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Perturbative Casimir shifts of nondispersive spheres at finite temperature

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The quantum-electrodynamic Helmholtz free energy of binding, at temperature T, is determined perturbatively to order $(n\alpha)^2$ for atomic solid spheres of radius a, having dielectric constant $\varepsilon \simeq 1+4\pi n\alpha$ and magnetic susceptibility either $\mu=1$ or $\mu=1/\varepsilon \simeq 1-4\pi n\alpha$. Here n is the number density, and the atomic polarizabilty α is taken as independent of frequency. The perturbative shifts are regularized by disallowing atomic separations below some minimum λ; they are renormalized by dropping components proportional to the volume and surface area, and the renormalized shifts $\Delta B/(n\alpha)^2$ are expressed in terms of moments of the interatomic potential W at given T, quoted from the preceding paper. Such shifts are always dominated by (nominally) divergent components of order $-\hbar c/\lambda$, independent of T and a. For $kTa/\hbar c \gg 1$, the convergent terms are of order $-kT\ln(kTa/\hbar c)$; for $kTa/\hbar c \ll 1$, they are of order $-(kTa/\hbar c)^3(\hbar c/a)$ when $\mu=1$ and of order $-(kTa/\hbar c)^4(\hbar c/a)$ when $\mu=1/\varepsilon$. There is no compelling reason why these convergent terms should be exactly the same as the shifts determined by recent normal-mode summations; nevertheless, agreement is complete for $\mu=1/\varepsilon$ and almost complete for $\mu=1$.

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I. INTRODUCTION

Consider the quantum-electrodynamic (QED) Helmholtz free energy of binding—call it B—for a single macroscopic body modeled as a continuum, as opposed to the interaction between mutually disjoint bodies. Limited but worthwhile insights are obtainable from the simplified case where the material is optically dilute, say an atomic solid mimicked by a nondispersive (frequency-independent) dielectric function $\varepsilon=1+4\pi n\alpha$, with α is the electrostatic polarizability and n the number density of the atoms, and with $(n\alpha)^2 \ll 1$. Physically, constant α can be viewed as an approximation in problems where the important frequencies are far below typical atomic excitation frequencies Ω; formally, it corresponds to taking the *nondispersive limit* $\Omega \to \infty$ from the outset. To order $(n\alpha)^2$, that is, to leading order of perturbation theory [1], B as derived from QED is identically the same as the binding calculated from the dipole approximation to the properly retarded two-body potential $W(\rho)$ at separations ρ, defined so that the interatomic force is $-\nabla W$. Nondispersively and dispersively, the equivalence is demonstrated elsewhere ([2] and [3], cited as I and II, respectively).

Here we shall study B for nondispersive solid spheres of radius a, made of material whose magnetic permeability $\mu=1+4\pi n\chi$ may also differ from unity. Then the QED Hamiltonian reads [4]

$$H=H_{\mathrm{rad}}+H_{\mathrm{int}}, \quad H_{\mathrm{int}}=-\tfrac{1}{2}\int_{r<a} d^3r\{n\alpha\mathbf{E}^2(\mathbf{r})+n\chi\mathbf{B}^2(\mathbf{r})\}, \tag{1}$$

where H_{rad} is the Hamiltonian for the free (body-absent) electromagnetic field and H_{int} is treated as a perturbation. Specifically, we consider two scenarios, the first pure dielectric with $\chi=0$ and the second wholly artificial, but of some interest in field theory (see Sec. IV), with $\varepsilon\mu=1$, i.e., with $\chi=-\alpha$. The appropriate potentials have been derived from the atomic version of Eq. (1) in the preceding paper [5] (cited as III). We write them as

$$W=-\alpha^2 f(\rho,T), \tag{2}$$

naturally with different functions f in the two cases, and generally suppressing the argument T. Thus, to order $(n\alpha)^2$, QED yields

$$B=-\frac{1}{2}(n\alpha)^2\int_{r<a}\int_{r'<a} d^3r\, d^3r'\, f(\rho), \quad \rho=|\mathbf{r}-\mathbf{r}'|. \tag{3}$$

The underlying theory is developed at zero temperature and applied to several simple shapes in I and II, and is extended to dispersive spheres at $T>0$ elsewhere [6] (cited as V). We need several earlier conclusions, especially from II.

(i) Physically the most persuasive way to *regularize* B, i.e., to rid it of ambiguities stemming from the singularity of W at coincident $\mathbf{r}$ and $\mathbf{r}'$, is to restrict the integral (3) to $\rho > \lambda$, where $\lambda \ll a$ is a minimum separation enforced by the (nonperturbative) repulsion when the atomic wave functions begin to overlap. This is preferable to the more traditional prescription [7], which substitutes for $W(\rho)$ another potential $W(\rho,\tilde{\lambda})$ generated by exchanging only virtual photons having wave numbers $k<1/\tilde{\lambda}$, with the cutoff distance $\tilde{\lambda}$ often understood to be of order c/Ω. The point is that B is sensitive to ρ comparable to λ, where $W(\rho,\tilde{\lambda})$ is totally unlike the true $W(\rho)$. Contributions to B that would diverge in the hypothetical limit $\lambda \to 0$ we call (*nominally*) *divergent*. Terms that remain finite in this limit we call *convergent*; terms that vanish with λ we shall drop without further comment. Dispersion would cure no divergences, but would reduce their order in $1/\lambda$ (e.g., linear to logarithmic, quadratic to linear, etc.).

(ii) It proves convenient to write

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$$B = Vu + S\sigma + \Delta B, \tag{4}$$

where V and S are the volume and surface area of the body, u is the free energy of binding per unit volume in unbounded material, and σ the surface tension, i.e., the free energy per unit surface area of a half-space, while we call ΔB the *renormalized* [8] *Casimir free energy*. Nondispersively, the leading terms of u and σ are of orders $1/\lambda^4$ and $1/\lambda^3$, respectively, while for a nondispersive sphere, we shall find that $\Delta B/(n\alpha)^2$ subdivides into a (linearly) divergent term featuring only λ, but neither T nor a, plus a convergent function of T and a, which we call the *pure Casimir term*. The divergent part of ΔB will prove important as a probe for the differences between the physics of our approach and of mode summations; it has a long if intermittent history dating back to Candelas [9], and II discusses it in some detail.

(iii) The total (i.e., the regularized but unrenormalized) free energy B is dominated by its most divergent components $Vu + S\sigma$, which ensure that it always tends to induce collapse rather than expansion, regardless of ΔB, which is smaller by many orders of magnitude. From this point of view, the sign of ΔB is merely a detail, incapable of generating paradoxes whether it is positive or negative, and the same is true, *a fortiori*, of the pure Casimir term, whose magnitude is smaller still. Conversely, it bears stressing that *all three terms in* Eq. (4) *are essential: if one disregards* $Vu + S\sigma$, *then one can no longer make sense of the physics*.

(iv) By contrast, there is a genuine paradox in the fact that the perturbatively determined two-body potential W affords sensible estimates of all the nominally divergent contributions regardless of the value of $(n\alpha)^2$, whereas it can yield the far-subdominant pure Casimir term only when $(n\alpha)^2$ is small enough to make all many-atom (nonadditive) interactions truly negligible. This is precisely why the pure Casimir term represents such a challenge to nonperturbative field-theory techniques (e.g., to summations over normal modes); and the complications attending such techniques are, in turn, the reason why it is worth viewing them through the very different and relatively easily implemented perturbative calculations that we report. The motivations evidently run full circle.

(v) The nondispersive limit $\Omega \to \infty$ taken at the outset, as it is here, automatically entails $kT \ll \hbar\Omega$: all atoms are in the ground state. It also confines one to the retarded regime where $a \gg c/\Omega$, and similarly for any other important distances. Thus, for pure dielectrics at $T=0$, it formally reduces W to the Casimir-Polder potential $-23\alpha^2/4\pi\rho^7$ for *all* $\rho > \lambda$. Unfortunately, this enormous technical simplification badly misrepresents the nominally divergent components: in order to get them right, one requires the proper dispersive calculations spelled out (for dielectrics [10]) in II and V. However, subject to $kT \ll \hbar\Omega$, paper V eventually finds that dispersion leaves the pure Casimir term unaffected. Though this might perhaps have been suspected in advance, this author believes that only the dispersive calculation is compelling: *prima facie* plausible guesses cease to be plausible once it is realized that eventually one must accommodate all the dimensionless cross ratios between as many as *four* energy parameters, namely, $\hbar c/a$, $\hbar c/\lambda$, $\hbar\Omega$, and kT. Meanwhile, the results reported here could be viewed as well defined if somewhat artificial preliminaries, more useful by hindsight than can be warranted beforehand, to the complete theory in V.

(vi) It is found in I and spelled out further in II and V that u, σ, and ΔB for a sphere may be expressed in terms of the moments

$$\mathcal{J}_N(\rho) \equiv \int_\rho^\infty d\rho'\, f(\rho')\rho'^N \quad (N=2,3), \quad \Delta\mathcal{J}_5 \equiv \int_\lambda^{2a} d\rho\, f\rho^5. \tag{5}$$

($\mathcal{J}_5$ would diverge for $T>0$, when $f \sim kT/\rho^6$.) Thus,

$$u = -(n\alpha)^2 2\pi\mathcal{J}_2(\lambda), \quad \sigma = (n\alpha)^2 \frac{\pi}{2}\mathcal{J}_3(\lambda), \tag{6}$$

$$\begin{aligned} \frac{B}{(n\alpha)^2\pi^2} &= -\tfrac{1}{2}\int_\lambda^{2a} d\rho\,\rho^2 f\left\{\frac{16a^3}{3} - 4a^2\rho + \frac{\rho^3}{3}\right\} \\ &= -\frac{1}{2}\left\{\frac{16a^3}{3}[\mathcal{J}_2(\lambda) - \mathcal{J}_2(2a)]\right. \\ &\qquad \left. - 4a^2[\mathcal{J}_3(\lambda) - \mathcal{J}_3(2a)] + \frac{1}{3}\Delta\mathcal{J}_5\right\}, \end{aligned} \tag{7}$$

whence eventually

$$\frac{\Delta B}{(n\alpha)^2\pi^2} = -\frac{1}{6}\Delta\mathcal{J}_5 + \frac{8a^3}{3}\mathcal{J}_2(2a) - 2a^2\mathcal{J}_3(2a). \tag{8}$$

Notice that $\Delta\mathcal{J}_5$ depends on $f(\rho)$ only for separations ρ that *can* be realized in the sphere, while $\mathcal{J}_{2,3}(2a)$ depend only on separations that *cannot* be. Thus nominal divergences can enter ΔB only through $\Delta\mathcal{J}_5$.

Through f, all the moments are functions of T as well as of the integration limits. Since B and ΔB are shifts in the Helmholtz free energy, the corresponding shifts in the energy and entropy read

$$\Delta S = -\partial\Delta B/\partial T, \quad \Delta E = \Delta B + T\Delta S, \tag{9}$$

and similarly for S and E in terms of B. Asymptotic approximations to these shifts emerge naturally in the regimes

$$\text{low } T: \quad 4\pi Ta/\hbar c \equiv A \ll 1, \tag{10}$$

$$\text{high } T: \qquad A \gg 1. \tag{11}$$

On the other hand, we take

$$4\pi kT\lambda/\hbar c \equiv \Lambda \ll 1, \tag{12}$$

anticipating a near approach to the limit $\lambda \to 0$, as formally we always do. Physically, Eq. (12) is enforced by the realistic inequality $\hbar\Omega \ll \hbar c/\lambda$ (cf. II) combined with $kT \ll \hbar\Omega$.

The rest of this paper is laid out as follows. Section II quotes the requisite potentials from III and arranges them conveniently for evaluating Eq. (8). Sections III and IV deal with $\chi = 0$ and $\chi = -\alpha$, respectively. Each has one subsection on low and another on high temperatures; two awkward integrals are relegated to the Appendix. These sections also compare our ΔB, briefly, to expressions already reported in the literature for related, but not necessarily identical, quantities, by Nesterenko, Lambiase, and Scarpetta [11] (cited as NLS) for dielectrics and by Klich, Feinberg, Mann, and Revzen [12] (cited as KFMR) for $\varepsilon\mu=1$. Section V comments on such comparisons more critically, and summarizes our conclusions.

II. THE POTENTIALS AND THE FUNCTIONS $f(\rho)$

It is shown in III that the potentials read [13]

$$W(\text{dielectric})=-\alpha^2 f,\quad f=\frac{kT}{\rho^6}Q\coth(2\pi kT\rho)\quad(\chi=0),\tag{13}$$

$$W(\varepsilon\mu=1)=-\alpha^2 f,\quad f=\frac{2kT}{\rho^6}R\coth(2\pi kT\rho)\qquad(\chi=-\alpha),\tag{14}$$

featuring the homogeneous zero-degree differential operators

$$Q\equiv 3-3\rho\partial+\frac{5}{4}\rho^2\partial^2-\frac{1}{4}\rho^3\partial^3+\frac{1}{16}\rho^4\partial^4,\quad \partial\equiv\frac{\partial}{\partial\rho},\tag{15}$$

$$R\equiv 3-3\rho\partial+\frac{3}{2}\rho^2\partial^2-\frac{1}{2}\rho^3\partial^3+\frac{1}{8}\rho^4\partial^4.\tag{16}$$

The asymptotics of $W(\rho)$ hinge on the expansions

$$\coth(z\to 0)=\frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+\frac{2z^5}{945}-\frac{z^7}{4725}+\cdots\tag{17}$$

and

$$\coth(z\to\infty)=1+2\{\exp(-2z)+\exp(-4z)+\cdots\},\tag{18}$$

as do eventually those of ΔB as a function of Λ and A. As already stated, we shall drop terms that vanish with λ and thereby with Λ. Also, *for large A we shall drop terms exponentially small in the sense that they are at most of order* $\exp(-A)$. Much use will be made of

$$Q\begin{pmatrix}1/\rho\\1\\\rho\\\rho^3\\\rho^5\end{pmatrix}=\begin{pmatrix}(23/2)(1/\rho)\\3\\0\\0\\(11/2)\rho^5\end{pmatrix},\quad R\begin{pmatrix}1/\rho\\1\\\rho\\\rho^3\\\rho^5\end{pmatrix}=\begin{pmatrix}15/\rho\\3\\0\\0\\3\rho^5\end{pmatrix},\tag{19}$$

and of

$$(Q,R)\exp(-2n\rho)=(\text{quartics})\times O(\exp(-2n\rho)).\tag{20}$$

Thus, from Eqs. (13)–(19), it is easy to verify for $\chi=0$ that

$$f(2\pi kT\rho\ll 1)=\frac{23}{4\pi\rho^7}+\frac{11(2\pi)^5}{945}\frac{(kT)^6}{\rho}+O(\rho(kT)^8),\tag{21}$$

$$f(2\pi kT\rho\gg 1)=\frac{kT}{\rho^6}\{3+O((kT\rho)^4\exp(-4\pi kT\rho))\},\tag{22}$$

and for $\chi=-\alpha$ that

$$f(2\pi kT\rho\ll 1)=\frac{15}{\pi\rho^7}+\frac{12(2\pi)^5}{945}\frac{(kT)^6}{\rho}+O(\rho(kT)^8),\tag{23}$$

$$f(2\pi kT\rho\gg 1)=\frac{kT}{\rho^6}\{6+O((kT\rho)^4\exp(-4\pi kT\rho))\}.\tag{24}$$

III. PURE DIELECTRIC MEDIA

This is the case treated dispersively in V. Here we must evaluate the moments needed in Eq. (8) using f from Eq. (13).

A. Low temperatures

Start with $\mathcal{J}_2(2a)$. Define

$$c(z)\equiv\coth(z),\quad c^{(n)}(z)\equiv\frac{d^n c(z)}{dz^n};\tag{25}$$

then, a change of variables and repeated integrations by parts yield

$$\begin{aligned}\mathcal{J}_2(2a)&=kT(4\pi kT)^3\frac{1}{8}\int_A^\infty\frac{dz}{z^4}Qc(z)\\&=\frac{kT}{8a^3}\left[-\frac{A^3c^{(3)}(A)}{16}+\frac{A^2c^{(2)}(A)}{4}\right.\\&\qquad\left.-Ac^{(1)}(A)+c(A)\right],\end{aligned}\tag{26}$$

$$\mathcal{J}_2(2a)=\frac{1}{\pi a^4}\left\{\frac{23}{256}+\frac{7A^4}{11520}-\frac{11A^6}{60480}+O(A^8)\right\}.\tag{27}$$

The simplicity of Eq. (26) stems from the fact that its expected component proportional to $\int_A^\infty dz\, c(z)/z^4$ turns up with coefficient zero, for no deep reason that this author can see.

There is no such luck with $\mathcal{J}_3(2a)$, for which one finds analogously to Eq. (26) that

$$\begin{aligned}\mathcal{J}_3(2a)=\frac{A^3}{16\pi a^3}&\left\{-\frac{1}{2}\mathcal{M}_3-\left[\frac{Ac^{(3)}(A)}{16}-\frac{5c^{(2)}(A)}{16}\right.\right.\\&\left.\left.+\frac{5c^{(1)}(A)}{4A}-\frac{7c(A)}{4A^2}\right]\right\},\end{aligned}\tag{28}$$

$$\mathcal{M}_3 \equiv \int_A^\infty \frac{dz\, c(z)}{z^3}. \tag{29}$$

The rather awkward integral $\mathcal{M}_3$ is dealt with in the Appendix, where one finds

$$\mathcal{M}_3 = \frac{1}{3A^3} + \frac{1}{3A} - \frac{\zeta(3)}{\pi^2} + \frac{A}{45} - \frac{2A^3}{2835} + \cdots. \tag{30}$$

Substituting into Eq. (28) and expanding the square brackets by powers of A, one eventually finds

$$\mathcal{J}_3(2a) = \frac{1}{\pi a^3}\left\{\frac{23}{96} + \frac{\zeta(3)A^3}{32\pi^2} - \frac{11A^6}{45360} + O(A^8)\right\}. \tag{31}$$

Three points are worth stressing. (i) In Eq. (31) the term with A^3, i.e., with T^3, is clearly an odd one out. It stems wholly from $\mathcal{M}_3$. (ii) This term is actually independent of the argument of $\mathcal{J}_3(2a)$, whence the corresponding terms cancel from the difference $[\mathcal{J}_3(\lambda) - \mathcal{J}_3(2a)]$ in the unrenormalized free energy B given by Eq. (7). (iii) The same applies [14] to the term with A^4, i.e., proportional to T^4, in Eq. (27).

By contrast to $\mathcal{J}_{2,3}(2a)$, it is easy to determine $\Delta\mathcal{J}_5$. We substitute f directly from Eq. (21) into Eq. (5), observe that we may set $\Lambda \to 0$ in the second term (the difference vanishes with Λ), and find

$$\Delta\mathcal{J}_5 = \Delta\mathcal{J}_{50} + \Delta\mathcal{J}_{51} + O(A^8/a), \tag{32}$$

where

$$\Delta\mathcal{J}_{50} \equiv \Delta\mathcal{J}_5(T=0) = \frac{23}{4\pi}\left[\frac{1}{\lambda} - \frac{1}{2a}\right], \quad \Delta\mathcal{J}_{51} = \frac{11A^6}{18\,900\pi a}. \tag{33}$$

Finally, on substitution from Eqs. (27), (31), (32), and (33), Eq. (8) yields

$$\frac{\Delta B}{(n\alpha)^2\pi^2} = -\frac{23}{24\pi\lambda} + \frac{1}{\pi a}\left\{\frac{23}{96} - \frac{\zeta(3)A^3}{16\pi^2} + \frac{7A^4}{4320} - \frac{11A^6}{113400} + O(A^8)\right\}. \tag{34}$$

As far as this author knows, the only other comparable calculation for this system at finite T is that of NLS. At low temperatures, a quantity F, which they call simply *the* Casimir free energy, is given by their Eq. (3.35), whose zero-temperature limit reproduces the correct pure Casimir shift $(n\alpha)^2 23\pi/96a$ if one identifies their symbol Δn^2 with our $4\pi^2(n\alpha)^2$. In our notation, their F then coincides with what one gets from our Eq. (34) by dropping the divergent component (with $1/\lambda$), and dropping also the component with A^3.

B. High temperatures

Here it is the integrals $\mathcal{J}_{2,3}(2a)$ that become trivial, because they involve $f(\rho)$ only for $kT\rho \geqslant kTa \gg 1$. Disregarding exponentially small terms as we do, we can by virtue of Eq. (22) replace $f \to 3kT/\rho^6$ and find

$$\frac{8a^3}{3}\mathcal{J}_2(2a) - 2a^2\mathcal{J}_3(2a) = -\frac{5kT}{12}. \tag{35}$$

Remarkably, this is independent of a.

The fifth moment is not so easy. A change of variables and repeated integration by parts leads to

$$\begin{aligned}\Delta\mathcal{J}_5 &= kT\int_\lambda^{2a}\frac{d\rho}{\rho}Q\coth(2\pi kT\rho) \\ &= kT\int_{\Lambda/2}^{A} dz\left\{\frac{3c}{z} - 3c^{(1)} + \frac{5zc^{(2)}}{4} - \frac{z^2c^{(3)}}{4} + \frac{z^3c^{(4)}}{16}\right\} \\ &= kT\{3\Delta\mathcal{M}_1 + \mathcal{N}_1\},\end{aligned} \tag{36}$$

$$\mathcal{N}_1 \equiv \left[\frac{z^3c^{(3)}}{16} - \frac{7z^2c^{(2)}}{16} + \frac{17zc^{(1)}}{8} - \frac{41c}{8}\right]_{\Lambda/2}^{A} \simeq \frac{17}{\Lambda} - \frac{41}{8}, \tag{37}$$

$$\Delta\mathcal{M}_1 \equiv \int_{\Lambda/2}^{A}\frac{dz\,c(z)}{z} \simeq \frac{2}{\Lambda} + \ln\left(\frac{A}{\pi}\right) + \gamma, \tag{38}$$

where $\gamma \simeq 0.577$ is Euler's constant. Our approximations in Eqs. (37) and (38) are, as always, accurate up to terms that either vanish with Λ or are at most of order $\exp(-A)$. While $\mathcal{N}_1$ is elementary, $\Delta\mathcal{M}_1$ like $\mathcal{M}_3$ is awkward and is, likewise, relegated to the Appendix, from which Eq. (38) has quoted. Substituting Eqs. (35)–(38) into Eq. (8), one obtains

$$\frac{\Delta B}{(n\alpha)^2\pi^2} \simeq -\frac{23}{24\pi\lambda} - \frac{1}{2}kT\{\ln(4kT\alpha) + \gamma\} + \frac{7}{16}kT. \tag{39}$$

The first term, here as in Eq. (34), is just the nominally divergent part of the nondispersive zero-temperature shift. The pure Casimir part may coincide with Eq. (3.41) given by NLS for their F at high T, which in our notation reads

$$\frac{F}{(n\alpha)^2\pi^2} = -\frac{1}{2}kT\{\ln(kTa) + C\} \quad (\mathrm{NLS}), \tag{40}$$

where C is a constant. [Their equation (3.42) quotes $C = \ln(4) + \gamma - 7/8$ from a forthcoming paper by Nesterenko, Pirozhenko, and Bordag, which does produce coincidence.]

IV. MEDIA WITH $\epsilon\mu = 1$

Brevik and Kolbenstvedt [15] were the first to point out that sums over normal modes for the Casimir energies of such media are immune to certain complications and ambiguities afflicting realistic cases where the speed of light is different inside and out. Recent references can be traced through KFMR. By contrast to mode summations, the integrations in our approach are easy or difficult to much the same degree for media of both types.

The moments required by Eq. (8) must now be evaluated using f from Eq. (14). We proceed as in Sec. III, merely replacing Q with $2R$.

A. Low temperatures

Analogously to Eqs. (26) and (27), one finds

$$\mathcal{J}_2(2a)=\frac{kT}{4a^3}\left[-\frac{A^3c^{(3)}(A)}{8}+\frac{A^2c^{(2)}(A)}{2}-Ac^{(1)}(A)+c(A)\right]=\frac{1}{\pi a^4}\left\{\frac{15}{64}-\frac{A^4}{2880}-\frac{A^6}{5040}+O(A^8)\right\}. \tag{41}$$

But in $\mathcal{J}_3$ the coefficient of the integral $\mathcal{M}_3$ now turns out to vanish, leaving us with

$$\mathcal{J}_3(2a)=\frac{A^3}{8\pi a^3}\left\{-\frac{Ac^{(3)}(A)}{8}+\frac{5c^{(2)}(A)}{8}-\frac{3c^{(1)}(A)}{2A}+\frac{3c(A)}{2A^2}\right\}=\frac{1}{\pi a^3}\left\{\frac{5}{8}-\frac{A^6}{3780}+O(A^8)\right\}. \tag{42}$$

The fifth moment can be found just as before. Alternatively, comparing the potential (23) with Eq. (21), one sees that $\Delta\mathcal{J}_5$ can be written down at once by adapting the dielectric result from Sec. III B: all we need do is (i) multiply the zero-temperature part by $2(R/\rho)/(Q/\rho)=60/23$ and (ii) multiply the A^6 part by $2(R\rho^5)/(Q\rho^5)=12/11$. Either way one finds

$$\Delta\mathcal{J}_5=\Delta\mathcal{J}_{50}+\Delta\mathcal{J}_{51}=\frac{15}{\pi}\left[\frac{1}{\lambda}-\frac{1}{2a}\right]+\frac{12A^6}{18\,900a}+\cdots. \tag{43}$$

Combining these results, we obtain

$$\frac{\Delta B}{(n\alpha)^2\pi^2}=-\frac{5}{2\pi\lambda}+\frac{1}{\pi a}\left\{\frac{5}{8}-\frac{A^4}{1080}-\frac{A^6}{9450}+O(A^8)\right\}. \tag{44}$$

By contrast to Eq. (34), there is no term in A^3, because $\mathcal{M}_3$ happens to have vanished from $\mathcal{J}_3(2a)$. To our accuracy the pure Casimir part of this coincides with Eq. (28) of KFMR, whose ξ^2 equals our $4\pi^2(n\alpha)^2$.

B. High temperatures

Comparing the leading terms of Eqs. (22) and (24), one finds that $\mathcal{J}_{2,3}(2a)$ are now double the expressions for pure dielectrics, while

$$\Delta\mathcal{J}_5=2kT\int_{\lambda}^{2a}\frac{d\rho}{\rho}Q\coth(2\pi kT\rho)=2kT\{3\Delta\mathcal{M}_1+\mathcal{N}(\varepsilon\mu=1)\}, \tag{45}$$

$$\mathcal{N}(\varepsilon\mu=1)\equiv\left[\frac{z^3c^{(3)}}{8}-\frac{7z^2c^{(2)}}{8}+\frac{13zc^{(1)}}{4}-\frac{25c}{4}\right]_{\Lambda/2}^{A}\simeq\frac{24}{\Lambda}-\frac{25}{4}. \tag{46}$$

Combining these results, we obtain

$$\frac{\Delta B}{(n\alpha)^2\pi^2}=-\frac{5}{2\pi\lambda}-kT\{\ln(4kTa)+\gamma\}+\frac{5}{4}kT. \tag{47}$$

Apart from the nominally divergent term $-5/2\pi\lambda$, this coincides with Eq. (42) of KFMR.

V. SUMMARY, COMPARISONS, AND COMMENTS

(i) The perturbative result for the renormalized free-energy shift ΔB is Eq. (8), where the moments (5) are to be evaluated with the potential function f given by Eq. (13) or (14) for $\mu=1$ or $\mu=1/\varepsilon$, respectively. For arbitrary temperature this must be done numerically. Convenient approximations for $4\pi kTa/\hbar c\equiv A\ll 1$ are given by Eq. (34) or (44) and, for $A\gg 1$, by Eq. (39) or (47).

(ii) When comparing ΔB with expressions F from mode summations, we have spoken of coincidence (or the lack of it) rather than of agreement, because it is evident that by *renormalization* such methods mean something physically quite different from our own definition, which amounts only to distinguishing between ΔB and $Vu+S\sigma$. To see that the difference is significant, it suffices to recall that ΔB retains a nominal divergence proportional to $1/\lambda$ though independent of a, while the mode summations we have cited subtract counterterms chosen so as to make F wholly convergent. In particular, without further study one cannot exclude the possibility that they might unwittingly have assigned to F some convergent components of $Vu+S\sigma$.

Accordingly, what should surprise one is not the occasional discrepancy between F and the pure Casimir parts of ΔB, but the large measure of coincidence that does exist. Though coincidence happens to be complete for nondispersive media ($\Omega\rightarrow\infty$) at $T=0$ (cf. I), the scope for differences evidently widens with the proliferation of input parameters when the theory is extended to $T>0$ (as here) or to finite Ω (cf. II), or to both (cf. V). Clearly, it would be interesting to devise a renormalization scheme for mode summations whose physics is demonstrably the same as ours, and then to explore its precise relation to the schemes used hitherto; but pending such studies, this author believes that comparisons like those in Secs. III and IV cannot fruitfully be exploited much further.

(iii) Meanwhile, ambiguities are least troublesome for $\varepsilon\mu=1$. Numerically, the zero-temperature pure Casimir shift appears already in the pioneering paper by Brevik and Kolbenstvedt [15]: to order $(n\alpha)^2$, their Eq. (2.59) implies

$$\frac{0.092\,35}{2}\times 4\times\left[1+\frac{0.311}{4}\right]=0.1991$$

for the exact coefficient $5\pi/8 = 0.1989$ visible in our Eq. (44). [The closed form $5\pi/8$ follows trivially from the form $23\pi/96$ for $\mu = 1$: comparing Eqs. (21) and (23), we see that one need merely multiply by the ratio 60/23 of the potentials at $T = 0$.]

Sections IV A and IV B have already noted that, at low and high T, respectively, our pure Casimir terms coincide with F in KFMR's Eqs. (28) and (42). However, for diagnosing renormalization prescriptions, this is less significant than it might seem, because KFMR's expressions are intended as differences between one sphere of finite and another of infinite radius. On the other hand, their F at high T, being the same as the pure Casimir part of our Eq. (47), seems meant to carry more information than its component $-kT\ln[(kTa) \times (\text{arbitrary constant})]$, which is the most that such a difference could warrant.

(iv) For pure dielectrics, Sec. III B has registered that ΔB coincides with NLS's F at high T, but Eq. (34) shows that at low T it differs from F by the addend $-(n\alpha)^2(4\pi kTa)^3\zeta(3)/16\pi a$. In view of paragraph (ii) just above, we cannot at present say anything conclusive about the physics (if any) underlying this discrepancy, but it might be worth recalling, from the paragraph following Eq. (31), that the total (i.e., the regularized, but unrenormalized) free energy B has no terms proportional either to A^3 or A^4. Indeed, it is straightforward to verify that, at low T,

$$\frac{B}{(n\alpha)^2\pi^2} = (\text{divergent terms}) + \frac{1}{\pi a}\left\{\frac{23}{96} - \frac{11A^6}{113400} + O(A^8)\right\} \quad (\chi = 0). \tag{48}$$

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APPENDIX: TWO INTEGRALS

1. The low-temperature integral $\mathcal{M}_3$

Section III A requires, with $A \ll 1$,

$$\mathcal{M}_3 \equiv \int_A^\infty \frac{dz}{z^3}\coth(z) = \int_A^\infty \frac{dz}{z^3}\left[\frac{2}{(2^{2z}-1)} + 1\right]$$

$$= \mathcal{N}_3 + \frac{1}{2A^2}, \quad \mathcal{N}_3 \equiv 8\int_{2A}^\infty \frac{dx}{x^3(e^x-1)}. \tag{A1}$$

To determine $\mathcal{N}_3$ we cite, by hindsight and with the limit $\eta \to 0$ in mind, the standard integral

$$L \equiv \int_0^\infty \frac{dx\, x}{(x^2+\eta^2)(e^x-1)} = \frac{1}{2}\left[\ln\left(\frac{\eta}{2\pi}\right) - \frac{\pi}{\eta} - \psi\left(\frac{\eta}{2\pi}\right)\right]; \tag{A2}$$

differentiate under $\int dx \cdots$ with respect to η,

$$L' \equiv -\frac{1}{2\eta}\frac{\partial L}{\partial \eta} = \int_0^\infty \frac{dx\, x}{(x^2+\eta^2)^2(e^x-1)}; \tag{A3}$$

and expand on the right of Eq. (A2) to obtain

$$L'(\eta \to 0) = \frac{\pi}{4\eta^3} - \frac{1}{4\eta^2} + \frac{\pi}{48\eta} - \frac{\zeta(3)}{8\pi^2} + (\text{terms vanishing with } \eta). \tag{A4}$$

The crucial trick is to split $L' = \int_0^\infty dx \cdots = (\int_0^{2A} + \int_{2A}^\infty)dx \cdots \equiv L_1' + L_2'$ on the right of Eq. (A3). In the limit L_2' reduces to $\mathcal{N}_3/8$, whence

$$\mathcal{N}_3 = 8\lim_{\eta \to 0}\{L' - L_1'\}. \tag{A5}$$

To calculate L_1' we expand $x/(e^x-1)$ appropriately to small x and integrate term by term:

$$L_1' \equiv \int_0^{2A} \frac{dx\, x}{(x^2+\eta^2)^2(e^x-1)} = \int_0^{2A} \frac{dx}{(x^2+\eta^2)^2}\left[1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \cdots\right], \tag{A6}$$

$$L_1'(\eta \to 0) = \frac{\pi}{4\eta^3} - \frac{1}{4\eta^2} + \frac{\pi}{48\eta} - \frac{1}{24A^3} + \frac{1}{16A^2} - \frac{1}{24A} - \frac{A}{360} + \frac{A^3}{11340} + \cdots. \tag{A7}$$

On substituting from Eqs. (A4) and (A7) into Eq. (A5), and thence Eq. (A1), one obtains the approximation quoted in Eq. (30).

2. The high-temperature integral $\Delta\mathcal{M}_1$

Section III B requires, with $A \gg 1$ and $\Lambda \ll 1$,

$$\Delta\mathcal{M}_1 \equiv \int_{\Lambda/2}^A \frac{dz\,\coth(z)}{z} = \int_{\Lambda/2}^A \frac{dz}{z}\left[\frac{2}{(2^{2x}-1)} + 1\right]$$

$$= 2\Delta\mathcal{N}_1 + \ln\left(\frac{2A}{\Lambda}\right), \tag{A8}$$

$$\Delta\mathcal{N}_1 \equiv \int_\Lambda^{2A} \frac{dx}{x(e^x-1)}. \tag{A9}$$

Again we start from Eq. (A2), splitting it into

$$L = \left\{\int_0^\Lambda + \int_\Lambda^{2A} + \int_{2A}^\infty\right\}\frac{dx\, x}{(x^2+\eta^2)(e^x-1)} \equiv L_1 + L_2 + L_3. \tag{A10}$$

We drop L_3 because it is exponentially small, and recognize that L_2 reduces to $\Delta\mathcal{N}_1$ as $\eta \to 0$. Hence,

$$\Delta\mathcal{M}_1=\ln\left(\frac{2A}{\Lambda}\right)+2\lim_{\eta\to 0}(L-L_1)+O[\exp(-2A)]. \tag{A11}$$

To calculate L_1 we note that it involves only $x\ll 1$, expand $x/(e^x-1)$ as on the right of Eq. (A6), and integrate term by term:

$$L_1=\frac{1}{\eta}\tan^{-1}\left(\frac{\Lambda}{\eta}\right)-\frac{1}{4}\ln\left(\frac{\Lambda^2+\eta^2}{\eta^2}\right)$$
$$+(\text{terms vanishing with }\Lambda\text{ even when }\eta=0). \tag{A12}$$

Now expand this for small η, expand similarly on the right Eq. (A2), combine the results, and take the limit

$$\lim_{\eta\to 0}(L-L_1)=\lim\left\{\frac{1}{2}\left[\ln\left(\frac{\eta}{2\pi}\right)-\frac{\pi}{\eta}+\left(\gamma+\frac{2\pi}{\eta}\right)\right]-\frac{1}{\eta}\left[\frac{\pi}{2}-\frac{\eta}{\Lambda}+\cdots\right]+\frac{1}{2}\ln\left(\frac{\Lambda}{\eta}\right)\right\}=\frac{1}{\Lambda}+\frac{1}{2}\ln\left(\frac{\Lambda}{2\pi}\right)+\frac{\gamma}{2}. \tag{A13}$$

Substitution into Eq. (A11) then yields the approximation already quoted in Eq. (38).

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[1] The total free energy contains also terms of order $n\alpha$, but these stem from the self-energies (Lamb shifts) of the individual atoms and are irrelevant to the binding (cf. [3] and references given therein).

[2] G. Barton, J. Phys. A **32**, 525 (1999), cited as I.

[3] G. Barton, J. Phys. A **34**, 4083 (2001), cited as II.

[4] We use natural units $\hbar=1=c$, apart from occasional highlighting. For the Maxwell field, we use unrationalized Gaussian units, as in III, whereas I and II use rationalized units.

[5] G. Barton, preceding paper, Phys. Rev. A **64**, 032102 (2001), cited as III.

[6] G. Barton, J. Phys. A (to be published), cited as V.

[7] In this respect this author would amend the calculation reported in I.

[8] Section V will comment on other and significantly different prescriptions for regularizing and renormalizing. Between such options one must choose on physical grounds, depending on what, precisely, one wishes to calculate.

[9] P. Candelas, Ann. Phys. (N.Y.) **143**, 241 (1982).

[10] This author has not tried to combine dispersion with a magnetic in addition to a dielectric response, because these responses disperse so differently.

[11] V. V. Nesterenko, G. Lambiase, and G. Scarpetta, e-print hep-th/0006121v2, revised 14 March 2001, cited as NLS.

[12] I. Klich, J. Feinberg, A. Mann, and M. Revzen, Phys. Rev. D **62**, 045017 (2000), cited as KFMR.

[13] Recall again that everything from here on applies in the nondispersive limit: for instance, the pure dielectric f here is the same as f_{nd} in V.

[14] Both these points are peculiar to the nondispersive limit.

[15] I. Brevik and H. Kolbenstvedt, Ann. Phys. (N.Y.) **143**, 179 (1982).