Reconstruction of SU(1,1) states

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We show how group symmetries can be used to reconstruct quantum states. The method we propose is presented in the context of the two-mode SU(1,1) states of the radiation field. In our scheme for SU(1,1) states, the input field passes through a nondegenerate parametric amplifier and one measures the probability of finding the output state with a certain number (usually zero) of photons in each mode. The density matrix in the Fock basis is retrieved from the measured data by the least-squares method after singular value decomposition of the design matrix followed by Tikhonov regularization. Several illustrative examples involving the reconstruction of a pair coherent state, a Perelomov coherent state, and a coherent superposition of pair coherent states are considered.

I. INTRODUCTION

The problem of the reconstruction of quantum states was first considered by Pauli [1] and Fano [2]. Since a quantum system is completely described by its density matrix, the task is essentially to reconstruct the density matrix of a system from information obtained by a set of measurements performed on an ensemble of identically prepared systems. To that end, the seminal work of Vogel and Risken [3] showed that for a single-mode *optical* field, the histograms of quadrature amplitude distributions measured by *homodyne* detection are just the Radon transform (or *tomography*) of the corresponding Wigner function. One can thus obtain the Wigner function by taking the inverse Radon transform of the data. Finally, the density matrix in the position representation is obtained from the Wigner function by Fourier transformation. This is the basis of *optical homodyne tomography* [3–6]. The technique was experimentally realized by Smithey *et al.* [4], who obtained the Wigner function and the density matrix of vacuum and quadrature-squeezed states of a mode of the electromagnetic field by using balanced homodyne detection. Much progress has been achieved in this field over the past few years [6]. It is now well known, for example, that one can determine the density matrix directly from the measured quadrature distribution without having to evaluate the Wigner function. Additionally, parallel tomographic schemes such as symplectic tomography [7] and photon number tomography [8] have been suggested for the reconstruction of quantum states of the light field which can even be multimode [9]. Other quantum systems for which reconstruction procedures were proposed include one-dimensional wave packets [10], harmonic and anharmonic molecular vibrations [11], motional states of atom beams [12], motional state of a trapped atom [13], Bose-Einstein condensates [14], cyclotron states of a trapped electron [15], atomic Rydberg wave functions [16], atoms in optical lattices [17], systems with a finite-dimensional state space (e.g., for spin) [18], and states in cavity QED [19,20]. Experimental reconstructions

were reported for electronic angular momentum states of hydrogen [21], vibrational quantum states of a diatomic molecule [22], and motional states of a single trapped atom [23]. Vasilyev *et al.* [24] have reported tomographic measurement of joint photon statistics of the two-mode quantum state produced in parametric amplification.

While extensive work has been done on states of a two-mode field, there are very many physical situations in which the state to be reconstructed has certain group symmetry. For example, in the process of downconversion, the two photons are produced together. In this case, the difference in the photon number in the two modes is conserved and the state has the symmetry property of the SU(1,1) group. Clearly, one could benefit considerably from the use of the group symmetry properties in the reconstruction of the state [25]. In a previous publication, one of us discussed how the underlying SU(2) symmetry of a state can be utilized very efficiently for its reconstruction [26]. In this paper, we consider reconstruction of states whose symmetry group is SU(1,1) [27]. The method we propose is demonstrated in the context of two-mode states of the radiation field. Note that propagation in free space, characterized by the Hamiltonian αp^2, is also an exmple of SU(1,1) symmetry and so is the more general Hamiltonian of the form $\alpha p^2 + \beta x^2 + \gamma xp$, which can be written as linear combinations of SU(1,1) generators. Thus SU(1,1) ideas will directly be applicable to, for example, atom optics. Furthermore, a variety of SU(1,1) coherent states for trapped ions [28] and for phonons [29] have been constructed.

The plan of the paper is as follows. In Sec. II, we present a group theoretic perspective of a general reconstruction procedure for quantum states. In Sec. III, we apply our method to reconstruct some important SU(1,1) states. The paper ends with concluding remarks in Sec. IV.

II. USING GROUP SYMMETRIES FOR STATE RECONSTRUCTION

Let us first recall the principles of photon number tomography. Several workers have suggested a procedure whereby the initial state of the radiation field described by the density matrix, $\rho^{(in)}$, is displaced by different amounts,

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$$\rho^{(in)} \rightarrow \rho^{(out)} = \mathcal{D}^\dagger(\alpha) \rho^{(in)} \mathcal{D}(\alpha),$$

$$\mathcal{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \tag{1}$$

One then measures the distribution of photons in the displaced field. The photon count in the output field is used to reconstruct the s-ordered distribution function of the input field. This method was very successfully used to measure the vibrational state of a trapped ion [23]. There is a related suggestion in the context of cavity QED, which yields the characteristic function of the radiation field [20]. In both of these situations, one measures atomic populations with rather high efficiency. Though a direct photon-counting measurement suffers from questions of poor efficiency of photodetectors, there exist several proposals on how to overcome the problem [30].

For the two-mode field with SU(2) symmetry, one can displace the state using the corresponding unitary operator for the SU(2) group. This has been shown to enable one to reconstruct the states of spin systems, states of polarization, etc. [26]. This is also closely related to a proposal in the context of Bose-Einstein condensates [14]. The displacement of the state is physically realized (say) by using external fields in the case of two-level atoms or spins. In the case of radiation fields, such a displacement is realized by optical components such as waveplates [31].

We next consider the case in which the underlying symmetry of the state is of the SU(1,1) group. In a two-mode realization of this group, the generators are

$$K_+ = a^\dagger b^\dagger, \quad K_- = ab, \quad K_z = (a^\dagger a + b^\dagger b + 1)/2, \tag{2}$$

where $a^\dagger a - b^\dagger b = \text{const} = q$ (say). Without any loss of generality, one can assume that $q \geq 0$. In that case, the vacuum state is given by the two-mode Fock state $|q,0\rangle$ with the property $K_-|q,0\rangle = 0$. The displacement operator for this group is the well-known squeezing operator parametrized by a complex quantity z:

$$\mathcal{S}(z) = \exp(z a^\dagger b^\dagger - z^* ab). \tag{3}$$

Acting on the $|0,0\rangle$ state, it produces the squeezed vacuum state

$$|z\rangle_0 = \mathcal{S}(z)|0,0\rangle. \tag{4}$$

It should be noted that even though we are dealing with the two-mode field, the underlying symmetry makes $\mathcal{S}(z)$ different from the product $\mathcal{D}(\alpha)\mathcal{D}(\beta)$ of the displacement operators. We can now proceed in the spirit of earlier constructions for the Heisenberg-Weyl and the SU(2) groups. We consider the operator defined by

$$\rho^{(out)} = \mathcal{S}^\dagger(z) \rho^{(in)} \mathcal{S}(z) \tag{5}$$

and the measurement of (say) q photons in mode a and *no* photons in mode b, i.e., the quantity
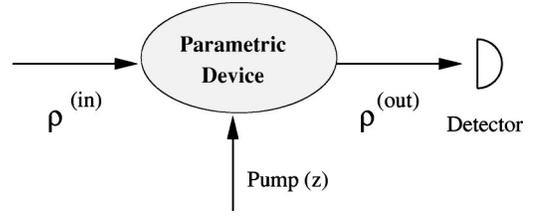
FIG. 1. Schematic of the reconstruction procedure.

$$
\begin{aligned}
p^{(out)}(q,0) &= \langle q,0 | \rho^{(out)} | q,0 \rangle \\
&= \langle q,0 | \mathcal{S}^\dagger(z) \rho^{(in)} \mathcal{S}(z) | q,0 \rangle \\
&= {}_q\langle z | \rho^{(in)} | z \rangle_q \equiv \mathcal{Q}(q,z),
\end{aligned} \tag{6}
$$

where $|z\rangle_q$ is defined in analogy to Eq. (4) with $|0,0\rangle$ replaced by $|q,0\rangle$. We would now like to demonstrate how measurements of $\mathcal{Q}(q,z)$ for a range of values of z can be used to reconstruct the input state $\rho^{(in)}$. In this case, as indicated in Fig. 1, $\rho^{(out)}$ can be obtained from $\rho^{(in)}$ by passing the input state through a nondegenerate parametric amplifier whose action is described by the Hamiltonian $H = \lambda a^\dagger b^\dagger + \text{H.c.}$, where λ is related to the nonlinear susceptibility. The operator $\mathcal{S}(z)$ is simply the evolution operator for this Hamiltonian with $z = i\lambda t$.

Using the disentangling theorem for $\mathcal{S}$ and substituting in the expression for $\mathcal{Q}(q,z)$, we can write this probability as a function of two auxiliary, *experimentally controlled* parameters,

$$y = \tanh^2 |z|, \quad \phi = i \ln\left(\frac{z}{|z|}\right). \tag{7}$$

After some algebra, one obtains

$$
\begin{aligned}
\mathcal{Q}(q,z) &\equiv \mathcal{Q}(q,y,\phi) \\
&= \frac{(1-y)^{q+1}}{q!} \sum_{m,n=0}^{\infty} \sqrt{\frac{(m+q)!(n+q)!}{m!n!}} \\
&\quad \times e^{i(m-n)\phi} y^{(m+n)/2} \rho_{n,m}(q).
\end{aligned} \tag{8}
$$

For the sake of clarity, we have used the notation $\langle n+q,n | \rho^{(in)} | m+q,m \rangle = \rho_{n,m}(q)$. At this point, we make the physically reasonable assumption that

$$\rho_{n,m}(q) \approx 0 \quad \text{for} \quad m,n > n_{max} \tag{9}$$

if n_{max} is suitably large [32]. Next we introduce the Fourier transform of the probability data with respect to the phase angle ϕ:

$$g_k(q,y) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{ik\phi} \mathcal{Q}(q,y,\phi), \tag{10}$$

and construct the quantity

$$f_k(q,y) = \frac{g_k(q,y) y^{-k/2}}{(1-y)^{q+1}}. \tag{11}$$

The construction is legitimate since the potentially singular points $y=0$ and $y=1$ are inaccessible to the experimenter. The point $y=0$ corresponds to "doing nothing" to the input state, whereas $y=1$ would correspond to $|z|$ (and hence either the pump amplitude or the duration of the experiment) being infinity.

The integration over ϕ yields a Kronecker δ function and one obtains a simple power series expansion for $f_k(q,y)$:

$$f_k(q,y) = \sum_{m=0}^{n_{max}-k} B_{mk}(q)\rho_{m+k,m}(q)y^m, \qquad (12)$$

where

$$B_{mk}(q) = \frac{1}{q!}\sqrt{\frac{(m+k+q)!(m+q)!}{m!(m+k)!}}. \qquad (13)$$

The task now is to obtain the density-matrix elements from tabulated values of $f_k(q,y)$. This can be done, in principle, by least-squares inversion [32,33].

A. Least-squares method

We write $f_k(q,y)$ in the form $f_k(q,y) = \sum_{j=1}^{M} a_j^{(M)}\phi_j(y)$, where $\phi_j(y) = y^{j-1}$ are the basis functions, $a_j^{(M)} = B_{j-1,k}(q)\rho_{j-1+k,j-1}(q)$ contain the unknown density-matrix elements, and $M = n_{max}-k+1$. Here the superscript (M) denotes that the coefficients depend in general on the number of basis functions included in the approximation. Let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_N$ be a set of points at which the values of $f_k(q,y)$ are measured. We denote by $\tilde{f}_i$ the measured value at $\tilde{y}_i$ with an error $\tilde{f}_i - f_k(q,\tilde{y}_i)$. It is generally assumed that the error at different points is uncorrelated. The design matrix G is an $N\times M$ matrix whose ijth element is given by $G_{ij} = \phi_j(\tilde{y}_i)$. We introduce two vectors $\vec{a} = \{a_1,a_2,\ldots,a_M\}$ and $\vec{b} = \{\tilde{f}_1,\tilde{f}_2,\ldots,\tilde{f}_N\}$. In the least-squares method, the coefficients a_j are determined by minimizing the quantity $\chi^2 = |G\vec{a}-\vec{b}|^2$.

Although the method of least squares is used extensively in the literature, it will give meaningful values for the coefficients $\rho_{m+k,m}(q)$ only for small values of m. This is so because for large values of m, the corresponding normal equations become ill-conditioned. Hence we cannot expect to solve them unless very high precision arithmetic is used. Even then, a slight change in the data (due, for example, to round-off error) may change the solution significantly. This ill-conditioning can be traced to the fact that for large values of m, the basis functions y^m are not really independent in the sense that there will be little difference between terms of (say) y^9 and y^{10} if the precision in the measured data is unable to resolve it. In such cases, one usually proceeds with the singular value decomposition (SVD) of the design matrix in which one works directly with the design matrix G rather than with G^TG (as in the least-squares method without SVD). Thus the ill-conditioning becomes very reduced. The design matrix G is written in the form $G = U\Sigma V^T$, where U is an $N\times M$ matrix, Σ is an $M\times M$ diagonal matrix with diagonal elements σ_1, σ_2, $\ldots$, σ_M, and V is an $M\times M$

orthogonal matrix so that $U^TU = V^TV = VV^T = I_M$, the $M\times M$ unit matrix. The matrix U consists of M orthonormalized eigenvectors associated with the M largest eigenvalues of GG^T, and the matrix V consists of the orthonormalized eigenvectors of G^TG. The diagonal elements of Σ are the non-negative square roots of the eigenvalues of G^TG and are called the *singular values*. If $\vec{u}_i$ and $\vec{v}_i$ are the ith columns of U and V, respectively, then the solution can be written as $\vec{a} = \sum_{i=1}^{M}(\vec{u}_i\cdot\vec{b}/\sigma_i)\vec{v}_i$. The variance in the estimated parameters a_j can be written as $\sigma^2(a_j^{(M)}) = \sum_{i=1}^{M} v_{ji}^2/\sigma_i^2$. It can thus be seen that the error will be rather large for small σ_i, and dropping such terms will reduce the errors at the cost of increasing the mean-square deviation slightly. The columns of V corresponding to small σ_i identify the linear combination of variables, which contribute little towards reducing χ^2 but make a large contribution in the standard deviation. Thus even if some of the singular values are not small enough to cause round-off problems, they can have a huge effect on the least-squares solution in the presence of noise.

A systematic way of giving lower weight to small singular values is via Tikhonov regularization [34]. Tikhonov regularization is a widely used technique for regularizing discrete ill-posed problems. One introduces a regularization parameter λ and filter factors T_i that depend on λ and the singular values σ_i as

$$T_i = \frac{\sigma_i^2}{\sigma_i^2+\lambda^2} \approx \begin{cases} 1 & \text{if } \sigma_i \gg \lambda, \\ \sigma_i^2/\lambda^2 & \text{if } \sigma_i \ll \lambda. \end{cases} \qquad (14)$$

The regularized least-squares solution is given by $\vec{a}_\lambda = \sum_{i=1}^{M} T_i(\vec{u}_i\cdot\vec{b}/\sigma_i)\vec{v}_i$. Comparison with the "naive" or unregularized least-squares solution ($\lambda=0$) shows that the filter factors essentially filter out the contributions to $\vec{a}_\lambda$ corresponding to the small singular values, while they leave the SVD components corresponding to large singular values virtually unaffected. A possible choice of λ is based on the L curve, which is a log-log plot of $|\vec{a}_\lambda|^2$ versus $|G\vec{a}_\lambda-\vec{b}|^2$ for different values of λ. The points on the horizontal branch correspond to large noise, whereas the points on the vertical branch correspond to large data misfit. The optimum choice of λ corresponds to points near the corner of the L curve [35].

III. RESULTS AND DISCUSSION

In this section, we will reconstruct the density matrix from a simulation of the corresponding probability data for a pair coherent state [36], a Perelomov [37] coherent state, and a coherent superposition of pair coherent states.

In a real experiment, the parameters y and ϕ can take only a finite (but large) number of values. In the absence of any *a priori* knowledge about the input state, we choose a set of values of ϕ equally distributed between 0 and 2π [32,38]: $\phi_s = 2\pi s/(N_\phi-1)$, and a set of values of y that are equispaced between $y_{min} = 0.01$ and $y_{max} = 0.26$: $\tilde{y}_n = y_{min} + (y_{max} - y_{min})(n-1)/(N-1)$. Then the Fourier transform with re-
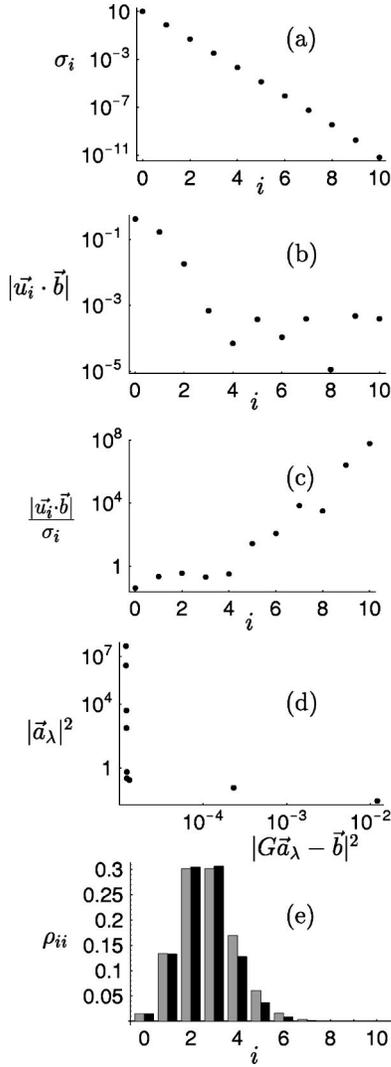
FIG. 2. Reconstruction of the diagonal density-matrix elements ρ_{ii} of the pair coherent state $|\Phi(3,0)\rangle$ [see Eqs. (16) and (17)] by the least-squares method. The truncation parameter was set at $n_{\max}=10$. The singular values σ_i, the values of $|\vec{u}_i \cdot \vec{b}|$, and the ratios $|\vec{u}_i \cdot \vec{b}|/\sigma_i$ are plotted in (a), (b), and (c), respectively. The corresponding L curve for the problem is plotted in (d). The regularization parameter λ has the values 10^{-8} (top left), 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3} (corner), 10^{-2}, 10^{-1}, and unity (bottom right). In (e), we plot the bar charts for the exact values (shaded gray) of the diagonal elements and the computed values (shaded black) with $\lambda = 0.001$.

spect to ϕ in Eq. (10) is approximated by a discrete Fourier transform

$$g_k(q,\tilde{y}_i) \rightarrow \frac{1}{N_\phi} \sum_{s=0}^{N_\phi-1} e^{2\pi i k s/N_\phi} \mathcal{Q}(q,\tilde{y}_i, 2\pi s/N_\phi). \quad (15)$$

Thus apart from truncation error due to the assumption (9), one will also have to deal with error due to *discretization* of the variables y and ϕ. The systematic error due to phase discretization can be reduced to zero by choosing $N_\phi \geqslant 2n_{\max}+1$ [32,38], whereas the error in the discretization of
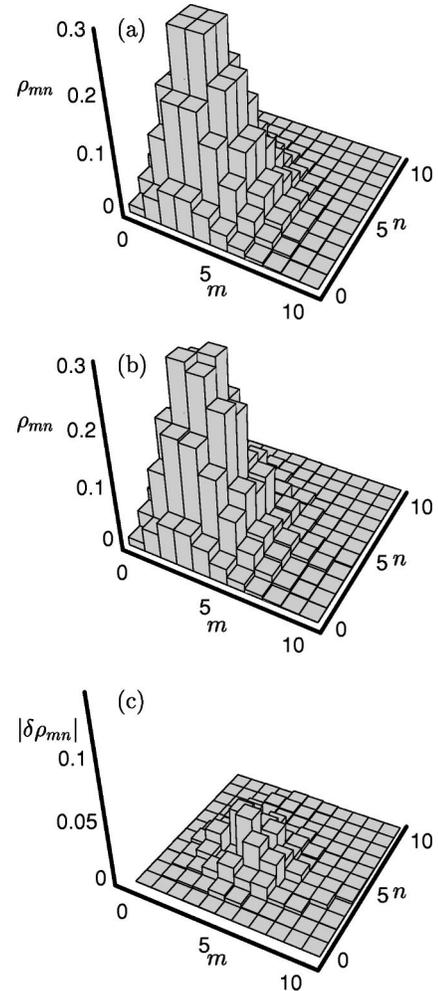
FIG. 3. Reconstruction of the density-matrix elements ρ_{mn} of the pair coherent state $|\Phi(3,0)\rangle$ by the least-squares method. The truncation parameter was set at $n_{\max}=10$. (a) Exact values; (b) reconstructed values; (c) the absolute difference between the exact and the reconstructed values.

y is of order N^{-2} and can be made arbitrarily small by taking a sufficiently large value of N. We have set $N_\phi=21$, $N=101$, and $n_{\max}=10$ in the calculations to follow. The data were simulated in the following way. We add to the exact probability data $f_k(q,\tilde{y}_i)$ an error term $\delta f_k(q,\tilde{y}_i) = R\mathcal{G}(f_k(q,\tilde{y}_i))\sqrt{f_k(q,\tilde{y}_i)/\tau}$, where R is a real random number uniformly distributed between -1 and 1, $\mathcal{G}$ is a Gaussian distribution with zero mean and unit variance [39], and $\tau = 20\,000$ is the number of trials at $y = \tilde{y}_i$. All our calculations have been carried out using the software package MATHEMATICA. For the record, the random numbers were generated with a seed value of 45.

A. Reconstruction of a pair coherent state

Pair coherent states of the radiation field can be generated via the competition of four-wave mixing and two-photon absorption in a nonlinear medium [36]. Pair coherent states can also be realized for the motion of a trapped ion [28]. One
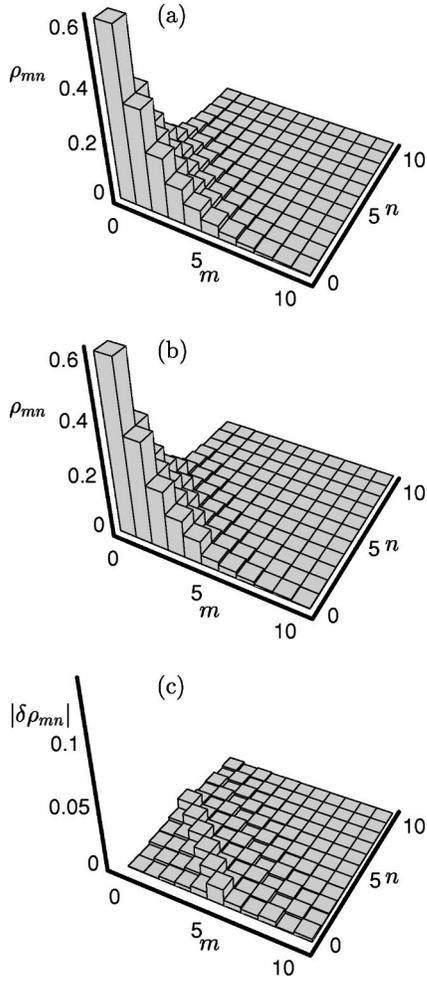
FIG. 4. Same as in Fig. 3 but for a Perelomov coherent state $\Psi(0.5,0)$ [see Eqs. (18) and (19)].

FIG. 5. Same as in Fig. 3 but for a coherent superposition of pair coherent states $|\Phi(\pm 3,0)\rangle$ [see Eq. (20)].

drives the ion with a laser on resonance and two other lasers with appropriately chosen directions of propagation and tuned to the second lower vibrational side band. In the Lamb-Dicke limit, the ion is found in a pair coherent state.

The state vector for a pair coherent state has the form

$$|\Phi(\xi,p)\rangle = N(\xi,p) \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!(n+p)!}} |n+p,n\rangle,$$

$$N(\xi,p) = \left[\sum_{n=0}^{\infty} \frac{|\xi|^{2n}}{n!(n+p)!} \right]^{-1/2}. \quad (16)$$

Here ξ is a complex parameter and $p \geq 0$ is an integer. The corresponding exact density matrix elements in the Fock basis are given by

$$\rho_{n,m}(p) = |N(\xi,p)|^2 \frac{\xi^n \xi^{*m}}{\sqrt{n!(n+p)!m!(m+p)!}}. \quad (17)$$

Note that $\rho_{m,n}(p) = \rho_{n,m}^*(p)$, and for real values of ξ the density matrix is symmetric.

The least-squares reconstruction from the simulated data fails in this case (some of the diagonal elements assume absolute values of the order of 10^3 or so) even with SVD when the tolerance parameter is set to its default value of 10^{-p+2}, where p is the machine precision. The failure is due to *overfitting*, that is, the use of a higher degree polynomial for $f_0(0,\tilde{y}_i)$ than necessary. As a result, the design matrix becomes ill-conditioned and some of the diagonal elements of Σ become very small. We mention parenthetically that the default tolerance removes none of these singular (or almost singular) values.

For a better understanding of the problem, we plot the singular values σ_i, the values of $|\vec{u}_i \cdot \vec{b}|$, and the ratios $|\vec{u}_i \cdot \vec{b}|/\sigma_i$ in Figs. 2(a), 2(b), and 2(c), respectively. It is seen that the values of $|\vec{u}_i \cdot \vec{b}|$ reach a noise floor of about 10^{-4} after $i=3$. The singular values continue to decay. Consequently, the ratios $|\vec{u}_i \cdot \vec{b}|/\sigma_i$ increase rapidly. It is clear from the plot that we cannot expect to obtain useful information from the singular values beyond $i=3$. In Fig. 2(d), we plot the L curve for various values of the regularization parameter λ. The point on the corner of the L curve corresponds to $\lambda = 0.001$, which indeed lies between σ_3 and σ_4. The regular-

ized solution is then given by $\vec{a}_\lambda$ where $\lambda = 0.001$. The result is in excellent agreement with the exact results as seen in Fig. 2(e). Proceeding as above, one can obtain a regularized solution for other values of k as well. The reconstruced density-matrix elements are in reasonable agreement with the exact results as seen in Fig. 3.

B. Reconstruction of a Perelomov coherent state

It is well known that Perelomov coherent states can be produced in parametric interactions. The state vector for a Perelomov coherent state is given by

$$|\Psi(\eta,q)\rangle = \frac{(1-|\eta|^2)^{(q+1)/2}}{\sqrt{q!}} \sum_{p=0}^{\infty} \eta^p \sqrt{\frac{(p+q)!}{p!}}|p+q,p\rangle,$$
(18)

where η is, in general, a complex parameter with $|\eta| < 1$, and $q \geq 0$ is an integer. The corresponding exact density-matrix elements in the Fock basis have the expression

$$\rho_{n,m}(q) = \frac{(1-|\eta|^2)^{q+1}}{q!} \sqrt{\frac{(n+q)!(m+q)!}{n!m!}}\eta^n\eta^{*m}.$$
(19)

For $q = 0$ and real values of η, the density matrix is not only symmetric but also has the following additional symmetries: $\rho_{n+2k,n}(0) = \rho_{n+k,n+k}(0)$ and $\rho_{n+2k+1,n}(0) = \rho_{n+k+1,n+k}(0)$. Consequently, only $f_0(0,y)$ and $f_1(0,y)$ need to be measured and modeled. We choose $q = 0$, $\eta = 0.6$, and set $n_{max} = 10$. Proceeding as before, we plot the exact density-matrix elements in Fig. 4 along with the computed elements reconstructed by the least-squares method with singular value decomposition. Once again, the recon-

struction is found to be satisfactory. Using the method of optical homodyne tomography, Vasilyev *et al.* [24] have, for the first time, reconstructed the diagonal elements of the two-mode Perelomov coherent state produced by a parametric amplifier. Their experiment also demonstrates how well the SU(1,1) symmetry holds in parametric amplification.

C. Reconstruction of a coherent superposition of pair coherent states

Our final example is the reconstruction of a coherent superposition of pair coherent states $|\Phi(\pm3,0)\rangle$:

$$|\psi\rangle = \tfrac{1}{2}[|\Phi(3,0)\rangle + |\Phi(-3,0)\rangle].$$
(20)

It can be easily shown that the nmth density-matrix element of $|\psi\rangle$ will be nonzero only when both n and m are even, in which case its value will equal the nmth density-matrix element of $|\Phi(\pm3,0)\rangle$. As a result, only *even* values of k and *even* powers of y appear in the modeling of $f_k(0,y)$. As shown in Fig. 5, satisfactory agreement is obtained between the exact and reconstructed density-matrix elements.

IV. CONCLUSION

We have suggested a scheme [40] for the reconstruction of two-mode SU(1,1) states using parametric amplifiers. The probability of the output state being in a certain two-mode number state is measured. The probability data are then "inverted" to extract the density matrix of the input state by taking advantage of certain symmetries in the input state. We have shown that this inversion can be achieved by the least-squares method after singular value decomposition of the design matrix followed by Tikhonov regularization.

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[1] W. Pauli, in *Handbuch der Physik*, edited by H. Geiger and K. Scheel (Springer, Berlin, 1933), Vol. 24/1, p. 83 [reprinted in *Encyclopedia of Physics* (Springer, Berlin, 1958), Vol. 5, p. 17].

[2] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).

[3] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).

[4] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993); D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. **T48**, 35 (1993); M. Munroe, D. Boggavarapu, M. E. Anderson, and M. G. Raymer, Phys. Rev. A **52**, R924 (1995).

[5] G. M. D'Ariano, C. Macchiavello, and M. G. A. Paris, Phys. Lett. A **195**, 31 (1994); Phys. Rev. A **50**, 4298 (1994); H. Kühn, D. -G. Welsch, and W. Vogel, J. Mod. Opt. **41**, 1607 (1994); A. Wünsche, *ibid.* **44**, 2293 (1997); U. Leonhardt, M. Munroe, T. Kiss, Th. Richter, and M. G. Raymer, Opt. Commun. **127**, 144 (1996); U. Leonhardt, H. Paul, and G. M. D'Ariano, Phys. Rev. A **52**, 4899 (1995); U. Leonhardt and M. Munroe, *ibid.* **54**, 3682 (1996).

[6] For a survey, see U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, 1997); D. -G. Welsch, W. Vogel, and T. Opatrný, in *Progress in Optics*,

edited by E. Wolf (North-Holland, Amsterdam, 1999), Vol. 39, p. 63.

[7] D. F. V. James and G. S. Agarwal, J. Opt. Soc. Am. B **12**, 704 (1995); S. Mancini, V. I. Man'ko, and P. Tombesi, Quantum Semiclassic. Opt. **7**, 615 (1995); G. M. D'Ariano, S. Mancini, V. I. Man'ko, and P. Tombesi, *ibid.* **8**, 1017 (1996); V. I. Man'ko, J. Russ. Laser Res. **17**, 579 (1996).

[8] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996); S. Wallentowitz and W. Vogel, Phys. Rev. A **53**, 4528 (1996); T. Opatrný and D. -G. Welsch, *ibid.* **55**, 1462 (1997); T. Opatrný, D. -G. Welsch, S. Wallentowitz, and W. Vogel, J. Mod. Opt. **44**, 2405 (1997); S. Mancini, P. Tombesi, and V. I. Man'ko, Europhys. Lett. **37**, 79 (1997).

[9] G. S. Agarwal and S. Chaturvedi, Phys. Rev. A **49**, R665 (1994); G. M. D'Ariano, M. Vasilyev, and P. Kumar, *ibid.* **58**, 636 (1998); H. Kühn, D. -G. Welsch, and W. Vogel, *ibid.* **51**, 4240 (1995); M. G. Raymer, D. F. McAlister, and U. Leonhardt, *ibid.* **54**, 2397 (1996); T. Opatrný, D. -G. Welsch, and W. Vogel, Opt. Commun. **134**, 112 (1997); Th. Richter, J. Mod. Opt. **44**, 2385 (1997); Phys. Rev. A **55**, 4629 (1997); H. Paul, P. Torma, T. Kiss, and I. Jex, J. Mod. Opt. **44**, 2395 (1997).

[10] A. Royer, Phys. Rev. Lett. **55**, 2745 (1985); Found. Phys. **19**, 3

(1989); U. Leonhardt and M. G. Raymer, Phys. Rev. Lett. **76**, 1985 (1996): D. S. Kraehmer and U. Leonhardt, Appl. Phys. B: Lasers Opt. **65**, 725 (1997); U. Leonhardt and S. Schneider, Phys. Rev. A **56**, 2549 (1997); M. G. Raymer, J. Mod. Opt. **44**, 2565 (1997).

[11] U. Leonhardt, Phys. Rev. A **55**, 3164 (1997); T. Opatrný, D. -G. Welsch, and W. Vogel, *ibid.* **56**, 1788 (1997); L. Davidovich, M. Orszag, and N. Zagury, *ibid.* **57**, 2544 (1998).

[12] U. Janicke and M. Wilkens, J. Mod. Opt. **42**, 2183 (1995); T. Pfau and Ch. Kurtsiefer, *ibid.* **44**, 2551 (1997); D. A. Kokorowski and D. E. Pritchard, *ibid.* **44**, 2575 (1997); Ch. Kurtsiefer, T. Pfau, and J. Mlynek, Nature (London) **386**, 150 (1997); S. H. Kienle, D. Fischer, W. P. Schleich, V. P. Yakovlev, and M. Freyberger, Appl. Phys. B: Lasers Opt. **65**, 735 (1997).

[13] M. Freyberger, Phys. Rev. A **55**, 4120 (1997). For an interesting application of the underlying method in a different context, see D. G. Fischer and M. Freyberger, Opt. Commun. **159**, 158 (1999).

[14] S. Mancini and P. Tombesi, Europhys. Lett. **40**, 351 (1997); E. L. Bolda, S. M. Tan, and D. F. Walls, Phys. Rev. Lett. **79**, 4719 (1997); Phys. Rev. A **57**, 4686 (1998).

[15] S. Mancini and P. Tombesi, Phys. Rev. A **56**, 3060 (1997).

[16] X. Chen and J. A. Yeazell, Phys. Rev. A **56**, 2316 (1997).

[17] A. Zucchetti, S. Wallentowitz, W. Vogel, and N. P. Bigelow, Phys. Rev. A **61**, 011405 (2000).

[18] U. Leonhardt, Phys. Rev. Lett. **74**, 4101 (1995); Phys. Rev. A **53**, 2998 (1996); W. Band and J. L. Park, Found. Phys. **1**, 133 (1970); **1**, 211 (1971); **1**, 339 (1971); Am. J. Phys. **47**, 188 (1979); I. D. I. D. Ivanović, J. Math. Phys. **24**, 1199 (1983); W. K. Wootters, Found. Phys. **16**, 391 (1986); W. K. Wootters and B. D. Fields, Ann. Phys. (N.Y.) **191**, 363 (1989); V. V. Dodonov and V. I. Man'ko, Phys. Lett. A **229**, 335 (1997); G. S. Agarwal, Phys. Rev. A **57**, 671 (1998); M. G. Benedict and A. Czirjak, *ibid.* **60**, 4034 (1999); S. M. Chumakov, A. B. Klimov, and K. B. Wolf, *ibid.* **61**, 034101 (2000).

[19] M. Freyberger and A. M. Herkommer, Phys. Rev. Lett. **72**, 1952 (1994); P. J. Bardroff, C. Leichtle, G. Schrade, and W. P. Schleich, *ibid.* **77**, 2198 (1996).

[20] M. S. Kim and G. S. Agarwal, Phys. Rev. A **59**, 3044 (1999); M. S. Kim, G. Antesberger, C. T. Bodendorf, and H. Walther, *ibid.* **58**, R65 (1998); L. G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78**, 2547 (1997).

[21] J. R. Ashburn, R. A. Cline, P. J. M. van der Burgt, W. B. Westerveldt, and J. S. Risley, Phys. Rev. A **41**, 2407 (1990).

[22] T. J. Dunn, J. N. Sweetser, I. A. Walmsley, and C. Radzewicz, Phys. Rev. Lett. **70**, 3388 (1993); T. J. Dunn, I. A. Walmsley, and S. Mukamel, *ibid.* **74**, 884 (1995).

[23] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996).

[24] M. Vasilyev, A. K. Sood, J. F. Whitaker, and R. Merlin, Science **275**, 1638 (1997); M. Artoni and J. L. Birman, Opt. Commun. **104**, 319 (1994); X. Hu and F. Nori, Phys. Rev. Lett. **76**, 2294 (1996).

[25] G. M. D'Ariano, Phys. Lett. A **268**, 151 (2000).

[26] G. S. Agarwal, Phys. Rev. A **57**, 671 (1998); C. Brif and A. Mann, e-print quant-ph/9904054, (1999).

[27] J. Banerji and G. S. Agarwal, Phys. Rev. A **59**, 4777 (1999).

[28] S. C. Gou, J. Steinbach, and P. L. Knight, Phys. Rev. A **54**, R1014 (1996).

[29] G. A. Garrett, A. G. Rojo, A. K. Sood, J. F. Whitaker, and R. Merlin, Science **275**, 1638 (1997); M. Artoni and J. L. Birman, Opt. Commun. **104**, 319 (1994); X. Hu and F. Nori, Phys. Rev. Lett. **76**, 2294 (1996).

[30] See, for example, G. M. D'Ariano, U. Leonhardt, and H. Paul, Phys. Rev. A **52**, R1801 (1995); and Welsch *et al.* in Ref. [6].

[31] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Chap. VI.

[32] T. Opatrný, D. -G. Welsch, and W. Vogel, Phys. Rev. A **56**, 1788 (1997).

[33] See, for example, C. L. Lawson and R. J. Hanson, *Solving Least Squares Problems* (Prentice-Hall, Englewood Cliffs, NJ, 1974).

[34] A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-posed Problems* (Winston/Wiley, Washington, DC, 1977.)

[35] P. C. Hansen, SIAM Rev. **34**, 561 (1992); P. C. Hansen and D. P. O'Leary, SIAM J. Sci. Comput. (USA) **14**, 1487 (1993).

[36] G. S. Agarwal, J. Opt. Soc. Am. B **5**, 1940 (1988).

[37] A. M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin 1986); C. M. Caves and B. L. Shumaker, Phys. Rev. A **31**, 3068 (1985); **31**, 3094 (1985).

[38] T. Opatrný, D. -G. Welsch, S. Wallentowitz, and W. Vogel, J. Mod. Opt. **44**, 2405 (1997); D. F. McAlister and M. G. Raymer, *ibid.* **44**, 2359 (1997); U. Leonhardt, *ibid.* **44**, 2273 (1997).

[39] See, for example, Bolda *et al.*, in Ref. [14].

[40] Note that our method is based on the group property and the subsequent use of the conservation law, though these might be somewhat affected by experimental noise.