# Discrete time in quantum mechanics

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The possibility that time can be regarded as a discrete dynamical variable is reexamined. We study the dynamics of the free particle and find in some cases superluminal propagation.

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## I. INTRODUCTION

The purpose of any physical theory, such as the one under discussion in this paper, is to set up a mathematical model that enables us to correlate some empirical phenomena. A physical theory is considered satisfactory if we can make quantitative predictions of physical data. Such data usually involve the measurements of certain quantities that are expressed in a system of fundamental units. The choice of these units is to some extent arbitrary with respect to magnitude as well as with respect to kind. Such a choice of units will therefore be guided primarily by considerations of convenience.

A great simplification is introduced in quantum mechanics if we use as the fundamental units certain physical quantities that are constants of nature. Two constants of this sort are $c$, the speed of light in vacuum, and $\hbar$, Planck's bar constant. We usually choose these constants as two of the fundamental units of our systems. As the third unit, we use the second or centimeter as the conventional and arbitrary unit of time or length. Here we shall mention that this choice is guided by the fact that the theory under discussion arises from an intimate relation of special relativity characterized by the constant $c$ and quantum mechanics characterized by $\hbar$. To our knowledge, at present there exists no theory that involves in its fundamental laws either a universal time or a universal length, which would make a natural choice of the third basic unit. The need for such a theory involving a fundamental time (length) has been the subject of much speculation in the past and present, but it seems safe to say that we are far from understanding the role of such a unit in existing theories.

Throughout the development of quantum mechanics, time always appears as a continuous parameter. Take the example of a nonrelativistic particle. In Feynman's path integration formulation, the probability amplitude for the particle to be at the position $\mathbf{q}(t_i)$ and at $\mathbf{q}(t_f)$ is given by the amplitude sum over all paths $\mathbf{q}(t)$ connecting $\mathbf{q}(t_i)$ and $\mathbf{q}(t_f)$, apart from a normalization constant. Clearly, the position of the particle $\mathbf{q}$ is not treated on the same basis as the (real) time $t$: at a given time, the path integration can be viewed as that over the whole range of eigenvalues of the position operator. This then underlies the familiar difference between $\mathbf{q}$ as an operator and $t$ as a parameter.

In fact, this asymmetry can be made out in classical mechanics. The classical trajectory of a particle is determined by the extremity of the action, which is a functional of $\mathbf{q}(t)$. While $\mathbf{q}$ is the dynamical variable, $t$ appears only as a continuous parameter. By setting the variational derivative, we obtain the usual Lagrange equation of motion, whose solution gives the classical path. In relativistic quantum field theory, space and time have to be treated symmetrically due to Lorentz invariance. The usual approach is to regard $q_i$ and $t$ all as parameters; the operators are then field variables.

The purpose here is not to replace a continuous dynamical evolution parameter with a discrete parameter. Our interest is in the construct of a self-consistent discrete complex-time quantum mechanics with well-specified equations of motion. This is motivated by the notion that at some small scale, time is really discrete. This has echoes in theories such as relativistic quantum mechanics with a time associated to the electron's Compton wavelength ($10^{-22}$ s) and string theory, where the Planck time ($10^{-43}$ s) sets a scale at which conventional notions of space and time break down.

There are various circumstances in physics where it is convenient or necessary to replace the continuous time (temporal evolution) parameter with a discrete parameter. There have been various attempts to construct classical and quantum-mechanical theories based on this notion, such as the work of Caldirola [1] and Lee [2]. The work of Yamamoto *et al.* [3], Hashimoto *et al.* [4], Klimek [5], Jaroszkiewicz and Norton [6], and Milburn [7,8] show that the subject continues to receive attention.

The underlying postulate is that on sufficiently short time steps, the system does not develop continuously under a mixture of unitary and nonunitary evolution but rather in a sequence of *identical* transformations. The inverse of this time step is the mean frequency of the steps, $\delta t$, which turns into an expansion parameter. If the time step is large enough, the evolution appears approximately continuous on laboratory time scales. To zeroth order, the Schrödinger equation is recovered.

One feature of this model is that constants of motion remain constants of motion and thus stationary states remain stationary states. Whether or not these consequences are observable depends on the size of $\delta t$.

In the following, we wish to explore some alternative possibilities. First, in place of treating time as a real parameter, we may consider time as a continuous *complex* parameter (analytical continuation formulation). Second, time can be treated as a *discrete* complex parameter (discrete complex-time formulation). As we shall see, both possibilities can be realized. The result is that in this new formalism, our usual idea of continuous-time structure will appear only as an approximation.

 

## II. CONTINUOUS-TIME EVOLUTION

Let us consider a quantum system whose time evolution is given by the complex-time propagator

$$U(s)=\exp(sH), \tag{1}$$

where $s=(-i/\hbar)(t+iv)$ with $t$ and $\nu$ real parameters. In the above the Hamiltonian, $H$ is assumed to be Hermitian and time-independent. In a particular physical system, we look for a complete set of commuting observables $\hat{\alpha}$. We then can take their simultaneous eigenkets as basic kets: $|\Psi_\alpha(s)\rangle$, $\alpha=(\alpha_1,\alpha_2,\ldots)$, where $\alpha_i$ is the eigenvalue of the observable $\hat{\alpha}_i$. The Schrödinger equation for the system is then

$$H|\Psi_\alpha(s)\rangle=\frac{d}{ds}|\Psi_\alpha(s)\rangle=E_\alpha|\Psi_\alpha(s)\rangle. \tag{2}$$

The reason for looking at the propagator (1) will be clearer as we go along. First let us note that we can write down the formula for it at once:

$$U(s)=\sum_\alpha |\Psi_\alpha(s)\rangle\langle\Psi_\alpha(s)|\exp(sE_\alpha). \tag{3}$$

The main point to note is that even though the time is now complex, the eigenvalues and eigenfuctions that enter into the formula for $U(s)$ are the usual ones. Conversely, if we knew $U(s)$, we could extract the former.

The Hamiltonian is assumed to be represented by a self-adjoint operator. According to the basic principles of QM, one defines a Hilbert space $\mathcal{H}$ for each QM system. Every measurable quantity or ''observable'' is represented by a self-adjoint operator. The state of the system at time $s$ is given by a vector $|\Psi(s)\rangle\in\mathcal{H}$, which is analytic in the complex plane defined by $s$. Note that

$$\begin{aligned}\frac{d}{ds}|\Psi(s)\rangle&=\frac{\partial t}{\partial s}\frac{\partial}{\partial t}|\Psi(s)\rangle=i\hbar\frac{\partial}{\partial t}|\Psi(s)\rangle,\\ \frac{d}{ds}|\Psi(s)\rangle&=\frac{\partial v}{\partial s}\frac{\partial}{\partial v}|\Psi(s)\rangle=\hbar\frac{\partial}{\partial v}|\Psi(s)\rangle.\end{aligned} \tag{4}$$

The $\Psi$ are then normalizable, i.e.,

$$\langle\Psi(s)|\Psi(s)\rangle=\int_{\mathcal{V}}\Psi^\dagger(\mathbf{q},s)\Psi(\mathbf{q},s)d^3q=1, \quad \forall s, \tag{5}$$

where $\mathcal{V}$ is the volume where the system is contained. Given a basis $\{|\Psi_j(s)\rangle\}$ of $\mathcal{H}$, a self-adjoint operator is defined as satisfying

$$\langle\Psi_i(s)|O|\Psi_j(s)\rangle=\langle\Psi_j(s)|O|\Psi_i(s)\rangle^*. \tag{6}$$

The stationary states can be written in the form

$$\langle\mathbf{q}|\Psi_{E_\alpha}(s)\rangle=\Psi_{E_\alpha}(\mathbf{q},s)=e^{sE_\alpha}u_{E_\alpha}(\mathbf{q}), \tag{7}$$

with

$$Hu_{E_\alpha}=Eu_{E_\alpha}. \tag{8}$$

The (conformal) mapping $s\rightarrow\exp(sE_\alpha)$, which has no zeros and no singularities in the entire complex plane, turns out to possess an essential singularity at infinity.

## III. GENERAL EVOLUTION

The feature of quantum mechanics that most distinguishes it from classical mechanics is the coherent superposition of distinct physical states. This feature is at the heart of the less intuitive aspects of the theory. It is the basis for the concern about measurement in quantum mechanics, and it is the explanation for the nonappearance of chaos in systems that classically would be chaotic. Apparently, however, the superposition principle does not operate on macroscopic scales, although nothing in the present formulation of quantum mechanics would indicate this.

We now consider an ad hoc time distribution $0,s_1,s_2,\ldots,s_N$ in the complex plane, where $s_i-s_j=\delta s$ is a fundamental time interval. Thus in the Schrödinger equation we need to introduce a *discrete derivative* associated with the given time distribution, namely

$$H_D|\Psi_\alpha(s)\rangle=\frac{\delta_\lambda}{\delta s}|\Psi_\alpha(s)\rangle, \tag{9}$$

with $H_D$ in the $s$ representation, where $s$ is a given $s_j$, and

$$\begin{aligned}\delta_\lambda|\Psi_\alpha(s)\rangle\equiv\lambda^{-1}(s,\delta s)[&|\Psi_\alpha(s+\delta s)\rangle\\ &-|\Psi(s+\delta s-\delta s\lambda(s,\delta s))\rangle]\end{aligned} \tag{10}$$

is a state difference where $\lambda$ is a *holomorphic* function of $s$ and $\delta s$ in the whole $\lambda$-complex plane, with $\delta s$ a given finite difference in the complex *time* plane. This is to be interpreted as a more general time evolution, the continuous evolution being a limited case.

In the above,

$$s=-\frac{i}{\hbar}(t+iv), \quad \delta s\equiv-\frac{i}{\hbar}(\delta t+i\delta v), \tag{11}$$

and $|\Psi_\alpha(s)\rangle$ is *analytic* at $s$, therefore it can be expanded in a Laurent series. Thus

$$\frac{\delta_\lambda}{\delta s}|\Psi_\alpha(s)\rangle=\frac{d}{ds}|\Psi_\alpha(s)\rangle+\sum_{n=2}^\infty\frac{\lambda_s^{n-1}}{n!}\frac{d^n}{ds^n}|\Psi_\alpha(s)\rangle, \tag{12}$$

i.e.,

$$H_D=\frac{2}{\lambda_s\delta s}e^{[1-(\lambda_s/2)]\delta sH}\sinh\left(\frac{\lambda_s}{2}\delta sH\right), \tag{13}$$

where $\lambda_s\equiv\lambda(s,\delta s)$. For $\lambda_s$ a constant function of $s$, the stationary states are of the form

$$\Psi_{E_D}(\mathbf{q},s)=e^{s\epsilon}u_E(\mathbf{q})=e^{1/\hbar(t\epsilon_I+v\epsilon_R)}e^{-i/\hbar(t\epsilon_R-v\epsilon_I)}u_E(\mathbf{q}), \tag{14}$$

with $E$ the eigenvalues of $H$ and $\epsilon \equiv \epsilon_R + i\epsilon_I$. In other words, we assume that $H$ and $H_D$ act on the same Hilbert space. Furthermore, if

$$H_D u_E(\mathbf{q}) = E_D u_E(\mathbf{q}), \tag{15}$$

we find that

$$E_D = \frac{2}{\lambda_s \delta s} e^{[1-(\lambda_s/2)]\delta s E} \sinh\left(\frac{\lambda_s}{2}\delta s E\right). \tag{16}$$

We must require some basic physical conditions upon the energy eigenvalues. Particularly, they must take real values, which means that $\mathrm{Im}\, E_D = 0$, a condition we have to impose on Eq. (16).

## IV. EXAMPLES

We shall consider two particular cases where the Hamiltonian $H_D$ is Hermitian.

Case (a) regards $\lambda_s = 1$, with $\delta s = (1/\hbar)\tau$, where $\tau = \tau_1$ is a finite time element. This case corresponds to an intrinsic loss of information. The discrete derivative is

$$\frac{\delta_\lambda}{\delta s} = H_D = \frac{2\hbar}{\tau} e^{(\tau/2)i(\partial/\partial t)} \sinh\left(\frac{i\tau}{2}\frac{\partial}{\partial t}\right) = \frac{\hbar}{\tau}(e^{\tau i(\partial/\partial t)} - I). \tag{17}$$

Thus

$$H_D|\Psi_\alpha(s)\rangle = \frac{\delta_\lambda}{\delta s}|\Psi_\alpha(s)\rangle = \sum_{n=1}^{\infty} \frac{(\delta s)^{n-1}}{n!}\frac{d^n}{ds^n}|\Psi_\alpha(s)\rangle. \tag{18}$$

To determine the commutator $[H_D, s]$, we evaluate this commutator operating on $\Psi(s)$, i.e.,

$$[H_D, s] = \exp\left(\frac{\tau}{\hbar}H\right) = i\hbar\left(I + \frac{\tau}{\hbar}H_D\right), \tag{19}$$

which involves a modification to the standard time-energy commutation relation.

Next let us consider a relativistic spin-0 free particle. The Hamiltonian becomes

$$H_D = \frac{\hbar}{\tau}\left[\exp\left(\frac{\tau}{\hbar}H\right) - 1\right], \tag{20}$$

where $H = \sqrt{c^2\mathbf{p}^2 + m^2c^4}$, $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$. Therefore,

$$\frac{\delta\hat{q}_j}{\delta s} = \frac{i}{\hbar}[H_D, \hat{q}_j] = H^{-1}\exp\left(\frac{\tau}{\hbar}H\right)c^2 p_j, \tag{21}$$

where

$$\hat{q}_j \equiv q_j, \quad \hat{p}_j \equiv \frac{H_D}{c^2}\frac{\delta\hat{q}_j}{\delta s} = H^{-1}H_D \exp\left(\frac{\tau}{\hbar}H\right)p_j \tag{22}$$

are canonical conjugate coordinates, with $p_j = -i\hbar\,\partial/\partial q_j$. Furthermore,

$$\frac{\delta\hat{q}_j}{\delta s} = H^{-1}\exp\left(\frac{\tau}{\hbar}H\right)c^2 p_j \tag{23}$$

corresponds to the *group velocity* of the quantum waves. Thus from Eq. (22) we get

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0,$$

$$\begin{aligned}[\hat{q}_i, \hat{p}_j] &= i\hbar\left(I + \frac{\tau}{\hbar}H_D\right)\Bigg\{H^{-1}H_D\delta_{ij} \\ &\quad + \left[1 - \left(1 - \frac{2\tau H}{\hbar}\right)H_D H^{-1}\right]c^2H^{-2}p_ip_j\Bigg\} \\ &= i\hbar\left(1 + \frac{3}{2}\frac{\tau}{\hbar}H_D\right)\delta_{ij} + i\frac{3}{2}\frac{\tau c^2}{\hbar}H_D^{-1}p_ip_j + O\left[\left(\frac{\tau}{\hbar}\right)^2\right].\end{aligned} \tag{24}$$

Let us now postulate the existence of a relativistic invariant mass $M$ in this context. To this end, we impose the condition

$$E_D^2 - c^2\hat{\mathbf{p}}^2 = E_D^2\left\{1 - \exp\left(\frac{2\tau E}{\hbar}\right)\left[1 - \left(\frac{mc^2}{E}\right)^2\right]\right\} = M^2c^4. \tag{25}$$

For $\mathbf{p} = \mathbf{0}$, we get

$$M = \frac{\hbar}{\tau c^2}\left[\exp\left(\frac{\tau}{\hbar}mc^2\right) - 1\right] = m + \frac{\tau}{2\hbar}m^2c^2 + O\left[\left(\frac{\tau}{\hbar}\right)^2\right], \tag{26}$$

which represents a shift in value of the inertial mass $m$.

For a massless particle,

$$E_D = \frac{\hbar}{\tau}\left[\exp\left(\frac{\tau c}{\hbar}|\mathbf{p}|\right) - 1\right], \quad v_j(\tau) = c\frac{e^{(\tau c/\hbar)|\mathbf{p}|}}{|\mathbf{p}|}p_j. \tag{27}$$

As is apparent, even for very small time step $\tau$, this model predicts the possibilities of superluminal propagation provided that $|\mathbf{p}| \gtrapprox 0$.

Case (b) addresses $\lambda_s = 2$, with $\delta s = (-i/\hbar)\tau_0$ a step time element, i.e., unitary evolution. The generalized Schrödinger equation is

$$H_D|\Psi_\alpha(s)\rangle = \frac{\delta_\lambda}{\delta s}|\Psi_\alpha(s)\rangle, \tag{28}$$

with

$$\frac{\delta_\lambda}{\delta s}|\Psi_\alpha(s)\rangle = \frac{1}{2\,\delta s}[|\Psi(s+\delta s)\rangle - |\Psi(s-\delta s)\rangle], \tag{29}$$

the so-called *symmetric derivative*. The Hamiltonian takes the form [1]

$$H_D = \frac{\hbar}{\tau_0}\sin\left(\frac{\tau_0}{\hbar}H\right) = \frac{\hbar}{\tau_0}\sin\left(i\tau_0\frac{\partial}{\partial t}\right). \tag{30}$$

Therefore, the velocity operator is now given by

$$\frac{\delta\hat{q}_j}{\delta s}=\frac{i}{\hbar}[H_D,\hat{q}_j]=H^{-1}\cos\left(\frac{\tau_0}{\hbar}H\right)c^2p_j. \qquad (31)$$

The position and momentum operators are then

$$\hat{q}_j\equiv q_j, \quad \hat{p}_j\equiv\frac{H_D}{c^2}\frac{\delta\hat{q}_j}{\delta s}=\frac{\sin(2\tau_0E/\hbar)}{2\tau_0E/\hbar}p_j, \qquad (32)$$

with $p_j=-i\hbar\,\partial/\partial q_j$ as before. Thus from Eq. (32) we get

$$[\hat{q}_i,\hat{q}_j]=[\hat{p}_i,\hat{p}_j]=0,$$

$$\begin{aligned}[\hat{q}_i,\hat{p}_j]&=\delta_{ij}i\hbar\frac{\sin(2\tau_0E/\hbar)}{2\tau_0E/\hbar}\\&\quad+i\hbar\left(\cos(2\tau_0E/\hbar)-\frac{\sin(2\tau_0E/\hbar)}{2\tau_0E/\hbar}\right)\frac{c^2p_ip_j}{E^2} \qquad (33)\\&=\delta_{ij}i\hbar\left[I-\frac{2}{3}\left(\frac{\tau_0}{\hbar}\right)^2H_D^2\right]-i\hbar\frac{4}{3}\left(\frac{\tau_0}{\hbar}\right)^2c^2p_ip_j\\&\quad+O\left[\left(\frac{\tau_0}{\hbar}\right)^3\right].\end{aligned}$$

For a massless particle,

$$E_D=\frac{\hbar}{\tau_0}\sin\left(\frac{c\tau_0}{\hbar}p\right), \quad v_j(\tau_0)=c\cos\left(\frac{c\tau_0}{\hbar}p\right)\frac{p_j}{p}. \qquad (34)$$

For bound states, Eq. (30) says that a maximum value for the energy of the excited states exists: $E_D=\hbar/\tau_0$.

## V. FINAL COMMENTS

To conclude, on sufficiently small time scales we conjecture the system evolves by a sequence of unitary timelike steps generated by the Hamiltonian. The Schrödinger equation can be obtained to $n$th order in the expansion (complex) parameter $\delta s$. This timelike discretization involves a modification both to the standard time-energy commutation relation and to the $q,p$ canonical commutation relations. Particularly, for the case (a) even for small time step $\delta s$, the model predicts the possibility of superluminal velocities, for instance, for a massless particle with nonvanishing momentum.

If the physical evolution time scale is much larger than the discrete time scale, then the evolution resembles a quantum stochastic process. This process could be studied using the quantum version of the Ito and Stratonovich stochastic calculus [9].


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