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Multiplayer quantum games

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Recently the concept of quantum information has been introduced into game theory. Here we present the first study of quantum games with more than two players. We discover that such games can possess an alternative form of equilibrium strategy, one which has no analog either in traditional games or even in two-player quantum games. In these ''coherent'' equilibria, entanglement shared among multiple players enables different kinds of cooperative behavior: indeed it can act as a contract, in the sense that it prevents players from successfully betraying one another.

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Game theory is a mature field of applied mathematics. It formalizes the conflict between competing agents, and has found applications ranging from economics through to biology [1,2]. Quantum information is a young field of physics. At its heart is the realization that information is ultimately a physical quantity, rather than a mathematical abstraction [3]. It is known that various problems in this field can be usefully thought of as games. Quantum cryptography, for example, is readily cast as a game between the individuals who wish to communicate, and those who wish to eavesdrop [4]. Quantum cloning has been thought of as a physicist playing a game against nature [5], and indeed even the measurement process itself may be thought of in these terms [6]. Furthermore, Meyer [7,8] has pointed out that the algorithms conceived for quantum computers may be usefully thought of as games between classical and quantum agents. Against this background, it is natural to seek a unified theory of games and quantum mechanics [9–11].

Formally a *game* involves of a number of agents or *players*, who are allowed a certain set of moves or *actions*. The *payoff function* \$() specifies how the players will be rewarded after they have performed their actions. The i^{th} player's *strategy*, s_i, is her procedure for deciding which action to play, depending on her information. The *strategy space*, $S=\{s_i\}$, is the set of strategies available to her. A *strategy profile* $s=(s_1,s_2,\ldots,s_N)$ is an assignment of one strategy to each player. We will use the term *equilibrium* purely in its game theoretic sense, i.e., to refer to a strategy profile with a degree of stability; for example, in a Nash equilibrium no player can improve her expected payoff by unilaterally changing her strategy. The study of equilibria is fundamental in game theory [1]. The games we consider here are *static*: they are played only once so that there is no history for the players to consider. Moreover, each player has complete knowledge of the game's structure. Thus the set of allowed actions corresponds directly to the space of deterministic strategies.

Our procedure for quantizing games is a generalization of the elegant scheme introduced by Eisert *et al.* [12,13]. We reason as follows. Game theory, being a branch of applied mathematics, defines games without reference to the physical universe. However, quantum mechanics is a physical theory, and must be applied to a physical system. We therefore begin by creating a physical model for the games of interest. A very natural way to do this is by considering the flow of information; see Fig. 1(a). This classical physical model is then to be quantized. Our quantization procedure is the most natural one that meets the following requirements: (a) The classical information carriers (bits) are to be generalized to quantum systems (qubits), (b) these qubits are to be mutually entangled [14]; and (c) the resulting game must be a *generalization* of the classical game: the identity operator $\hat{I}$ should *correspond* to ''don't flip,'' and the bit-flipping operator $\hat{F}=\hat{\sigma}_x$ should *correspond* to ''flip'' [15], in the sense that when all the players restrict themselves to choosing from $\{\hat{F},\hat{I}\}$, then the payoffs of the classical game are recovered. To simultaneously meet requirements (b) and (c), we employ a pair of entangling gates as shown in Fig. 1(b), and insist that $\hat{J}$ commutes with any operator formed from $\hat{F}$ and $\hat{I}$ acting in the subspaces of different qubits. If we restrict ourselves to unitary, maximally entangling gates [16] that act symmetrically on ones and zeros, then we may specify $\hat{J}$ without loss of generality [17]: $\hat{J}=1/\sqrt{2}(\hat{I}^{\otimes N}+i\hat{F}^{\otimes N})$.

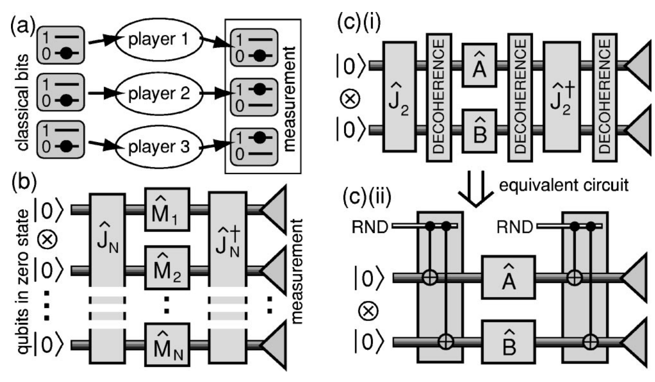

FIG. 1. (a) A physical model for a game in which each player has two possible actions: we send each player a classical two-state system (a bit) in the zero state. They locally manipulate their bit in whatever way they wish: under classical physics their choices are really just to flip, or not to flip. They then return the bits for measurement, from which the payoff is determined. (b) Our N-player quantized game. Throughout this paper, ''measurement'' means measurement in the computational basis, $\{|0\rangle,|1\rangle\}$. (c) The effect of introducing total *decoherence* of the quantum information. RND denotes a random classical bit, the vertical lines denote CONTROL-NOT.

The representation in Fig. 1(b) allows one to regard quantized games as simple quantum algorithms. The games we consider below could in fact be realized in a quantum computer possessing very few qubits (between one and three qubits per player, depending on the generality of the strategy space); NMR quantum computers are already adequate for this purpose [18]. Far more speculatively, one may envisage a market negotiating its trades quantum mechanically. Our results below suggest that it is *possible* that a well-designed quantum scheme could help to prevent certain types of negative "herd" behavior without the ad hoc imposition of distortative trading rules.

In comparing the quantum and classical games, the choice of strategy space is fundamental. The classical game is to be embedded in the quantum game, therefore the space should include playing the "classical" actions $\{\hat{I},\hat{F}\}$, but in principle we could choose any superset of this classical space. Previous studies have considered two-player games, and have employed strategy sets of limited generality. For example, in Ref. [8] Meyer explored the consequences of giving one player a full unitary strategy space while constraining the other to use only the "classical" space. Meyer has provided [7] an interesting interpretation of such one-sided games wherein the players are a quantum computer and its operator. In a second approach [13], Eisert *et al.* permitted both players the same strategy set, but introduced a constraint into that set [19], which amounted to permitting a certain strategy Q while forbidding the logical counter strategy. As one might intuitively expect, Q emerges as the ideal strategy. In contrast to these earlier approaches, throughout the present paper we allow all of our players to perform any action on their qubits that is quantum mechanically possible. This includes adjoining arbitrarily large ancillas, performing measurements and applying operations conditioned on the outcomes of those measurements. We believe this to be the most natural generalization of our classical model, where the only restrictions on the actions of the players were those imposed by classical physics. General quantum operations are represented by trace-preserving, completely-positive maps, and we denote the space of strategies corresponding to all such operations by S_{TCP}.

In traditional game theory, there is a fundamental distinction between so-called "pure" strategies, in which players choose their actions deterministically, and "mixed" strategies, which can involve probabilistic choices. An important consequence of adopting a general quantum model is that the players can implement any probabilistic strategy entirely deterministically through the use of ancillary qubits. For example, such qubits could function as a random number generator controlling the operations applied to the primary qubit. Even so, there is a subset of S_{TCP} that is in many ways analogous to the classical deterministic strategies, namely the set of all strategies that correspond directly to a unitary action. Strategies from this subset, which we label S_U, imply coherent manipulations of the local qubits, i.e., manipulations without the addition of ancillary qubits. Another way of identifying the set S_U is that they are precisely the strategies that do not destroy any of the entanglement introduced by the $\hat{J}$ gate [20].

In the multiplayer games below, we discover that equilibria for *all* of S_{TCP} can consist of strategies drawn *only* from S_U. We will refer to these special equilibria as *pure*, or *coherent*. They are *fundamentally* quantum mechanical, in that they disappear when the quantum correlations implicit in the entangled states are replaced with classical correlations, as in Fig. 1(c). In analogous two-player games (where both players are permitted S_{TCP}), it is impossible for "pure" equilibria [19] to occur; instead, equilibria exist only when the players choose to degrade the entanglement. Unsurprisingly therefore, those equilibria do persist in the Fig. 1(c) variant.

Consider the classical N-player Minority Game [21]. Here each player privately chooses between two options, say "0" and "1." The choices are then compared and the players who have made the minority decision are rewarded (by one point, say). If there is an even split, or if all players have made the same choice, then there is no reward. The structure of this game reflects many common social dilemmas, for example, choosing a route in rush hour, choosing which evening to visit an overcrowded bar, or trading in a financial market. We can immediately quantize this game as discussed above.

We begin with $N=3$. Does quantization introduce new equilibria? Yes: for example, the players can coordinate their actions simply by measuring their qubits and exploiting the classical correlations. However, such strategies are of limited interest in the present context, since they also function in the decoherent circuit of Fig. 1(c). In fact, we can quickly prove that all pure quantum strategies simply reduce to classical strategies. The most general pure strategy for player i can be written $s_i=\alpha_i^A(\beta_i^A i\sigma_x+\beta_i^B i\sigma_y)+\alpha_i^B(\gamma_i^A I+\gamma_i^B i\sigma_z)$, where all the α,β, and γ coefficients are real and $(\alpha_i^A)^2+(\alpha_i^B)^2=(\beta_i^A)^2+(\beta_i^B)^2=(\gamma_i^A)^2+(\gamma_i^B)^2=1$. Then simply by applying the J gates and deriving measurement probabilities in the standard fashion, we find that the β and γ terms disappear, yielding PROB(player 1 in minority)= $(\alpha_1^B\alpha_2^A\alpha_3^A)^2+(\alpha_1^A\alpha_2^B\alpha_3^B)^2$, and similarly for players 2 and 3. But these are just the probabilities that occur in the classical game when player i flips with probability $(\alpha_i^A)^2$, reducing the quantum game to the classical game.

Surprisingly, the situation is completely different in the four-player Minority Game. Classically, the players have no better strategy than to choose randomly between the 0 and 1 actions. The expected payoff for each player is then one eighth of a point, i.e., the game only 'pays out' half the time. But when we quantize the game, for the first time we discover fully coherent equilibria. One example [22] is the profile $s=(a,a,a,a)$, where $a=1/\sqrt{2}\cos(\pi/16)(I+i\sigma_x)-1/\sqrt{2}\sin(\pi/16)(i\sigma_y+i\sigma_z)$. Then the final state prior to measurement is, up to a global phase,

$$2^{-3/2}(|1000\rangle+|0100\rangle+|0010\rangle+|0001\rangle-|1110\rangle-|1101\rangle-|1011\rangle-|0111\rangle).$$

Thus, each player has expected payoff $\frac{1}{4}$, which is twice the performance of the classical game and the logical maximum for a cooperative solution. The reasoning below proves that the profile s is a true Nash equilibrium: even though the players are allowed the full generality of S_{TCP}, no player can improve her expected payoff by unilaterally defecting from *s*.

(i) (i) Note that the Minority Game has the special property that the same expected payoffs result whether or not we apply the second gate, $\hat{J}^\dagger$, prior to measurement. This can be seen by noting that $\hat{J}^\dagger$ transforms any basis vector $|abcd\rangle$ only within the subspace spanned by vectors $\{|abcd\rangle,|\overline{abcd}\rangle\}$, where $\overline{x}=NOT(x)$, which have the same payoff value.

(ii) Because of (i), we can focus attention on the state prior to $\hat{J}^\dagger$. This state has the property that measurement of any three of the four qubits will yield one of the eight outcomes, (000), (001), . . . ,(111), with *equal* probability. This must remain true regardless of any local action performed on the fourth qubit. Note that violation of this physical principle would permit superluminal communication.

(iii) Six of these eight outcomes are *unwinnable* by the fourth player: if, for example, measurement of the first three qubits yields (001), then neither a 0 or a 1 will put the fourth player in the minority. Thus, because of the equal weighting of the outcomes, her expected payoff cannot exceed $\frac{1}{4}$. *But this is just the payoff each player has with the originally proposed strategy profile.*

Thus in moving from the $N=3$ to the $N=4$ player Minority Game, a fundamentally nonclassical equilibrium becomes available. This equilibrium is optimal and fair: the game always pays out the maximum amount *and* the expected payoff for each of the players is the same. In the classical Minority Game, this *can* be achieved, but only by sharing additional classical information [23]. We are therefore led to ask, are there games with pure quantum equilibria whose performance *cannot* be matched classically *even* in the presence of free communication? Surprisingly, the answer is yes. To demonstrate, we exploit the concept of ‘‘dominant’’ strategies.

A player has a dominant strategy if this strategy yields a higher payoff than any alternative, *regardless* of the strategies adopted by other players. A rational player will inevitably adopt such a strategy, even if players freely converse before playing (unless we introduce some kind of binding contract, which amounts to switching to another payoff table entirely). If every player has a dominant strategy, then the game's inevitable outcome is the *dominant-strategy equilibrium*. The famous Prisoner's Dilemma, shown in Fig. 2(a), has the dominant-strategy equilibrium (‘‘defect,’’ ‘‘defect’’). As noted above, no maximally entangled two-player quantum game can have equilibria in the strategy space S_U. Thus, quantization of Prisoner's Dilemma removes the dominant-strategy equilibrium [12], but does not provide alternative coherent equilibria that might offer better payoffs.

To investigate the multiplayer case, we quantize the game of Fig. 2(b). We find that coherent equilibria *do* occur. The classically inevitable outcome, now written as $(\hat{F},\hat{F},\hat{F})$, becomes a Nash equilibrium – but other, radically superior equilibria emerge. For example, the profile $s=(\hat{I},\ 1/\sqrt{2}(\hat{\sigma}_x+\hat{\sigma}_z),\hat{\sigma}_x)$, with expected payoffs (5,9,5), is a Nash equilibrium (and is *strict* for players A and C: any unilateral deviation necessarily leads to *reduction* in their expected payoffs). Note that there is no in-principle difficultly with the asymmetry [24] of the profile, since in this game we are permitting players to communicate with each other prior to playing. The proof that this profile is a Nash equilibrium runs as follows.

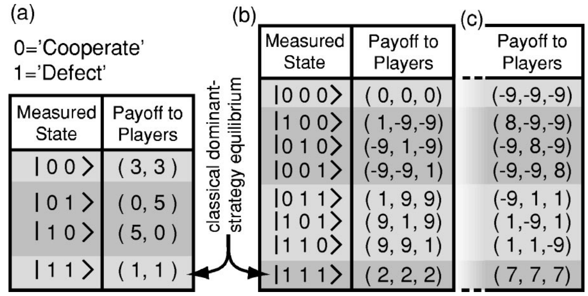

FIG. 2. Games possessing a dominant-strategy equilibrium: (a) The Prisoner's Dilemma. Each player reasons thus: 'If my partner were to cooperate, my best action would be to defect. If he were to defect, my best action is still to defect. So I have a *dominant* strategy: ‘‘always defect.’’ ' (b) A three-player dilemma game. Classically, each player has the dominant strategy ‘‘choose 1.’’ Consequently, each player's payoff is just two points, despite the existence of strategy profiles, such as 'choose 1 with probability 80%', where all the players have greater expected payoffs. (c) A game where quantum players do *less* well than their classical counterparts.

Let $|\psi\rangle=(\hat{I}\otimes 1/\sqrt{2}(\hat{\sigma}_x+\hat{\sigma}_z)\otimes\hat{I})\hat{J}|000\rangle$be the state after the actions of players A and B, and suppose that player C applies a general open quantum operation $\mathcal{R}$, i.e., a completely positive, trace-preserving map on density operators. We can, therefore, write $\mathcal{R}(\rho)=\Sigma_k\hat{M}_k\rho\hat{M}_k^\dagger$, with the restriction $\Sigma_k\hat{M}_k^\dagger\hat{M}_k=\hat{I}$ [25]. We may think of this expansion as representing a *k*-outcome measurement, where it is allowed to perform unitary operations conditioned on the outcome of the measurement. The state-change corresponding to outcome *k* is given by $|\psi\rangle\mapsto(\langle\psi|\hat{M}_k^\dagger\hat{M}_k|\psi\rangle)^{-1/2}\hat{M}_k|\psi\rangle$. Since player C only applies local operations, the most general $\hat{M}_k=\hat{I}\otimes\hat{I}\otimes\hat{C}_k$, where $\hat{C}_k$ is any 2×2 matrix. But it is then simple to show, by applying this $\hat{M}_k$ followed by the gate $\hat{J}^\dagger$, that player C's expected payoff is maximized only if $\hat{C}_k\propto\hat{\sigma}_x$. Thus, the only strategy for player C which maximizes her expected payoff for every one of her measurement outcomes is, up to global phase, σ_x. Similar arguments for players A and B verify that *s* is indeed a Nash equilibrium for the full quantum strategy space S_{TCP}.

We have seen that superior quantum coherent equilibria occur in some games (the three-player Dilemma and the 4 player Minority Game), but are absent in others (the three-player Minority, and any maximally entangled fair two-player game). But do quantum players always fare at least as well as their classical counterparts? No. Figure 2(c) is the payoff table for a game with a very high-performing dominant strategy; since all other outcomes have much lower total payoffs, this classically inevitable outcome is optimal. But in the quantized game this profile, now written as $(\hat{F},\hat{F},\hat{F})$, is no longer even a Nash equilibrium: any player can unilaterally improve her payoff by switching to $s=\sigma_y$, with severe consequences for the other players. Hence any equilibria in the quantum game will be inferior to the classical equilibrium: in this game, entanglement ''spoils'' classical cooperation.

To conclude, we have performed the first investigation of multiplayer quantum games, finding that such games can exhibit forms of pure quantum equilibrium that have no analog in classical games, or even in two-player quantum games. In the Minority Game, we found that the players were able to exploit entanglement to overcome the frustration in the classical variant, and so play the game ''perfectly.'' More dramatically, in our Dilemma game the quantum players can escape the classical Dilemma entirely: they can play cooperatively knowing that no player can successfully ''defect'' against the others. In this respect, quantum entanglement fulfills the role of a contract. A subsequent analysis [26] has examined the impact of a noisy environment on our Dilemma game.

S.B. was initially attracted to this topic by several inspiring discussions with Neil Johnson, who conjectured that alternative forms of equilibria would result from the complexity of the $N>2$ player minority game. We also acknowledge helpful discussions with Julia Kempe, Art Pittenger, Guy Steele Jr., and Vlatko Vedral. The authors were supported by EPSRC, the Rhodes Trust, and the EU QAIP project under Contract No. EC-IST-1999 11234.

[1] For an introduction see, e.g., E. Rasmusen, *Games and Information* (Blackwell, Oxford, 1995).

[2] M. A. Nowak and K. Sigmund, Nature (London) **398**, 367 (1999).

[3] For a review see, e.g., C. H. Bennett and D. P. DiVincenzo, Nature (London) **404**, 247 (2000).

[4] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[5] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).

[6] D. Deutsch, Proc. R. Soc. London, Ser. A **455**, 3129 (1999).

[7] D. A. Meyer, e-print http://xxx.lanl.gov/abs/ quant-ph/0004092.

[8] D. A. Meyer, Phys. Rev. Lett. **82**, 1052 (1999).

[9] P. Ball, Nature (London) Science Update, 18 Oct. 1999.

[10] I. Peterson, Science News **156**, 334 (1999).

[11] G. Collins, Sci. Am. Int. Ed. Jan. 2000.

[12] J. Eisert and M. Wilkens, e-print http://xxx.lanl.gov/abs/quant-ph/0004076.

[13] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. **83**, 3077 (1999).

[14] Without entanglement, the quantized form of Fig. 1(a) remains equivalent to the classical probabilistic game.

[15] Adopting the general bit-flip operator $F=\cos(\theta)\hat{\sigma}_x+\sin(\theta)\hat{\sigma}_y$ would simply result in a trivial rotation of the features that we discover.

[16] Since we are interested in purely multipartite entanglement, we call a (pure) state maximally entangled if it is equivalent via local unitary operations to the GHZ-type state, $1/\sqrt{2}(|00\cdots 0\rangle+|11\cdots 1\rangle)$.

[17] Any other $\hat{J}$ meeting these conditions would be equivalent, via local unitary operations, to our $\hat{J}$, and would therefore induce equilibria corresponding to ours.

[18] J. Du *et al.*, e-print quant-ph/0104087.

[19] S. C. Benjamin and P. M. Hayden, Phys. Rev. Lett. (to be published).

[20] If the action is not drawn from S_U, then the resulting state is necessarily not equivalent via local unitary operations to a GHZ state.

[21] For a survey of recent work on the classical game, see http://www.unifr.ch/econophysics/minority/.

[22] Interestingly, the complexity of the four player game is such that several Nash equilibria occur. Players may use criteria, such as maximum projection into the subspace of 'classical' moves $\{\hat{I},\hat{F}\}$, to establish a focal point, but this becomes a psychological question.

[23] If we generate a pair of random classical bits, and send duplicates of these bits to each player, then Nash equilibria exist wherein the players use these bits to decide which player should be in the minority.

[24] There is a focal symmetric strategy profile in this game, where each player adopts $s=1/\sqrt{2}(I+i\sigma_y)$. However this profile is not a Nash equilibrium unless the players are actually constrained to unitary moves. One could presumably construct a similar game wherein this strategy profile does form a full Nash equilibrium—e.g., the game obtained by replacing the payoff columns in Fig. 2(b) by (8,8,8), (1,−9,−9), (−9,1,−9), (−9,−9,1), (0,4,4), (4,0,4), (4,4,0), (1,1,1) (top to bottom) looks promising in this respect.

[25] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, Lecture Notes in Physics Vol. 190 (Springer-Verlag, Berlin, 1983).

[26] N. F. Johnson, Phys. Rev. A **63**, 020302(R) (2001).