid for

$$t\ll\frac{1}{\Omega}=\left(\frac{m}{g^2}\right)^{1/3}. \tag{C11}$$

Expansion (C9), when substituted into Eq. (C5), produces the coordinate as a sum of two terms: The first one is due to the free motion in the potential $V(x)$, and the second one represents a deterministic correction arising from interaction with the apparatus. The interaction with the apparatus should be switched off before this term becomes comparable with the first term in Eq. (C9). In particular, if t is so small as $t\ll 1/\omega_0$ we obtain

$$f(t)=t+\frac{\Omega^3}{6}\int_0^t dt' t'^3\sin(\alpha[t-t'])e^{-\gamma(t-t')}. \tag{C12}$$

When this equation is substituted into Eqs. (C5) and (C6), it is seen that in all terms besides the second term in the RHS of Eq. (C6) the correction to $f(t)=t$ in Eq. (C12) can be dropped under the condition (C11). If one will take additionally: $x(0)\gg p(0)t/m$, Eqs. (C5), (C6), and (C12) produce Eq. (3.13).

It is important to notice that in this approximation there is a deterministic influence of the bath on the momentum. This is just friction, which may enhance or reduce $\langle p\rangle$. However, there is no such a systematic influence on the coordinate. It was therefore legitimate to assume in the bulk of this paper

that in the Heisenberg representation we have $x(t)\simeq x(0)$ in the time interval $0,\theta$ in spite of the large value reached by $\langle p\rangle$ at the time θ due to friction (Sec. IV B 2).

3. Relaxation of the tested particle

Although Eq. (C9) has been sufficient for our purposes, we will mention here how the full $f(t)$ behaves. This will allow us to understand the long-time behavior of the tested particle. One has to obtain the roots $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ of the following equation

$$[\alpha^2+(\gamma_k-\gamma)^2][\gamma_k^2+\omega_0^2]-\alpha\Omega^3=0 \tag{C13}$$

which provide

$$f(t)=\sum_{k=1}^{4} A_k e^{-\gamma_k t},\quad A_k=\mathrm{Lim}_{s\to-\omega_k}(s+\gamma_k)\hat{f}(s). \tag{C14}$$

Using the smallness of Ω the above roots can be obtained approximately as

$$\gamma_1=\gamma+i\alpha+\delta_1,\quad \gamma_2=\gamma_1^*, \tag{C15}$$

$$\gamma_3=i\omega_0+\delta_2,\quad \gamma_4=\gamma_3^*, \tag{C16}$$

where, after using $\alpha\gg\gamma,\alpha\gg\omega_0$, we find

$$\delta_1=\frac{\Omega^3}{2i}\frac{1}{(\gamma+i\alpha)^2+\omega_0^2}+O(\Omega^6)\simeq-\frac{\Omega^3\gamma}{\alpha^3}+\frac{i\Omega^3}{2\alpha^2}, \tag{C17}$$

$$\delta_2=\frac{\alpha\Omega^3}{2i\omega_0}\frac{1}{(i\omega_0-\gamma)^2+\alpha^2}+O(\Omega^6)\simeq\frac{\Omega^3\gamma}{\alpha^3}-\frac{i\Omega^3}{2\alpha\omega_0}. \tag{C18}$$

As seen from Eqs. (C15) and (C16) the particle has two relaxation times: $1/\gamma$ (which is also the relaxation time of the apparatus) and a much longer one $1/(\mathrm{Re}\,\delta_2)=\alpha^3/\Omega^3\gamma$. They are widely separated since $\alpha^3/\Omega^3\gg1$. After the time $\alpha^3/\Omega^3\gamma=\alpha^3 g^2/m\gamma$ all information about the initial state will be forgotten by the particle if it still interacts with the apparatus, and it will relax to its equilibrium state imposed by the bath temperature.

APPENDIX D: WEAK-COUPLING LANGEVIN EQUATION

In the present appendix we will derive from a consistent Hamiltonian formulation the Langevin equation (2.21) for the apparatus-bath dynamics. More information on weakly coupled (weakly damped) dissipative systems can be found in [24].

For simplicity we deal only with the lowest energy level of the apparatus, the Hamiltonian of which reduces to the chemical potential term $-\mu a^\dagger a=\hbar\alpha a^\dagger a$; as in Eq. (2.28) we drop the index 0. The dynamics of the excited state operators a_i would be obtained by substituting $\alpha+\omega_i$ to α. The considered mode is coupled to the thermal bath, which is also a noninteracting Bose gas having single-particle levels m with creation operators $\xi_m^\dagger$ and energies $\hbar\Omega_m$ (including the chemical potential of the bath). This bath is equivalent to the dense set of harmonic oscillators usually considered in quantum Brownian motion [20, 24, 25], but here the coupling should account for transfers of bosons between the bath and the apparatus proper, with conservation of the total particle number. We can thus describe the apparatus with its bath by the total Hamiltonian

$$H_{\mathrm{AB}}=\hbar\alpha a^\dagger a+\sum_m \hbar\Omega_m\xi_m^\dagger\xi_m+\sum_m \hbar(c_m a^\dagger\xi_m+c_m^*\xi_m^\dagger a). \tag{D1}$$

The resulting equations of motion in the Heisenberg picture are

$$\dot{a}=-i\alpha a-i\sum_m c_m\xi_m, \tag{D2}$$

$$\dot{\xi}_m=-i\Omega_m\xi_m-ic_m^* a. \tag{D3}$$

The initial state of the apparatus and the bath is assumed to be factorized at some remote initial time $t_0\ll0$. At that time the bath was in equilibrium at temperature β^{-1} while the apparatus was in an arbitrary state. Explicit integration of Eq. (D3), through

$$\xi_m(t)=\xi_m(t_0)e^{-i\Omega_m(t-t_0)}-ic_m^*\int_{t_0}^{t}dt'\,a(t')e^{i\Omega_m(t-t')}, \tag{D4}$$

allows us to eliminate the bath and to write a closed, exact equation of motion for the operator a. In the weak-coupling regime it is convenient to go to the rotating frame by means of the transformation

$$\tilde{a}(t)=a(t)e^{i\alpha(t-t_0)}. \tag{D5}$$

From Eqs. (D2), (D4), and (D5) we obtain

$$\frac{d\tilde{a}(t)}{dt}=-\sum_m |c_m|^2\int_{t_0}^{t}dt'\tilde{a}(t')e^{i(\alpha-\Omega_m)(t-t')}+\zeta(t), \tag{D6}$$

where $\zeta(t)$ appears as a Gaussian quantum noise defined by

$$\zeta(t)=-i\sum_m c_m\xi_m(t_0)e^{i(\alpha-\Omega_m)(t-t_0)}. \tag{D7}$$

This noise is characterized by the properties

$$[\zeta(t),\zeta^\dagger(t')]=\sum_m |c_m|^2 e^{i(\alpha-\Omega_m)(t-t')},\quad [\zeta(t),\zeta(t')]=0, \tag{D8}$$

$$\langle\zeta^\dagger(t')\zeta(t)\rangle=\sum_m \frac{|c_m|^2 e^{i(\alpha-\Omega_m)(t-t')}}{e^{\beta\hbar\Omega_m}-1},\quad \langle\zeta(t)\zeta(t')\rangle=0. \tag{D9}$$

We are interested in the thermodynamic limit for the bath, where the number of bath modes goes to infinity while their frequencies Ω_m tend to a continuum. For simplicity, we assume that the values Ω_m have a constant spacing $\Delta \to 0$, and that the coupling $|c_m|$ is a vanishingly small constant:

$$|c_m| = \sqrt{\frac{\gamma\Delta}{\pi}}. \tag{D10}$$

We then have

$$\sum_m |c_m|^2 e^{i(\alpha-\Omega_m)(t-t')} = 2\gamma\delta(t-t'). \tag{D11}$$

(If the spectrum of Ω_m is bounded, the δ-function stands for a narrow peak, with a width smaller than all the characteristic times of the problem if the range of Ω_m is sufficient.) Equation (D11) ensures that the retardation effect can be neglected in Eq. (D6), which becomes

$$\frac{d\tilde{a}(t)}{dt} = -\gamma\tilde{a}(t) + \zeta(t), \tag{D12}$$

and that the properties of the noise are simplified into

$$[\zeta(t), \zeta^\dagger(t')] = 2\gamma\delta(t-t'), \quad [\zeta(t), \zeta(t')] = 0, \tag{D13}$$

$$\langle \zeta^\dagger(t')\zeta(t)\rangle = \frac{\gamma}{\pi}\int \frac{d\omega\, e^{i\omega(t-t')}}{e^{\beta\hbar(\alpha-\omega)}-1}, \quad \langle \zeta(t)\zeta(t')\rangle = 0. \tag{D14}$$

Apart from the very short time-scale involved in the noise $\zeta(t)$, two time-scales, α^{-1} and γ^{-1}, enter the dynamical equations (D5), (D12) for the operator $a(t)$ generated by the apparatus-bath Hamiltonian $H_{\mathrm{A,B}}$. Other, longer time scales will also be induced by the interaction with the tested system. We assume that the *dynamical frequency* α *is the largest* characteristic frequency of the problem. The transformation (D5) then accounts for the fast, oscillatory motion of $a(t)$. The evolution of $\tilde{a}(t)$ takes place on larger time scales, of order γ^{-1} or more. We can thus expand Eq. (D14) for large α according to

$$\langle \zeta^\dagger(t')\zeta(t)\rangle \approx \frac{\gamma}{\pi}\int d\omega\, e^{i\omega(t-t')} \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!}\left(\frac{d}{d\alpha}\right)^n \frac{1}{e^{\beta\hbar\alpha}-1} \tag{D15}$$

$$= 2\gamma\sum_{n=0}^{\infty} \frac{i^n}{n!}\delta^{(n)}(t-t')\left(\frac{d}{d\alpha}\right)^n \frac{1}{e^{\beta\hbar\alpha}-1}. \tag{D16}$$

Since this expression will be integrated over a function of time which varies on the scale γ, its successive terms yield an expansion in powers of γ/α, and we can thus replace Eq. (D14) in the limit $\alpha \gg \gamma$ by

$$\langle \zeta^\dagger(t')\zeta(t)\rangle = 2\gamma\frac{\delta(t-t')}{e^{\beta\hbar\alpha}-1}, \quad \langle \zeta(t)\zeta(t')\rangle = 0. \tag{D17}$$

If the noise is redefined as

$$b_0(t) = \frac{1}{\sqrt{2\gamma}}\zeta(t)e^{-i\alpha(t-t_0)}, \tag{D18}$$

Eq. (D12) reduces to the equation of motion (2.21) for $a_0(t)$, and Eqs. (D13) and (D16) coincide with the conditions (2.23), (2.24), (2.25), on $b_0(t)$. We can likewise recover Eqs. (2.21)–(2.25) for i and $k \neq 0$ by adding ω_k to α and by coupling all the modes i of the apparatus with the modes m of the bath.

If we had not used the approximation $\alpha \gg \gamma$, we would have obtained the evolution of the average particle number $n_0(t) \equiv \langle a^\dagger(t)a(t)\rangle$ in the lowest level of the apparatus by integration of Eqs. (D5) and (D12), using Eq. (D14) instead of Eq. (D17). This yields

$$n_0(t) = n_0(t_0)e^{-2\gamma(t-t_0)} + \frac{\gamma}{\pi} \times \int \frac{d\omega}{(\omega-\alpha)^2+\gamma^2}\left|1-e^{-(\gamma+i\alpha-i\omega)(t-t_0)}\right|^2 \frac{1}{e^{\beta\hbar\omega}-1}, \tag{D19}$$

instead of Eq. (2.27). In particular, we find for arbitrary γ/α that $n_0(t)$ relaxes for large times towards

$$n_0(t) \to \frac{\gamma}{\pi}\int \frac{d\omega}{(\omega-\alpha)^2+\gamma^2}\frac{1}{e^{\beta\hbar\omega}-1}. \tag{D20}$$

It is only for $\gamma \ll \alpha$ that Eq. (D19) reduces to Eq. (2.27), and that $n_0(t)$ reaches the expected equilibrium Bose factor $1/(e^{\beta\hbar\alpha}-1)$ for large times.

Altogether the weak-coupling condition $\gamma \ll \alpha$, which will be enforced throughout this paper, ensures that the quantum Langevin equation (2.21) can be obtained from the apparatus-bath Hamiltonian (D1). Owing to this condition, while the coupling parameter γ governs the *dynamics* of the apparatus during its relaxation to equilibrium, the *equilibrium* properties of the apparatus remain unaffected by the presence of the bath. Note finally that the simplifying assumptions we made on the bath (equally spaced levels Ω_m, constant $|c_m|$) become irrelevant in the weak-coupling limit.

[1] N. Bohr, *New Theories in Physics* (International Institute of Scientific Cooperation, Paris, 1939).
[2] E. Wigner, Am. J. Phys. **31**, 6 (1963).
[3] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1980).
[4] J.A. Wheeler and W.H. Zurek, *Quantum Theory and Measurements* (Princeton University Press, Princeton, NJ, 1983).
[5] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955); reprinted in [4].

[6] R. Balian, Am. J. Phys. **57**, 1019 (1989).
[7] R. Balian and M. Véneroni, Ann. Phys. (N.Y.) **174**, 229 (1987).
[8] N.G. van Kampen, Physica A **153**, 97 (1988)
[9] D. Giulini *et al.*, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, New York, 1996).
[10] M. Namiki and S. Pascazio, Phys. Rep. **232**, 301 (1993).
[11] R. Omnès, *The Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1994); *Understanding Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1999).
[12] L. Rosenfeld, Commemoration issue, Suppl. Prog. Theor. Phys. 222 (1965).
[13] A. Daneri, A. Loinger, and G.M. Prosperi, Nucl. Phys. **33**, 297 (1962); reprinted in [4].
[14] K. Hepp, Helv. Phys. Acta **45**, 237 (1972).
[15] M. Cini, Nuovo Cimento Soc. Ital. Fis., B **73**, 27 (1983).
[16] J.S. Bell, Helv. Phys. Acta **48**, 93 (1975).
[17] Y. Nakamura, Yu.A. Pashkin, and J.S. Tsai, Nature (London) **398**, 786 (1999).
[18] J. Friedman, V. Patel, W. Chen, S.K. Tolpygo, and J.E. Lukens, Nature (London) **406**, 43 (2000).
[19] B. d'Espagnat, Nuovo Cimento, Suppl. **4**, 828 (1966); A. Fine, Phys. Rev. D **2**, 2783 (1970); A. Shimony, *ibid.* **9**, 2321 (1974).
[20] A.E. Allahverdyan and Th.M. Nieuwenhuizen, Phys. Rev. Lett. **85**, 1799 (2000); e-print cond-mat/0011389.
[21] H. Zeh, Found. Phys. **1**, 69 (1970); reprinted in [4].
[22] W.H. Zurek, Philos. Trans. R. Soc. London, Ser. A **356**, 1793 (1998); Phys. Today **44**, (October), 36 (1991).
[23] R. Balian, *From Microphysics to Macrophysics* (Springer, Berlin, 1992), Vol. II.
[24] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[25] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).
[26] M. Yanase, Phys. Rev. **123**, 666 (1961).
[27] A.E. Allahverdyan, R. Balian, and Th.M. Nieuwenhuizen, e-print cond-mat/0102255.