Collective modes in a dilute Bose-Fermi mixture

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Here we study the collective excitations of a dilute spin-polarized Bose-Fermi mixture at zero temperature, considering, in particular, the features arising from the interaction between the two species. We show that a propagating zero-sound mode is possible for the fermions even when they do not interact among themselves.

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Recent experimental progress in atomically trapped gases has led to a resurgence of interest in quantum fluids. A particular notable feature is the number of systems available, ranging from a single-component Bose gas in the original experiments, where Bose-Einstein condensation was first achieved [1] to a binary Bose mixture [2], a spinor condensate in optical traps [3], and a degenerate Fermi gas [4,5]. Other systems also received much recent attention, in particular a Bose-Fermi mixture. This last mentioned system occurs naturally if ''sympathetic cooling'' is employed to reduce the kinetic energy of the fermions [6]. There have already been several studies on the properties of this system. Questions addressed include stability against phase separation [7,8] and collective excitations [8].

Although Bose-Fermi mixtures have been studied intensively in low-temperature physics in the context of ^{3}He-^{4}He mixtures [9], atomically trapped gases offer many additional possibilities. By the choice of atoms, the concentration of the various components, or the control of interaction strength among them by external fields [10], one can unmask phenomena previously unobservable. In this paper, we shall study one example of this by considering the density oscillations of a Bose-Fermi mixture at low temperatures. We shall show that a variety of phenomena can arise due to the coupling between the two components for suitable parameters, such as the ratio of the sound velocity of the Bose gas to the Fermi velocity of the fermions. In particular, we shall show that it is possible to have a propagating fermionic sound mode even in the absence of interaction among the fermions themselves. Sound propagation was also considered in Ref. [8], which, however, did not investigate the effects being studied here. We shall comment on this later.

We shall then consider a mixture of weakly interacting Bose and Fermi gases at zero temperature. Both gases are assumed to be spin polarized, such as would usually be the case in magnetic traps. For a dilute mixture, interaction among the bosons themselves and between the bosons and fermions can be characterized by the scattering lengths a_{bb} and a_{bf} in the s-wave channels. However, the fermions do not interact among themselves since they are spin polarized. For simplicity we shall consider a uniform system. We shall further assume that the gas is stable against phase separation unless explicitly specified. We are interested in the density waves of this system. As we shall see, in general the modes may be damped. Also since the density oscillations are likely to be studied by exciting the systems with external potentials, we shall instead consider the density responses of the system under external perturbing potentials. Collective modes of the system will show up as resonances of these responses.

The Hamiltonian density is given by

$$H=\frac{\hbar^2}{2m_b}\nabla\psi_b^\dagger\nabla\psi_b-\mu_b\psi_b^\dagger\psi_b+\frac{\hbar^2}{2m_f}\nabla\psi_f^\dagger\nabla\psi_f-\mu_f\psi_f^\dagger\psi_f + \frac{1}{2}g_{bb}\psi_b^\dagger\psi_b^\dagger\psi_b\psi_b+g_{bf}\psi_b^\dagger\psi_f^\dagger\psi_f\psi_b+\psi_b^\dagger\psi_b V_b^{\rm ext} + \psi_f^\dagger\psi_f V_f^{\rm ext}, \tag{1}$$

where the subscripts b and f denote bosons and fermions, respectively; ψ_f and ψ_b are the field operators; m_b and m_f the masses; μ_b and μ_f are the chemical potentials; and $V_b^{\rm ext}$ and $V_f^{\rm ext}$ are the external potentials. All ψ's and $V^{\rm ext}$'s are implicitly at the same physical point $\vec{r}$ in space. The interaction parameters g_{bb} and g_{bf} are related to the scattering lengths a_{bb} and a_{bf} by $g_{bb}=4\pi\hbar^2 a_{bb}/m_{bb}$ and $g_{bf}=2\pi\hbar^2 a_{bf}/m_r$ where m_r is the reduced mass ($m_r^{-1}\equiv m_f^{-1}+m_b^{-1}$).

We shall treat the interactions g_{bb} and g_{bf} within the Bogoliubov and random-phase approximations, respectively [11]. The results can be written in the physically transparent forms

$$\begin{aligned}\delta n_b(q,\omega)&=-\chi_b[g_{bf}\delta n_f+V_b^{\rm ext}],\\ \delta n_f(q,\omega)&=-\chi_f[g_{bf}\delta n_b+V_f^{\rm ext}],\end{aligned} \tag{2}$$

expressing the response of the bosons and fermions to the potentials due to the other species and the external perturbations (the terms in the square brackets). Here $\delta n_b(q,\omega)$ and $\delta n_f(q,\omega)$ are the deviations of the bosonic and fermionic densities from equilibrium at wave vector q and frequency ω, and

$$\chi_b=-\frac{1}{g_{bb}}\left[\frac{c_b^2q^2}{\omega^2-c_b^2q^2-(q^2/2m_b)^2}\right] \tag{3}$$

and

$$\chi_f=N_f\left[1-\frac{\omega}{2v_fq}\ln\left(\frac{\omega+v_fq}{\omega-v_fq}\right)\right] \tag{4}$$

are the (q- and ω-dependent) responses of the pure boson and fermion systems, respectively, to effective external potentials. $N_f \equiv p_f m_f / 2\pi^2$ is the density of states for the fermions. ($p_f = (6\pi^2 n_f)^{1/3}$ is the Fermi momentum, $v_f = p_f/m_f$) For simplicity, in Eq. (4), I left out terms that are small if $q \ll p_f$. ω should be interpreted as having a small and positive imaginary part.

Equation (2) can be rearranged as

$$\begin{pmatrix} 1 & g_{bf}\chi_b \\ g_{bf}\chi_f & 1 \end{pmatrix}\begin{pmatrix} \delta n_b \\ \delta n_f \end{pmatrix} = -\begin{pmatrix} \chi_b V_b^{\rm ext} \\ \chi_f V_f^{\rm ext} \end{pmatrix}. \qquad (5)$$

Then, finally,

$$\begin{pmatrix} \delta n_b \\ \delta n_f \end{pmatrix} = -\frac{1}{1-g_{bf}^2\chi_b\chi_f}\begin{pmatrix} 1 & -g_{bf}\chi_b \\ -g_{bf}\chi_f & 1 \end{pmatrix}\begin{pmatrix} \chi_b V_b^{\rm ext} \\ \chi_f V_f^{\rm ext} \end{pmatrix}. \qquad (6)$$

In the case where $g_{bf}=0$, $\delta n_b = -\chi_b V_b^{\rm ext}$, and $\delta n_f = -\chi_f V_f^{\rm ext}$, and the responses thus reduce to those of the pure Bose and Fermi gases. The corresponding formulas for χ_b and χ_f were already given in Eqs. (3) and (4) above. Before we proceed, we shall recall the behavior of these responses [11], and thus the collective modes. For simplicity we shall restrict ourselves to small wave vectors, i.e., $q \ll m_b c_b$ and p_f, and without loss of generality $\omega > 0$. The bosonic response Im χ_b consists of a δ function at the excitation frequency $\omega = c_b q$. This is due to the Bogoliubov mode, which is purely propagating and undamped. For the fermions, however, there is no collective behavior. The absorptive part, Im χ_f, is finite for a whole range of frequencies $|\omega| < v_f q$, known as the particle-hole continuum, arising from the many possibilities of independent particle-hole excitations. Re χ_b is simple. It is given by g_{bb}^{-1} at $\omega=0$, and diverges to $\pm\infty$ as $\omega \to c_b q$ from below and above, respectively. Re χ_f is given by N_f at $\omega=0$. It decreases with increasing ω, changes sign at around $\omega \sim 0.83 v_f q$, and approaches $-\infty$ as $\omega \to v_f q$ from both above and below. For $\omega > v_f q$, it remains negative, with its magnitude gradually approaching zero as $\omega \to \infty$.

Now we return to the Bose-Fermi mixture. The response δn_b to an external potential $V_b^{\rm ext}$, acting on the bosons only, is given by $\chi_b/(1-g_{bf}^2\chi_b\chi_f)$. The existence and the dispersion of the bosonic collective mode are determined by the solution to the equation $(\chi_b)^{-1} - g_{bf}^2\chi_f = 0$, i.e.,

$$\left[-\omega^2 + c_b^2 q^2 + \left(\frac{q^2}{2m_b}\right)^2\right] - \left(\frac{g_{bf}^2}{g_{bb}}\right)(c_b^2 q^2)\chi_f = 0. \qquad (7)$$

It will be convenient to discuss the normalized response

$$\tilde{\chi}_b \equiv g_{bb}\chi_b/(1-g_{bf}^2\chi_b\chi_f), \qquad (8)$$

$\tilde{\chi}_b = 1$ in the static limit ($\omega=0, q\to 0$), when there is no boson-fermion interaction ($g_{bf}=0$).

Similiarly the fermionic response to an external potential acting on the fermions alone is $\chi_f/(1-g_{bf}^2\chi_b\chi_f)$. We shall discuss the behavior of the normalized quantity

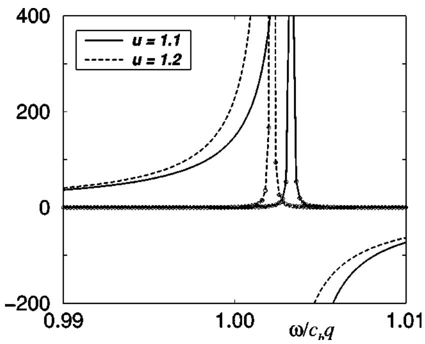

FIG. 1. Dimensionless bosonic responses Re $\tilde{\chi}_b$ and Im $\tilde{\chi}_b$ for $u \equiv c_b/v_f > 1$ and $W=0.01$. Lines for the imaginary parts are decorated with circles.

$$\tilde{\chi}_f \equiv N_f^{-1}\chi_f/(1-g_{bf}^2\chi_b\chi_f). \qquad (9)$$

The normalization is chosen such that $\tilde{\chi}_f = 1$ in the static limit ($\omega=0$, $q\to 0$), when there is no boson-fermion interaction ($g_{bf}=0$).

Before proceeding, let us first examine the responses at $\omega=0$. Stability requires that the density responses $\chi_b/(1-g_{bf}^2\chi_b\chi_f)$ and $\chi_f/(1-g_{bf}^2\chi_b\chi_f)$ be positive. Using the $\omega=0$ values of χ_b and χ_f above, these necessary conditions can be rewritten as $g_{bb}>0$ and $W \equiv N_f g_{bf}^2/g_{bb} < 1$. Using the expression of N_f given earlier, the last inequality gives $n_f^{1/3} g_{bf}^2 < \frac{2}{3} A g_{bb}$ where $A \equiv (\hbar^2/2m_f)(6\pi^2)^{2/3}$ as defined in Ref. [7]. These conditions were derived earlier in Refs. [7] and [8] using slightly different considerations. For bosons and fermions with similar masses, we shall see shortly that W, a dimensionless parameter, serves as a useful measure of the coupling between the bosons and fermions. If the bosons and fermions have similar masses, $|W|$ is of order $|a_{bf}^2/a_{bb} n_f^{-1/3}|$, and thus is typically small for dilute gases unless $|a_{bf}| \gg |a_{bb}|$. We shall limit ourselves only to the cases where $|W|$'s are small.

We shall discuss now the behaviors of $\tilde{\chi}_b$ and $\tilde{\chi}_f$ in turn. The results are qualitatively different depending on whether $c_b \gtrless v_f$. The velocity ratio $u \equiv c_b/v_f$ can be re-expressed as

$$u = \frac{m_f}{m_b}\frac{(4/3)^{1/3}}{\pi^{1/6}}\frac{(n_b a_{bb})^{1/2}}{n_f^{1/3}}.$$

The value of u can basically be arbitrary without violating any stability criterion (not only the linear stability condition above but also others derived in Ref. [7])

BOSONIC RESPONSE

(1) $c_b > v_f$: In this regime a propagating bosonic mode exists. It can be easily verified (e.g., graphically) that the mode frequency ω satisfies $\omega > c_b q$ ($> v_f q$). The original bosonic mode at $\omega = c_b q$ is pushed upward by the particle-hole "modes" lying below. Some examples are shown in Fig. 1. This mode "repulsion" is generally expected (cf. coupled harmonic oscillators). However, it is of interest to

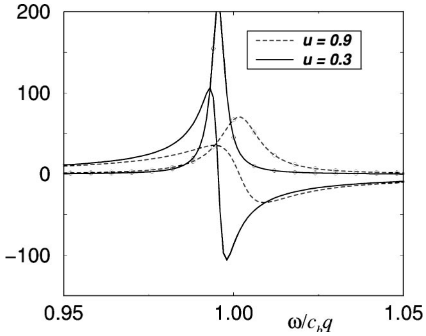

FIG. 2. Same as Fig. 1, but for $u \equiv c_b/v_f < 1$ and $W = 0.01$.

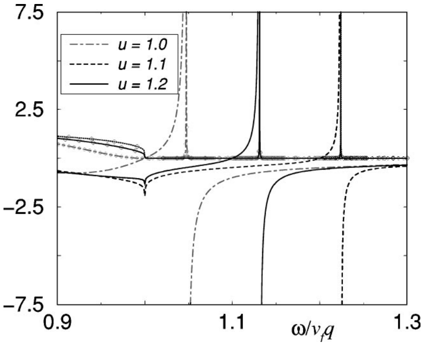

FIG. 3. Dimensionless fermionic responses Re $\tilde{\chi}_f$ and Im $\tilde{\chi}_f$ for $u \equiv c_b/v_f \geqslant 1$ and $W = 0.1$. The imaginary parts (lines decorated with circles) contain the particle-hole continua $\omega < v_f q$ and sharp spikes at the bosonic mode frequencies.

examine the microscopic nature of the mode. At the mode frequency both χ_b and χ_f are negative. Thus, e.g., if $g_{bf} > 0$, δn_b, and δn_f are of the same sign [see Eq. (5)]. The repulsion between the two species provides the enhanced restoring force and oscillation frequency. This frequency shift is typically small since usually $W \ll 1$.

(2) $c_b < v_f$: In this case the original bosonic mode lies inside the particle-hole continuum of the fermions. The bosonic mode is thus Landau damped. For weak coupling the damping, and thus the width of the response, can be estimated easily using Eq. (7) to be $\sim [\pi N_f g_{bf}^2/4 g_{bb}][c_b/v_f](c_b q)$. Examples are shown in Fig. 2. There is a small shift of the mode due to Re χ_f. The shift is toward higher frequency for u sufficiently close to 1, but opposite otherwise (Re $\chi_f < (>) 0$ for $\omega/v_f q > (<) 0.83$.)

(3) It is also of interest to study the bosonic mode for $g_{bb} < 0$. This is in fact the case for the ^{6}Li-^{7}Li mixture investigated in Ref. [6], where the ^{7}Li bosons have a negative scattering length of ≈ -1.5 nm. In this case the original bosonic system is unstable, and the Bogoliubov mode has an imaginary frequency for sufficiently small wave vector ($q < q_c = 2 m_b |c_b|/\hbar$, here $|c_b| \equiv [|g_{bb}| n_b/m_b]^{1/2}$). Since $N_f g_{bf}^2 > 0 > g_{bb}$, the system is still unstable in the presence of fermions [7] (also see above). Of interest is the effect of the fermions on the unstable mode. Now for imaginary frequencies $\omega = i\alpha$,

$$\chi_f(q, i\alpha) = N_f \left\{ 1 - \frac{\alpha}{v_f q} \left[\frac{\pi}{2} - \tan^{-1} \frac{\alpha}{v_f q} \right] \right\}$$

is purely real and positive. χ_f decreases monotonically with α from $\chi_f = N_f$ at $\alpha = 0$ to 0 as $\alpha \to \infty$. It can be easily verified that there is a *real* solution for α to the dispersion relation [cf. Eq. (7)]

$$[\alpha^2 - |c_b|^2 q^2] - \left(\frac{g_{bf}^2}{|g_{bb}|} \right) (|c_b|^2 q^2) \chi_f(q, i\alpha) = 0 \quad (10)$$

for sufficiently small q (which includes the physically most relevant region where α attains its maximum, i.e., the fastest growing instability). Thus the instability is *not* damped by the particle-hole degree of freedom. In fact it can be verified easily that, for given q, α is increased in the presence of the fermions. The system has become even more unstable. This mode has δn_b and δn_f of opposite signs, and corresponds to phase separation as expected.

FERMIONIC RESPONSE

(1) $c_b > v_f$: In this case the fermionic response for $0 < \omega < q v_f$ is only slightly modified. A feature appears near $\omega \sim c_b q > v_f q$ due to the coupling to the bosonic mode. An example is as shown in Fig. 3.

(2) $c_b < v_f$: In this regime there are two important features of the fermionic response. If $u = c_b/v_f$ is sufficiently close to 1, the imaginary part contains a sharp resonance at ω above the particle-hole continuum (Figs. 4 and 5). There are two ways of understanding this mode. It can be regarded as a continuation of the situation from $c_b > v_f$, i.e., it is due to the bosonic mode which is itself slightly pushed up in frequency (cf., Fig. 3; note, in particular, the result for $u = 1$). Alternatively, this mode can be considered as a zero-sound mode induced by the bosons. The form for $\tilde{\chi}_f$ in Eq. (9) is precisely that of an *interacting* Fermi gas with *s*-wave interaction g_{ff} [and therefore necessary with more than one spin species, where the response is given by $\chi_f/(1 + g_{ff}\chi_f)$], though with an effective *frequency-dependent* interaction $g_{ff} \to -g_{bf}^2 \chi_b$, i.e., an effective *s*-wave Landau parameter

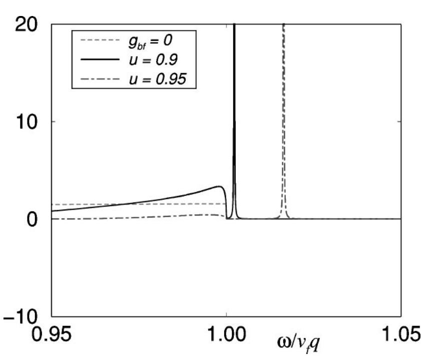

FIG. 4. Dimensionless fermionic response Im $\tilde{\chi}_f$ for $u \equiv c_b/v_f < 1$ showing the zero-sound modes induced by the bosons. $W = 0.1$. Also shown is Im χ_f for a pure Fermi gas ($g_{bf} = 0$) for comparison.

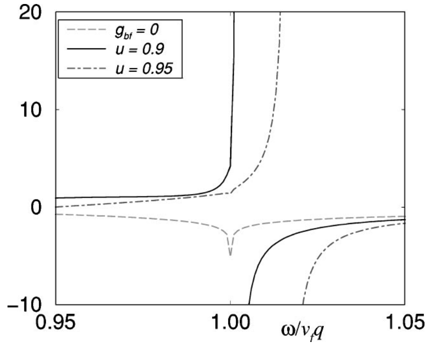

FIG. 5. Same as Fig. 4, except that now Re $\tilde{\chi}_f$ is shown.

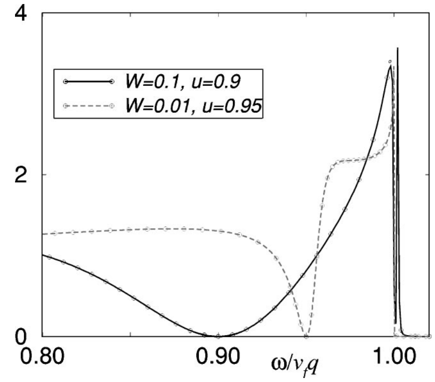

FIG. 6. Imaginary part of the dimensionless fermionic response Im $\tilde{\chi}_f$ for $u \equiv c_b/v_f < 1$, showing mainly the region $\omega < v_f q$.

given by $F_0 \to W/[(\omega/c_b q)^2 - 1]$. The bosonic mode $\omega \sim c_b q$, for c_b sufficiently close to but below v_f, will thus induce a zero-sound mode for the fermions, just as an interaction among the fermions will [12]. Note, however, that there cannot be a real *s*-wave interaction among the fermions, as they are of equal spin. Thus this mode *cannot* be obtained by considering the effective interaction among the fermions, as in Ref. [8].

The frequency of this propagating mode can be estimated by using the well-known dispersion relation of the zeroth sound $\omega/v_f q \approx 1 + 2e^{-2[1+(1/F_0)]}$, with the effective $F_0 \to W/[(1/u)^2 - 1]$ as suggested above. In order for the velocity of the mode to be, say, 1% above v_f, then c_b has to be within around 7% of v_f if $W = 0.1$. This estimate agrees with the numerical results of Fig. 4.

The second interesting feature is that, near the original bosonic mode frequency $\omega \sim c_b q$, there is a reduction in the absorptive part Im $\tilde{\chi}_f$ (see Fig. 6). In fact, Im $\tilde{\chi}_f \to 0$ as $\omega \to c_b q$. This, as well as the corresponding behavior of Re $\tilde{\chi}_f$, can be seen easily mathematically from Eq. (5) due to the resonance nature of χ_b at this frequency. Physically this can be regarded as due to mode-mode repulsion—the bosonic mode has pushed away the particle-hole "modes" near $\omega \sim c_b q$. This feature is present even for small coupling *W*. A larger *W* mainly increases the width of this "transparent" region. Thus, in fact, the frequency dependence of Re $\tilde{\chi}_f$ is actually *stronger* for smaller *W*'s.

The energy absorption by the Bose-Fermi mixture from an external perturbation acting on the fermions is thus substantially reduced for frequencies within this transparent region. The width of this region can be estimated by using the observation that the fermionic response is roughly reduced by the factor $1 + W[(c_b q)^2/\omega^2 - (c_b q)^2]$ for these frequencies. For the fermionic response to be reduced to, say, 1/2 of its bare value, then $|\omega - c_b q|/c_b q < W/2$. This estimate agrees very roughly with the numerical results in Fig. 6. In conclusion, I have investigated the collective modes of a Bose-Fermi mixture, and have shown that there is important mode-mode coupling effects, especially if $v_f \sim c_b$.

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[12] One can check (e.g., graphically) that there is a *real* solution for ω to the equation $\chi_f^{-1} - g_{bf}^2 \chi_b$, i.e., the mode is undamped.