Four-wave mixing in degenerate atomic gases

P. Villain, P. Öhberg, L. Santos, A. Sanpera, and M. Lewenstein
Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany
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We study the process of four-wave mixing (FWM) in ultracold degenerate atomic gases. In particular, we address the problem of FWM in boson-fermion mixtures. We develop an approximate description of such processes using asymptotic analysis of high-order perturbation theory taking into account quantum statistics. We perform also numerical simulations of FWM in boson-fermion mixtures and obtain an analytic and numerical estimate of the efficiency of the process.

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I. INTRODUCTION

In recent years, atom optics has become a flourishing subject. With the successful experiments on Bose-Einstein condensation (BEC) [1–3], new directions in linear [4] and nonlinear atom optics [5] with macroscopic wave packets have emerged. So far, the most spectacular experiment in nonlinear atom optics concerns the observation of four-wave mixing of bosonic matter waves [6]. In this process, three macroscopic matter wave packets interact and produce a fourth one. Also recently, Jin *et al.* [7] have trapped and cooled a sample of spin-polarized ^{40}K below the Fermi temperature. This has triggered an outburst of activities to understand the properties of ultracold fermionic systems [8], and has stimulated a great interest in the studies of fermion-boson mixtures [9]. In particular, the question "is nonlinear atom optics with fermions possible?" has been posed [10]. Recently, it has been argued that the answer to this question is positive [11,12]. The remaining open question is what role does the statistics play in determining the efficiency of the four-wave-mixing (FWM) process.

The recent experiment of Deng *et al.* [6] with BEC has demonstrated that FWM in bosonic gases is a truly macroscopic process with an efficiency of the order of 6%. In this experiment, Bragg pulses [13] were used to create three condensate clouds, which then interacted through collisions. Four-wave mixing of matter waves can also be described as Bragg scattering from a grating. In this picture, two counterpropagating matter waves create the grating from which the third wave scatters, generating a fourth one. In this paper, we investigate the process of FWM in fermion-boson mixtures and show that with an appropriate choice of parameters, it is possible to create a macroscopic efficiency for the production of the fermionic fourth wave. We investigate here two complementary regimes. (i) Using an asymptotic analysis of perturbation theory, we study the regime where every fermion fulfills the Bragg condition and the spread of the momentum is much smaller than the momentum of the grating. (ii) We study numerically a situation more feasible experimentally, where the momentum spread of the fermion cloud Δk becomes large so that the Bragg condition is not fulfilled for every fermion.

The paper is organized as follows. In Sec. II, we study analytically the scattering process using a perturbative treatment. Section III describes the numerical approach where the momentum spread is emphasized. Finally, the results are summarized in Sec. IV.

II. ANALYTICAL APPROACH

An intuitive understanding of Bragg scattering for bosons may be obtained considering a homogeneous condensate in a box of volume *V*. For such a case, the Hamiltonian has the form

$$\hat{H}=\sum_{\vec{k}} \epsilon_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} + \tfrac{1}{2} u_0 \sum_{\vec{k},\vec{k}',\vec{q}} a_{\vec{k}+\vec{q}}^{\dagger} a_{\vec{k}'-\vec{q}}^{\dagger} a_{\vec{k}'} a_{\vec{k}}, \tag{1}$$

where $u_0=4\pi\hbar^2 a/mV$ is the interaction strength proportional to the *s*-wave scattering length *a* and $\epsilon_{\vec{k}}$ is the kinetic energy. Let us consider an initial state $|i\rangle=|N_1,\vec{k}_1;N_2,\vec{k}_2;N_3,\vec{k}_3\rangle$ representing N_i particles of momentum $\vec{k}_i$, $i=1,2,3$. We assume for the moment that only one particle is scattered leading to the final state $|f\rangle=|N_1',\vec{k}_1;N_2',\vec{k}_2;N_3',\vec{k}_3;1,\vec{k}_4\rangle$ with $\vec{k}_4\neq\vec{k}_i$ for $i=1,2,3$. Among all the processes that conserve momentum and energy, the one corresponding to

$$\vec{k}_4=\vec{k}_1-\vec{k}_2+\vec{k}_3 \tag{2}$$

is particularly favorable and represents in fact the first-order Bragg scattering [14,15] (see Fig. 1). The above process

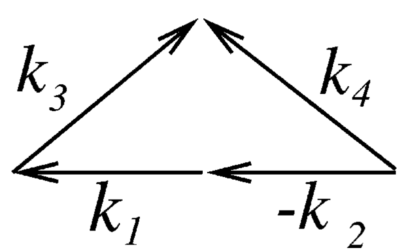

FIG. 1. In the reference frame in which $\vec{k}_1$ and $\vec{k}_2$ are collinear, the scattering process can always be described in a plane. The criterion $\vec{k}_4=\vec{k}_1-\vec{k}_2+\vec{k}_3$ together with the conservation of energy $|\vec{k}_4|=|\vec{k}_3|$ give the Bragg condition $k_{3x}=k_1$ with $\vec{k}_1=-\vec{k}_2$ chosen in the direction of the *x* axis.

corresponds to the term $a^{\dagger}_{\vec{k}_4} a^{\dagger}_{\vec{k}_2} a_{\vec{k}_1} a_{\vec{k}_3}$ in the Hamiltonian, which accounts for the creation of a particle in the already macroscopical ''grating state'' $\vec{k}_2$. This introduces a bosonic enhancement factor $\sqrt{N_2+1}$ in the transition amplitude of the process $|i\rangle \rightarrow |f\rangle$. From this point of view, due to bosonic enhancement, Bragg scattering is the most probable process.

Let us extend the previous analysis to the case of many particles. The final state we are interested in is of the form $|f\rangle = |N'_1,\vec{k}_1;N'_2,\vec{k}_2;N'_3,\vec{k}_3;N_4,\vec{k}_4\rangle$, where $\vec{k}_4$ is given by Eq. (2). This final state must respect the conservation of momentum

$$\sum_{i=1}^{3} N_i \vec{k}_i = \sum_{i=1}^{3} N'_i \vec{k}_i + N_4 \vec{k}_4, \tag{3}$$

and the conservation of the particle number

$$\sum_{i=1}^{3} N_i = \sum_{i=1}^{3} N'_i + N_4. \tag{4}$$

Equations (2), (3), and (4) imply conservation of energy. In the reference frame in which $\vec{k}_1$ and $\vec{k}_2$ are collinear, the scattering process is planar. Thus, we have five equations with six parameters $N'_i, N_4, \vec{k}_4$. The actual value of N_4 can be determined by maximizing the transition probability P_{if}. Using perturbation theory with respect to the off-diagonal elements of the Hamiltonian in Eq. (1), we calculate the transition amplitude T_{if}. For $N_1 = N_2 = N_3$, this quantity exhibits a divergence for $\tilde{\eta} = N_4/N_{\mathrm{tot}} = \frac{1}{6} \approx 16.7\%$, where N_{tot} is the total number of particles. This divergence appears due to the simplicity of the model and can be removed by using wave packets instead of plane waves. In any case, P_{if} will be strongly peaked at $N_4 \sim N_{\mathrm{tot}}/6$. The actual efficiency of the process—which in principle has to be calculated nonperturbatively—will be of the same order of magnitude. Furthermore, our analysis predicts a saturation of the efficiency with an increasing number of atoms, as a consequence of the Bose statistics. The results obtained from our simple model are in good agreement with the experimental results of Ref. [6] and with the more rigorous calculations of Refs. [16–18].

We turn now to the case of fermions. The previous analysis clearly stresses the role of bosonic enhancement; the macroscopic occupation of the states that form the grating select the Bragg scattering as the most favorable scattering process. From this point of view, the use of a fermionic grating would lead to a poor Bragg scattering. For this reason, we consider Bragg scattering of an incoming cloud of fermions on a bosonic grating. A pure fermionic grating leading to a well-defined scattering could in principle be created, as pointed out in Ref. [11], by considering a fermionic cloud that is in a single momentum state along the z axis but occupies many momentum states along the x and y axes.

In our model, fermions and bosons interact via two-body s-wave scattering. The incoming fermions must obey momentum and energy conservation as in the case of boson-boson scattering. For fermions, there will inevitably be a momentum spread due to Fermi statistics. In order to fulfill the Bragg condition, illustrated in Fig. 1, the momentum spread in the fermionic sample Δk must be restricted to $\Delta k \ll |\vec{k}_1 - \vec{k}_2|$. This condition can be achieved by trapping the fermions initially in a sufficiently shallow trap, which results in a well-localized momentum distribution.

In order to estimate the FWM efficiency, we develop a similar approach to the one outlined above, based on an asymptotic perturbative analysis using plane waves. We study the efficiency of the process starting with the fermionic state $|i\rangle = |1,\vec{k}_3+\vec{\chi}_1; \ldots ;1,\vec{k}_3+\vec{\chi}_{N_3}\rangle$ and consider a generic final state with N_4 states created around $\vec{k}_4$ that fulfill the Bragg condition. There are $j = \binom{N_3}{N_4}$ different orthogonal final states $|f_j\rangle$. In order to simplify the problem, we assume that the transition amplitude does not depend strongly on $|f_j\rangle$, which is a valid assumption as long as $\Delta k \ll |\vec{k}_1 - \vec{k}_2|$. We then evaluate the probability $P(N_4)$ of scattering N_4 particles by averaging the transition amplitude so that $P(N_4) \approx \binom{N_3}{N_4} |\overline{T_{if_j}}|^2$. The mean efficiency of the process $\eta = \langle N_4\rangle/N_3$ is calculated by using the distribution $P(N_4)$. We stress that the aim of this calculation is not to obtain an exact expression for the efficiency, but rather to check whether it can attain macroscopic values.

We begin by calculating T_{if_1} between the initial state $|i\rangle$ and the final state $|f_1\rangle$, which corresponds to depletion of N_4 states of the Fermi sea around $\vec{k}_3$. The macroscopically populated states of the bosonic grating provide a bosonic enhancement factor $\sqrt{(N_2+p)}$ $(p = 1,2, \ldots ,N_4)$. Let us define $U_{\mathrm{BF}} = 8\pi\hbar^2 a/mV$, where a denotes the fermion-boson s-wave scattering length and m the (same) mass for both fermions and bosons. Using a time-dependent perturbative method [21], one obtains for the N_4th order in U_{BF},

$$T_{if_1} \propto U_{\mathrm{BF}}^{N_4} \frac{\Gamma}{(E_i - E_{N_4}) \cdots (E_i - E_1)}, \tag{5}$$

where the bosonic enhancement amounts to

$$\Gamma = \frac{\sqrt{N_1!}\,\sqrt{(N_2+N_4)!}}{\sqrt{(N_1-N_4)!}\,\sqrt{N_2!}}. \tag{6}$$

The energies of the intermediate states corresponding to p fermions Bragg-scattered are given by

$$E_p = E_B + \frac{\hbar^2}{2m}\left(\sum_{j=p+1}^{N_3} (\vec{k}_3 + \vec{\chi}_j)^2 + \sum_{j=1}^{p} (\vec{k}_4 + \vec{\chi}_j)^2 \right) \tag{7}$$

with the initial energy $E_i = E_{p=0}$ and $\vec{k}_2 = k_2 \hat{e}_x = -k_1 \hat{e}_x$. We denote by E_B the energy of the bosons, which is constant during the process. This T_{if_1} is the transition amplitude corresponding to one of the $N_4!$ possible paths going from

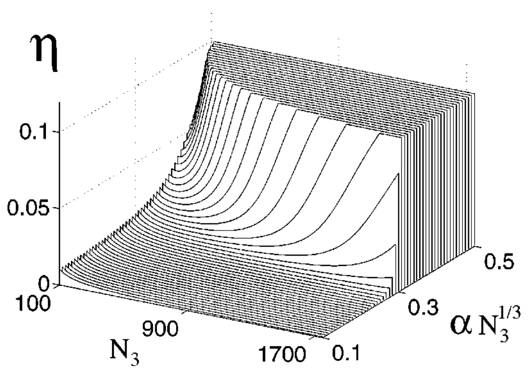

FIG. 2. Analytical estimate of the Bragg process efficiency. The model is limited by the assumption $\eta \ll 1$. For small values of α, the efficiency decreases with increasing N_3.

$|i\rangle$ to $|f_1\rangle$. Hence there are $N_4!$ different T_{if_1}'s and we estimate their contribution by taking an average value. To this aim, we consider the $\vec{\chi}$'s as independent random variables distributed between $-k_F$ and k_F with a top hat probability distribution $\propto \theta(k_F \pm \chi)$. Neglecting correlations between different $\vec{\chi}$'s is a valid approximation if $N_4 \ll N_3$. Consequently, we restrict ourselves to the values of $N_4 \leqslant 0.1 N_3$. The resulting probability of having N_4 Bragg-scattered fermions is

$$P(N_4) \propto \left(\frac{1}{\alpha^2}\right)^{N_3-N_4} \frac{1}{(N_3-N_4)!}, \tag{8}$$

where we have set $N_1 = N_2$. This Poissonian distribution depends on N_3 and on the parameter

$$\alpha = \frac{2\sqrt{10}\pi a n_b}{k_2 k_F}, \tag{9}$$

which is also N_3-dependent through the Fermi momentum k_F. The parameter α contains also the boson density n_b. In Fig. 2, we display the efficiency η, as a function of N_3 and the N_3-independent parameter $\alpha N_3^{1/3}$. We observe that for small values of α, the efficiency decreases with increasing N_3. On the contrary, with larger α the efficiency can reach larger values, limited only by the assumptions of the model, i.e., $\eta \leqslant 0.1$.

This simple analysis shows that under certain conditions, in particular if the number of fermions is sufficiently large, a macroscopic efficiency for the fermionic fourth wave can be created.

III. NUMERICAL RESULTS

The approximate results obtained above are a direct result of the Fermi-Bose statistics. Solving the complete many-body scattering problem for the fermions is a formidable task. We can simplify the situation assuming a polarized Fermi gas, i.e., noninteracting [19]. Second, we model the bosonic grating by a potential proportional to the local (in general time-dependent) BEC density. In fact, Fermi statistics appears in our model only through the initial state, which must obey the Pauli principle. In the simplest approach, we also neglect backaction from the fermions on the bosons. This approximation is valid in a situation where the boson density is much larger than the fermion density. We perform, however, also a full self-consistent simulation where the dynamics of the grating is taken into account. Due to numerical limitations, the latter could only be simulated for small numbers of fermions. In order to mimic the effect of high fermion densities, we used in this case a high value of the fermion-boson interaction strength. Finally, we restrict our numerical simulations to 2D. With these assumptions, we solve N_3 Schrödinger equations for the fermions with an external potential $V_f(x,y) = u_{\mathrm{BF}} n_b(x,y)$, with $u_{\mathrm{BF}} = 8\pi\hbar^2 a/mL$ and L the thickness of the cloud in the z direction, and $n_b(x,y)$ being the density of the condensate. For the bosons, the trap potential is a combination of a harmonic potential in the y direction and a periodic structure in the x direction created, for instance, by a standing-wave laser, $V_b(x,y) = \frac{1}{2} m \Omega_c^2 y^2 + U_l \cos^2(|\vec{k}_1 - \vec{k}_2| x)$. We have solved numerically the Gross-Pitaevskii (GP) equation with the potential $V_b(x,y)$ in order to determine the equilibrium bosonic density $n_b(x,y)$, which, in turn, results in a periodic potential $V_f(x,y)$ for the fermions. The backaction of fermions on bosons, where the fermions affect the dynamics of the bosons and the bosons affect the fermions, can be neglected provided $u_{\mathrm{BF}} n_f(x,y,t)/\mu \ll 1$, where μ is the chemical potential of the trapped condensate and $n_f(x,y,t)$ is the fermion density. In our simulation for the static case (no backaction), this ratio was kept smaller than 0.01. In the full self-consistent treatment, apart from N_3 equations for the fermions, we solve simultaneously the GP equation with the boson-fermion potential $u_{\mathrm{BF}} n_f(x,y,t)$ adjusted at each time step.

Initially, the fermions are trapped in a 2D potential of the form $V(x,y) = m\Omega^2(x^2 + y^2)$ centered at (x_0, y_0). The number of fermions is such that the Fermi level may not fulfill the condition $k_F \ll |\vec{k}_1 - \vec{k}_2|$. The trap is then removed and a momentum $\hbar \vec{k}_3$ is given to the fermions using a Bragg pulse [6,20]. Finally, we monitor the density of fermions to obtain the efficiency of the process.

The wave functions of the noninteracting fermions fulfill all the same Schrödinger equation,

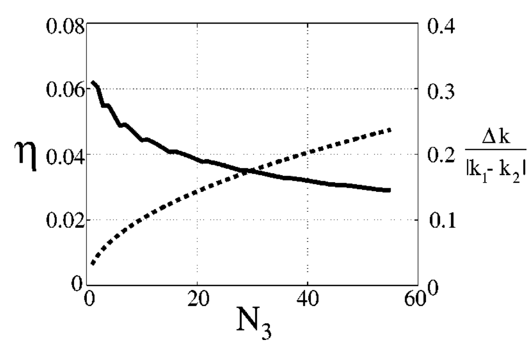

FIG. 3. Numerical estimate of the efficiency η versus the number of incoming fermions N_3. η decreases for larger values of N_3 due to the spread of the momentum, $\Delta k/|\vec{k}_1 - \vec{k}_2|$ on the right axis, for the incoming fermions.

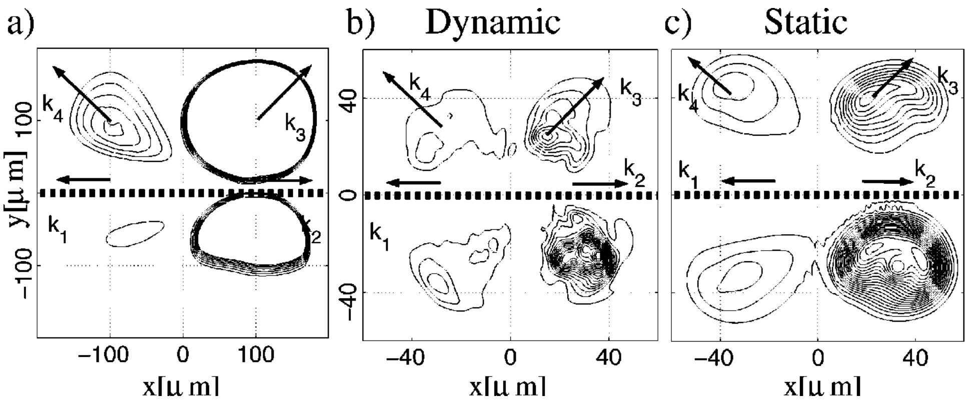

FIG. 4. A snapshot of the density of the fermionic cloud after the scattering. In (a), 3% of the atoms are found in the Bragg direction (see Fig. 3). The efficiency in (b), $\eta=0.091$ is still of the same order as in (c) with $\eta=0.149$. The grating is represented by the two counterpropagating waves $\vec{k}_1=-\vec{k}_2$.

$$i\hbar\frac{\partial}{\partial t}\Psi_i=\left[-\frac{\hbar^2}{2m}\nabla_i^2+V_f(x,y,t)\right]\Psi_i,\quad i=1,\ldots,N_3, \tag{10}$$

but with orthogonal initial conditions for each fermion. The initial states are the eigenstates of the displaced harmonic potential used for trapping the fermions,

$$\Psi_{n_x,n_y}(x,y)\propto e^{i\vec{k}_3\cdot\vec{x}}e^{-1/2(\tilde{x}^2+\tilde{y}^2)}H_{n_x}(\tilde{x})H_{n_y}(\tilde{y}), \tag{11}$$

where $\tilde{x}=(x-x_0)/\sqrt{\hbar/m\Omega}$, $\tilde{y}=(y-y_0)/\sqrt{\hbar/m\Omega}$, and H_{n_x} denotes the Hermite polynomials.

The scattering of the fermions can now be numerically simulated one by one in the static case and simultaneously for the self-consistent case. For the fermion cloud, we use ^{40}K atoms and a trap frequency of $\Omega/2\pi=10$ Hz, while for the bosonic trap we use a frequency of $\Omega_c=40\ \Omega$. This produces a narrow grating compared to the size of the fermionic cloud. Furthermore, we have taken the scattering length both for boson-boson and fermion-boson as $a=6.0$ nm and $L=30\ \mu$m with $N_c=2\times10^5$ bosons in the condensate. The grating has a wave number of $k_1=1.3\ \mu\text{m}^{-1}$ ($\vec{k}_1=-\vec{k}_2$) and $k_F/k_1<0.7$. The initial fermion cloud is positioned at $x_0=y_0=-40\ \mu$m and the momentum kick is settled to $\vec{k}_3=(1.3,1.3)\ \mu\text{m}^{-1}$.

In Fig. 3, we show the efficiency $\eta=N_4/N_3$ of the Bragg process as a function of the total number of fermions N_3 and the momentum spread Δk. With an increasing number of particles (N_3), the efficiency decreases due to the increase of the momentum spread of the fermions. Although this efficiency is very similar to the efficiency shown in Fig. 2, one should not be misled to compare the two results, since they correspond to two different regimes [22]. Figure 4 displays a snapshot of the cloud after the scattering. In Fig. 4(a), which corresponds to the situation in Fig. 3, we observe that approximately 3% of the cloud is scattered in the Bragg direction. In this figure, one can also see that a part of the fermionic cloud is reflected due to the chosen relation between the incoming fermion kinetic energy and the shape of the grating potential. Note also the appearance of the reflected wave packet in the direction $-\vec{k}_3$, which corresponds to a Bragg reflection. It is important to remember that the condensate is trapped in a harmonic trap in the y direction, which will consequently give a y component in the momentum from the grating. In Fig. 4(b), we present the self-consistent simulation where the dynamics of the grating is taken into account. Here we see that even if the densities for the bosons and fermions are of the same order, the effect is very similar to the situation with a static grating shown in Fig. 4(c). In these calculations, we have used only six fermions and a correspondingly low condensate density due to numerical limitations. The boson-fermion interaction energy $u_{\mathrm{BF}}n_{b,f}$, on the other hand, was similar to Fig. 4(a). Reflections are drastically reduced if the scattering length is negative since in this case the potential $V_f(x,y)$ becomes attractive. The Bragg scattering relies on the periodicity and contrast of the grating and it is present for both positive and negative scattering lengths. Also, for a fixed interaction time the efficiency of the Bragg process decreases with decreasing contrast of the grating.

The self-consistent simulation presented here offers a new tool to study nonlinear dynamical properties of boson-fermion mixtures. So far such static properties as different geometrical configurations for the ground state have been investigated [9]. With the time-dependent self-consistent approach, nonlinear phenomena such as spontaneous geometrical symmetry breaking of metastable configurations could be studied *in situ*. Also, the nonlinear shape oscillations for a boson-fermion mixture can be investigated by solving the coupled boson Gross-Pitaevskii equation and the fermion Schrödinger equations.

IV. CONCLUSIONS

In summary, we have discussed the effects of Bose and Fermi statistics on four-wave-mixing processes. For a pure bosonic process, the Bose statistics sets a fundamental limit for the efficiency. In the case of an incoming fermionic cloud, the result depends strongly on the various physical parameters involved in the problem. On the one hand, our numerical analysis shows that the momentum spread $\Delta k\ll|\vec{k}_1-\vec{k}_2|$ is a crucial parameter in order to obtain a macroscopic efficiency for the fourth wave. On the other hand, the analytical treatment, which assumes that the Bragg condition is always fulfilled, exhibits an interplay between the statistical and collisional effects leading to an efficiency decreasing for small N_3 and $\alpha N_3^{1/3} < 0.3$. For small α, the value of $\langle N_4 \rangle$ is negligible, meaning a small efficiency. On the contrary, for sufficiently large values of α, the efficiency can attain macroscopic values of the order of a few percent, indicating the creation of a macroscopic fermionic fourth wave.

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[22] Nevertheless, the efficiency curves for the numerical and analytical results show a similar behavior for $N_3 \approx 50$, but this is already at the border of validity for the analytical approach.