Violations of local realism with quantum systems described by N-dimensional Hilbert spaces up to $N=16$

T. Durt,[1] D. Kaszlikowski,[2,3] and M. Żukowski[4]
[1]*FUND, Free University of Brussels, Pleinlaan 2, B-1050 Brussels, Belgium*
[2]*Instytut Fizyki Doświadczalnej, Uniwersytet Gdański, PL-80-952 Gdańsk, Poland*
[3]*Department of Physics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore*
[4]*Instytut Fizyki Teoretycznej i Astrofizyki, Uniwersytet Gdański, PL-80-952, Gdańsk, Poland*
(Received 18 January 2001; published 10 July 2001)

Predictions for systems in entangled states cannot be described in local realistic terms. However, after admixing some noise such a description is possible. We show that for two quantum systems described by N-dimensional Hilbert spaces (quNits) in a maximally entangled state the minimal admixture of noise increases monotonically with N. The results are a direct extension of those of Kaszlikowski *et al.* [Phys. Rev. Lett. **85**, 4418 (2000)], where results for $N \leqslant 9$ were presented. The extension up to $N=16$ is possible when one defines for each N a specially chosen set of observables. We also present results concerning the critical detectors efficiency beyond which a valid test of local realism for entangled quNits is possible.

DOI: 10.1103/PhysRevA.64.024101 PACS number(s): 03.65.Ta, 42.50.Dv

In early 1990s Peres and Gisin [1] have shown that if one considers certain *dichotomic* observables applied to a maximally entangled state of two particles described by an N-dimensional Hilbert space (quNits), the violation of local realism, or more precisely of the CHSH inequalities, survives the limit of $N\rightarrow\infty$ and is maximal there. However, for any dichotomic quantum observables the CHSH inequalities give violations bounded by the Tsirelson limit [2], i.e., it is limited by the factor of $\sqrt{2}$. Therefore, the question whether the violation of local realism increases with growing N was still left open.

It has been recently shown [3] that one indeed observes an increase with N of the discrepancy between quantum and local realistic description of two maximally entangled quNits observed via unbiased multiport beam splitters [4]. The results presented in [3] have been obtained via a numerical method of linear optimization and have been limited to $N=9$ [5,6].

In the present paper we extend the computations up to $N=16$. In the case of the method presented in [3] for $N\geqslant 10$ the computational time was prohibitively long. We avoid this problem here by a careful choice of a fixed set of two pairs of observables for each N. As a result one can avoid the time-consuming search for optimal sets of observables, which was a part of the computer program used in [3].

Another critical parameter for any Bell-type test is the threshold efficiency of the detector to make it an unconditionally valid test of local realism. The efficiency of a detector is usually defined as its probability to fire when the quantum particle enters it. The procedure used in [3] can be easily adapted to also handle the question of inefficient detectors. We report here the threshold values of efficiency for N up to 16. It decreases with N; however, the decrease is very slow.

Let us consider two quNit systems described by the mixed states in the form

$$\rho_N(F_N)=F_N\rho_{noise}+(1-F_N)|\Psi^N_{max}\rangle\langle\Psi^N_{max}|, \qquad (1)$$

where $|\Psi^N_{max}\rangle$ is a maximally entangled two quNit state, $\rho_{noise}=(1/N^2)\hat{I}$, and the positive parameter $F_N\leqslant 1$ determines the "noise fraction" within the full state. The threshold minimal F^{tr}_N, for which the state $\rho_N(F_N)$ allows a local realistic model, will be our numerical value of the strength of violation of local realism by the quantum state $|\Psi^N_{max}\rangle$. The higher F^{tr}_N is the higher the minimum noise admixture is that is required to hide the nonclassicality of the quantum prediction.

To overcome the mentioned Tsirelson limit one has to use nondichotomic observables. Here, as in the previous work, we limit ourselves to observables defined by unbiased multiport beam splitters.

Unbiased $2N$-port beam splitters [7] are devices with the following property: if one photon enters into any single input port (out of the N), its chances of exit are equally split between all N output ports. The unbiased multiports are an operational realization of the concept of *mutually unbiased bases*, see [8]. Such bases are "as different as possible" [9], i.e., fully complementary. The 50-50 beam splitter is the simplest member of the family.

One can always build an unbiased multiport with the distinguishing trait that the elements of its unitary transition matrix $\mathbf{U}^N$ are *solely* powers of the Nth root of unity $\gamma_N=\exp(i2\pi/N)$, namely $\mathbf{U}^N_{ji}=(1/\sqrt{N})\gamma_N^{(j-1)(i-1)}$. Devices endowed with such a matrix were proposed to be called Bell multiports [10].

Let us now imagine spatially separated Alice and Bob who perform the experiment of Eq. (1). The maximally entangled state of the two quNits

$$|\Psi^N_{max}\rangle=\frac{1}{\sqrt{N}}\sum_{m=1}^{N}|m,A\rangle|m,B\rangle, \qquad (2)$$

where, e.g., $|m,A\rangle$ describes a photon in mode m propagating to Alice, can be prepared with the aid of parametric down conversion (see [10]). The two sets of N phase shifters at the inputs of the multiports, which are denoted as N-dimensional

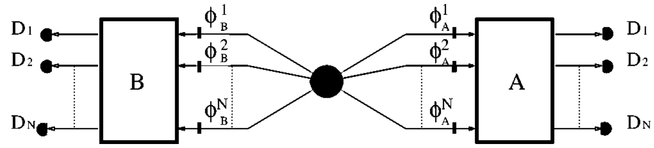

FIG. 1. The experiment of Alice and Bob with entangled quNits. Each of their measuring apparata consist of a set of N phase shifters (PS) just in front of a $2N$ port Bell multiport, and N photon detectors D_k, D_l (perfect, in the gedanken situation described here) which register photons in the output ports of the device. The phase shifters serve the role of the devices which set the free macroscopic, classical parameters which can be controlled by the experimenters. The source produces a beam-entangled two-particle state.

"vectors of phases" $\vec{\phi}_A=(\phi_A^1,\phi_A^2,\ldots,\phi_A^N)$ for Alice and $\vec{\phi}_B=(\phi_B^1,\phi_B^2,\ldots,\phi_B^N)$ for Bob, introduce phase factor $e^{i(\phi_A^m+\phi_B^m)}$ in front of the mth component of the initial state (2), where ϕ_A^m and ϕ_B^m denote the local phase shifts. Alice measures two observables A_1, A_2 defined by sets of phase shifts $\vec{\phi}_{A_1},\vec{\phi}_{A_2}$, whereas Bob measures two observables B_1, B_2 defined by sets of the phase shifts $\vec{\phi}_{B_1},\vec{\phi}_{B_2}$ (see Fig. 1).

Each set of local phase shifts constitutes the interferometric realizations of the "knobs" at the disposal of the observer controlling the local measuring apparatus which incorporates also the Bell multiport and N detectors. In this way the local observable is defined. Its eigenvalues refer simply to registration at one of the N detectors behind the multiport. The quantum prediction for the joint probability $P_{F_N}^{QM}(k,l)$ to detect a photon at the kth output of the multiport A and another one at the lth output of the multiport B calculated for the state (1) is given by

$$P_{F_N}^{QM}(k,l;\phi_A^1,\ldots\phi_A^N,\phi_B^1,\ldots\phi_B^N)$$
$$=\frac{1-F_N}{N}\left|\sum_{m=1}^{N}\exp[i(\phi_A^m+\phi_B^m)]\mathbf{U}_{mk}^N\mathbf{U}_{ml}^N\right|^2+\frac{F_N}{N^2}$$
$$=\frac{1-F_N}{N^3}\left(N+2\sum_{m>n}^{N}\cos(\Phi_{kl}^m-\Phi_{kl}^n)\right)+\frac{F_N}{N^2},\quad(3)$$

where $\mathbf{\Phi}_{kl}^m\equiv\phi_A^m+\phi_B^m+[m(k+l-2)](2\pi/N)$. The counts at a single detector, of course, do not depend upon the local phase settings: $P_{F_N}^{QM}(k)=P_{F_N}^{QM}(l)=1/N$.

The essential result of [3] is that quNits violate local realism more strongly than qubits in the following sense: the required minimal admixture of pure noise to the maximally entangled state, such that a local realistic description of the quantum predictions becomes possible, increases with N. This result has been obtained via numerical methods of linear optimization. Here we give a brief account of the method sending the reader for a more detailed description to [3].

It is well known (see, e.g., [11,14]) that the hypothesis of local hidden variables is equivalent to the existence of a (non-negative) joint probability distribution involving all four observables (A_1,A_2,B_1,B_2) from which it should be possible to obtain all the quantum predictions as marginals. Let us denote this hypothetical joint distribution by $P^{HV}(k_1,k_2,l_1,l_2)$, where k_1 and k_2 represent the outcome values for Alice′s measurement of observables A_1 and A_2, and l_1 and l_2 represent the outcome values for Bob's measurement of observables B_1 and B_2. In quantum mechanics one cannot even define such a distribution, since it involves mutually incompatible measurements. A given set of quantum predictions, here $P_{F_N}^{QM}(k_i,l_j|A_i,B_j)$, is reproducible by $P^{HV}(k_1,k_2,l_1,l_2)$, if and only if

$$P_{F_N}^{QM}(k_i,l_j|A_i,B_j)=\Sigma_{k_{i+1}}\Sigma_{l_{j+1}}P^{HV}(k_1,k_2,l_1,l_2),\quad(4)$$

where k_{i+1} and l_{j+1} are understood as modulo 2. The Bell theorem, within this context, says that there are quantum predictions, which for F_N below a certain threshold cannot be modeled by Eq. (4), i.e., there exists a critical F_N^{tr} below which one cannot have any local realistic model. The $4N^2$ linear equations (4) imposed on N^4 local hidden probabilities $P^{HV}(k_1,k_2,l_1,l_2)$ form the full set of necessary and sufficient conditions for the existence of local and realistic description of the experiment. This is a typical linear optimization problem with N^4+1 non-negative unknowns, $P^{HV}(k_1,k_2,l_1,l_2)$ and F_N, and $4N^2$ linear conditions (4).

In the previous work [3] an involved computer algorithm [12] was used to (i) solve the linear optimization problem for finding a minimal threshold F_N^{tr} for which, under specific chosen settings, Eq. (4) is satisfied, and (ii) find such settings for which (i) gives the highest possible value F_N^{tr} (the so-called "amoeba" procedure was used [13]). Since the task (ii) makes the computation, for high N, highly time consuming (since for each set of settings (i) has to be solved), the results of [3] reach only $N=9$.

Here we avoid this problem by dropping the point (ii) altogether. We search for F_N^{tr} for a specific single set of observables $A_1^{(N)}$, $A_2^{(N)}$, $B_1^{(N)}$, $B_2^{(N)}$ for each N. We have used the phase settings in the following form: $\vec{\phi}_{A_1}=(0,0,\ldots,0)$, $\vec{\phi}_{A_2}=(0,\pi/N,2\pi/N,\ldots,(N-1)\pi/N)$ for Alice and $\vec{\phi}_{B_1}=(0,\pi/2N,2\pi/2N,\ldots,(N-1)\pi/2N)$, $\vec{\phi}_{B_2}=-\vec{\phi}_{B_1}$ for Bob. For $N=2$ $\phi_{A_1}^2=0$, $\phi_{A_2}^2=\pi/2$, $\phi_{B_1}^2=\pi/4$, $\phi_{B_2}^2=-\pi/4$. These are the standard phases for the maximal violations of local realism in a two-qubit experiment (the first phase in each "phase vector" is irrelevant). For $N=3$, $\vec{\phi}_{A_1}=(0,0,0)$, $\vec{\phi}_{A_2}=(0,\pi/3,2\pi/3)$ and $\vec{\phi}_{B_1}=(0,\pi/6,\pi/3)$, $\vec{\phi}_{B_2}=(0,-\pi/6,-\pi/3)$ give maximal violation of local realism (a result of [3] discussed in [15]). For $N\geqslant4$ the phases were guessed. However, for up to $N=9$ these phases have given exactly the same results as that obtained with the second stage of optimization in [3]. Of course, we do not know if they are really optimal for $N\geqslant10$ because there is no data for comparison. Nevertheless, the violation of local realism obtained for these phases still grows with N as it is depicted in Fig. 2 and the growth has the same character as for $N\leqslant9$.

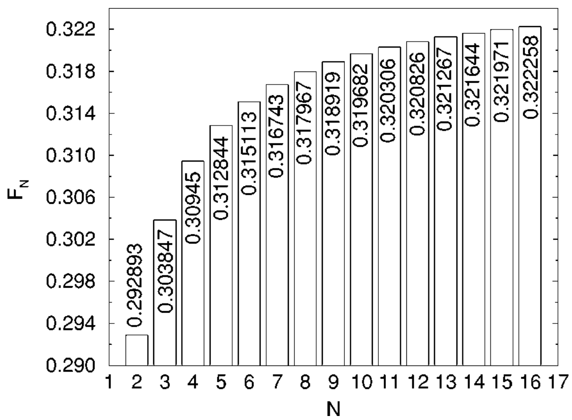

FIG. 2. Dependence of the critical noise admixture to the maximally entangled state (2) on the dimension of the Hilbert space of a single subsystem. For larger noise than shown here local realistic description exists. The increase of the value is interpreted as an objective measure of the increasing with N nonclassicality of pairs of entangled quNits.

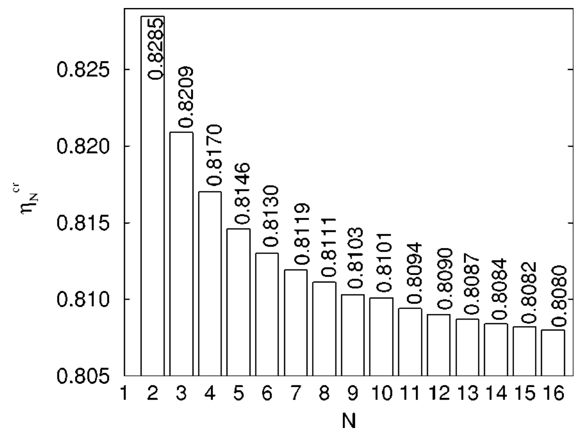

FIG. 3. Dependence of critical quantum efficiency of detectors η_N^{cr} versus the dimension of the Hilbert space N.

Another interesting question that may be raised here concerns the critical quantum efficiency of detectors below which there exists a local and realistic description of the system in the case without noise. It was showed [17] that for $N=2$ the critical efficiency equals $2\sqrt{2}-2(\approx 0.828)$. Taking into account that violation of local realism grows with N one may expect that for higher dimensions of Hilbert space the critical efficiency is lower than for two qubits. This problem has not been investigated in our previous work. Here we show that the presented method can be just as well applied to study this.

To this end it is necessary to modify the conditions (4) so as to take into account the probabilities of nondetection events, which are characterized by the quantum efficiency of detectors η $(0\leqslant\eta\leqslant 1)$ (for simplicity we assume that the efficiencies of all detectors are the same). This can be achieved as follows. To a local nondetection event we ascribe the additional value that differs from the values ascribed to the firings of detectors, say 0. In this case there are more local hidden probabilities and more linear constraints imposed on them for now the indices enumerating possible events extend from 0 to N (before the range was $1,\ldots,N$).

For non-ideal detectors, each endowed with identical inefficiency, the quantum probabilities $P^{QM}_{F_N,\eta}(k_i,l_j|A_i,B_j)$ of coincidences between detector k_i at Alice's side and detector l_j at Bob's side $(k_i,l_j\neq 0)$ while measuring observables A_i,B_j are equal to the corresponding probabilities with ideal detectors $(\eta=1)$ multiplied by η^2, i.e., $P^{QM}_{F_N,\eta}(k_i,l_j|A_i,B_j) =\eta^2 P^{QM}_{F_N}(k_i,l_j|A_i,B_j)$. The quantum probabilities $P^{QM}_{F_N,\eta}(0,l_j|A_i,B_j)$ and $P^{QM}_{F_N,\eta}(k_i,0|A_i,B_j)$ $(l_i\neq 0,k_i\neq 0)$ of events when one detector fails to fire at one of the sides of the experiment equals $(1/N)\,\eta(1-\eta)$ whereas the probability of the event when both detectors fail to fire $P^{QM}_{F_N,\eta}(0,0|A_i,B_j)$ is $(1-\eta)^2$. Replacing the left-hand side of Eq. (4) by appropriate quantum probabilities, i.e., $P^{HV}(k_i,l_j|A_i,B_j)=P^{QM}_{\eta,F_N}(k_i,l_j|A_i,B_j)$, one again obtains a linear optimization problem with respect to F_N, in which there are now $(N+1)^4$ local hidden probabilities and $4(N+1)^2$ linear constrains.

Due to the fact that η enters into equations quadratically it is not possible to optimize it by means of linear programing methods. The simple way of solving this difficulty is the following. One decreases the value of η (in our case by 1%) starting from $\eta=1$ and keeping the local phases fixed until the program returns $F_N=0$, which signals that for this efficiency there is already a local and realistic description. Of course, the critical efficiency applies only to the case of the observables chosen here. Once different observables or perhaps some nonmaximally entangled state (compare [16]) are chosen it may be lower. The results are depicted in Fig. 3. We see that critical efficiency decreases very slowly but continuously from the value obtained by Garg and Mermin [17] for two qubits $(N=2)$.

M.Z. and D.K. were supported by the KBN Grant No. 5 P03B 088 20; D.K. was supported by Fundacja na Rzecz Nauki Polskiej. T.D. is affiliated with the Fund for Scientific Research (FWO), Flanders, and is a member of the research group FUND (V.U.B.). This paper was written in the framework of the Flemish-Polish Scientific Collaboration Program No. 007.

[1] A. Peres, Phys. Rev. A **46**, 4413 (1992); N. Gisin and A. Peres, Phys. Lett. A **162**, 15−17 (1992).
[2] B. S. Tsirelson, Lett. Math. Phys. **4**, 93 (1980).
[3] Dagomir Kaszlikowski, Piotr Gnacinski, Marek Żukowski, Wieslaw Miklaszewski, and Anton Zeilinger, Phys. Rev. Lett. **85**, 4418 (2000).
[4] C. Mattle, M. Michler, H. Weinfurter, A. Zeilinger, and M. Żukowski, Appl. Phys. B: Lasers Opt. **60**, S111 (1995).
[5] The number of Bell inequalities grows extremely rapidly together with the increasing dimension N of Hilbert space [14,6]

and therefore their direct application becomes prohibitively difficult.

[6] Itamar Pitovsky and Carl Svozil, e-print quant-ph/0011060.

[7] A. Zeilinger, H.J. Bernstein, D.M. Greenberger, M.A. Horne, and M. Żukowski, in *Quantum Control and Measurement*, edited by H. Ezawa and Y. Murayama (Elsevier, New York, 1993); A. Zeilinger, M. Żukowski, M. A. Horne, H. J. Bernstein, and D.M. Greenberger, in *Quantum Interferometry*, edited by F. DeMartini and A. Zeilinger (World Scientific, Singapore, 1994).

[8] I. D. Ivanovic, J. Phys. A **14**, 3241 (1981); W. K. Wooters, Found. Phys. **16**, 391 (1986); J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **46**, 570 (1960).

[9] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).

[10] M. Żukowski, A. Zeilinger, and M. A. Horne, Phys. Rev. A **55**, 2564 (1997).

[11] A. Fine, J. Math. Phys. **23**, 1306 (1982).

[12] J. Gondzio, Eur. J. Oper. Res. **85**, 221 (1995); J. Gondzio, Computational Optimization and Applications **6**, 137 (1996).

[13] J. A. Nelder and R. Mead, Comput. J. (UK) **7**, 308 (1965).

[14] A. Peres, Found. Phys. **29**, 589 (1999).

[15] Dagomir Kaszlikowski, e-print quant-ph/0008086.

[16] P. H. Eberhard, Phys. Rev. A **47**, R747 (1993).

[17] A. Garg and N. D. Mermin, Phys. Rev. Lett. **49**, 901 (1982).