Optimal tests of quantum nonlocality

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We present a general method for obtaining *all* Bell inequalities for a given experimental setup. Although the algorithm runs slowly, we apply it to two cases. First, the Greenberger-Horne-Zeilinger setup with three observers each performing one of two possible measurements. Second, the case of two observers each performing one of *three* possible experiments. In both cases we obtain hundreds of inequalities. Since this is the set of all inequalities, the one that is maximally violated in a given quantum state must be among them. We demonstrate this fact with a few examples. We also note the deep connection between the inequalities and classical logic, and their violation with quantum logic.

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We shall present a general method for the derivation of *all* Bell inequalities for each given experimental setup. The validity of the method has been proved previously [1,2]. The purpose of this paper is to turn the method into a working algorithm and demonstrate its strength in two cases.

Consider some arbitrary elementary events A, B, C, . . . , such as *''the electron spin in the x direction is up,''* as well as some of the joints of these propositions; e.g., AB, AC, . . . , ABC, . . . ,. In order to be consistently interpretable, the probabilities of these events $P(A)$, $P(B)$, $P(C)$, . . . , $P(AB)$, $P(AC)$, . . . , $P(ABC)$, . . . , must satisfy some inequalities; for example: $P(A)+P(B)-P(AB)\leqslant 1$ or $P(A)-P(AB)-P(AC)+P(BC)\geqslant 0$. These inequalities are satisfied for every possible classical probability distribution P.

In the middle of the 19th century, George Boole [1,3–6] investigated these inequalities and referred to them as *conditions of possible experience.* The number and complexity of the inequalities increase fast as the number of events grow. Among them are the famous inequalities that arise in the Einstein-Podolsky-Rosen (EPR) experiment and its generalizations, in particular the Bell/(EPR) experiment and its generalizations, in particular, the Bell inequalities and Clauser-Horne (CH) inequalities [7–10].

Consider, for example, the latter. We have four events: A_1,A_2 that correspond to Alice's measurements on the left, and B_1,B_2 measured by Bob on the right. In order to derive the CH inequalities we list the $2^4=16$ extreme cases where the probability of the elementary events A_1,A_2,B_1,B_2 are set to be either zero or one. That is, we consider the *truth* Table I, where $t(A_i),t(B_j)\in\{0,1\}$. Assume that each of the sixteen rows in the truth table is a vector in an eight-dimensional real space. Denote by C the convex hull of the sixteen vectors taken as vertices. C is a *correlation polytope*. Now, let P be any classical probability distribution on the Boolean algebra generated by the events A_1,A_2,B_1,B_2. It is not hard to see that the vector

$$p=[P(A_1),\ P(A_2),\ P(B_1),\ P(B_2),\ P(A_1B_1),\ P(A_1B_2),\ P(A_2B_1),\ P(A_2B_2)] \tag{1}$$

is an element of C. Conversely, if $p\in C$, then there is a probability distribution P such that p has the representation (1) [1].

Every convex polytope has two representations: One as the convex hull of its vertices (the V representation), and the other as the intersection of a finite number of half spaces, each given by a linear inequality (the H representation). The problem of finding the inequalities when the vertices are known is called *the hull problem.*

Solving the hull problem for the CH case yields $0\leqslant P(A_iB_j)\leqslant P(A_i),\ P(B_j),\ 1\geqslant P(A_i)+P(B_j)-P(A_iB_j)$, $i,j=1,2$, as well as $-1\leqslant P(A_1B_1)+P(A_1B_2)+P(A_2B_2)-P(A_2B_1)-P(A_1)-P(B_2)\leqslant 0$, and permutations $(A_1\leftrightarrow A_2;\ B_1\leftrightarrow B_2)$ thereof.

A necessary and sufficient condition that a vector p is an element of C is that its coordinates satisfy these inequalities [1]. As is well known, some of the CH inequalities are violated by the relative frequencies measured in the EPR experiment. This fact can be taken as an indication that the underlying Boolean structure (classical propositional logic) should be replaced by the nondistributive quantum logic [1,11].

The above procedure can be applied to any number of events. If there are n elementary events, then we have 2^n vertices, and the dimension of the space is $n+k$, where k is the number of (pair, triple, . . . ,) intersections that we consider. There are algorithms to solve the hull problem but they run in exponential time in n. (In fact, deciding if a vector p is an element of the corresponding correlation polytope is NP complete [2].) However, for small enough cases the problem can be solved fairly quickly by one of the available algorithms.

We have chosen the CDD package [12] which is an efficient implementation of the double description method [13] due to Fukuda, Prodon, and Rosta [14–16], as well as the LPOLY package due to Kreuzer and Skarke [17]. We have

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TABLE I. Truth table corresponding to the CH inequalities.

A_1	A_2	B_1	B_2	A_1B_1	A_1B_2	A_2B_1	A_2B_2
$t(A_1)$	$t(A_2)$	$t(B_1)$	$t(B_2)$	$t(A_1)t(B_1)$	$t(A_1)t(B_2)$	$t(A_2)t(B_1)$	$t(A_2)t(B_2)$

selected two examples by which to demonstrate the method and the violation of the inequalities by quantum frequencies. The first is the Greenberger-Horne-Zeilinger (GHZ) case of three particles and two possible measurements on each particle. The second is the case of two particles and *three* possible measurements on each one. This last case may be of particular interest to experimentalists. Here, one obtains a considerable improvement of the results (in the strength of violation of the inequalities, and in the *number* of inequalities that are violated) without an intractable increase in the complexity of the experiment.

In the Mermin version [18,19] of the GHZ case [20,21], the relevant propositions involve three particles, denoted by A,B,C, and two properties, denoted by 1,2, respectively. The set of 26 propositions involve all three-particle events and is given by $\{A_1, A_2, B_1, B_2, C_1, C_2, A_1B_1, A_1C_1, A_1B_2, A_1C_2, A_2B_1, A_2C_1, A_2B_2, A_2C_2, B_1C_1, B_1C_2, B_2C_1, B_2C_2, A_1B_1C_1, A_1B_1C_2, A_1B_2C_1, A_1B_2C_2, A_2B_1C_1, A_2B_1C_2, A_2B_2C_1, A_2B_2C_2\}$.

The resulting correlation polytope is 26-dimensional and has 64 vertices and 53 856 faces corresponding to an equal amount of Boole-Bell type inequalities. For a complete listing of all Boole-Bell type inequalities, see Ref. [22]. Many of these inequalities are trivial; e.g., $P(A_1B_1) \geqslant P(A_1B_1C_1) \geqslant 0$ or $P(A_1)+P(A_1B_1C_1) \geqslant P(A_1B_1) + P(A_1C_1)$. Many inequalities can be reduced to others by the symmetries. There are two types of symmetries. One kind is obtained by permuting the events, the second type by *complementing* the events. If an inequality is valid for an event A then it is also valid for its complement $\overline{A}$. Thus, we can substitute $P(\overline{A})=1-P(A)$ instead of $P(A)$ in the inequality, substitute $P(\overline{A}B)=P(B)-P(AB)$ instead of $P(AB)$, and replace $P(ABC)$ by $P(\overline{A}BC)=P(BC) - P(ABC)$. Each event can be complemented in this way resulting in additional $2^6=64$ symmetry operations. Inequalities which have been discussed in this context by Larsson and Semitecolos [23] and by de Barros and Suppes [24] have similar counterparts in the enumeration. See also Kaszlikowski *et al.* [25] for a related approach. We stress here that our method produces *optimal* Boole-Bell inequalities in the sense that they represent the *best possible upper bounds* for the conceivable classical probabilities. In what follows, we shall enumerate some of the new Boole-Bell inequalities.

$$\begin{aligned} 2\geqslant & -P(A_1)+2P(A_2)+P(B_1)+P(B_2)-P(C_1)+2P(C_2)-P(A_1B_1)+P(A_1C_1)+2P(A_1B_2)+P(A_1C_2)\\ & -P(A_2B_1)+P(A_2C_1)-2P(A_2B_2)-3P(A_2C_2)+P(B_1C_1)-P(B_2C_1)-P(B_1C_2)-2P(B_2C_2)+2P(A_1B_1C_1)\\ & -2P(A_2B_1C_1)-2P(A_1B_2C_1)-2P(A_1B_1C_2)+2P(A_2B_2C_1)+2P(A_2B_1C_2)-P(A_1B_2C_2)+3P(A_2B_2C_2), \end{aligned} \tag{2}$$

$$\begin{aligned} 3\geqslant & +2P(A_2)+3P(B_2)+2P(C_2)+2P(A_1C_1)-P(A_1C_2)+P(A_2B_1)-P(A_2C_1)-3P(A_2B_2)-P(A_2C_2)+P(B_1C_2)\\ & -3P(B_2C_2)+P(A_1B_1C_1)-2P(A_2B_1C_1)-3P(A_1B_2C_1)-2P(A_1B_1C_2)+2P(A_2B_2C_1)-2P(A_2B_1C_2)\\ & +2P(A_1B_2C_2)+2P(A_2B_2C_2), \end{aligned} \tag{3}$$

$$\begin{aligned} 0\geqslant & -3P(A_1)-2P(B_1)-P(C_1)+2P(A_1B_1)+P(A_1C_1)+3P(A_1B_2)+3P(A_1C_2)+2P(A_2B_1)+P(A_2C_1)\\ & -2P(A_2B_2)-P(A_2C_2)+P(B_1C_1)+P(B_2C_1)+2P(B_1C_2)-2P(B_2C_2)+P(A_1B_1C_1)-2P(A_2B_1C_1)\\ & -3P(A_1B_2C_1)-4P(A_1B_1C_2)+P(A_2B_2C_1)-P(A_2B_1C_2)-P(A_1B_2C_2)+3P(A_2B_2C_2), \end{aligned} \tag{4}$$

$$\begin{aligned} 0\geqslant & -P(A_1)-2P(B_1)-2P(C_1)+2P(A_1B_1)+2P(A_1C_1)+P(A_1B_2)+P(A_1C_2)+P(A_2B_1)+P(A_2C_1)\\ & -P(A_2B_2)-P(A_2C_2)+2P(B_1C_1)+2P(B_2C_1)+2P(B_1C_2)-2P(B_2C_2)-P(A_1B_1C_1)-2P(A_2B_1C_1)\\ & -3P(A_1B_2C_1)-3P(A_1B_1C_2)-P(A_2B_2C_1)-P(A_2B_1C_2)-P(A_1B_2C_2)+4P(A_2B_2C_2). \end{aligned} \tag{5}$$

Suppose the elementary experiences or propositions are clicks in a counter of a three particle interferometer as discussed by Greenberger, Horne, Shimony, and Zeilinger [21]. In the interferometric case [21], $P(A_i)=P(B_i)=P(C_i) =1/2$ and $P(A_iB_j)=P(A_iC_j)=P(B_iC_j)=1/4$, where $i,j =1,2$. The joint quantum probabilities of events depend on three angles ϕ_1,ϕ_2,ϕ_3 in each one of the detector groups A,B,C, respectively. They are given by $P(A_iB_jC_k) =(1/8)[1-\sin(\phi_{A,i}+\phi_{B,j}+\phi_{C,k})]$, where again $i,j,k=1,2$. For example, C_2 corresponds to the proposition, ''*the first detector of the detector group C at angle* $\phi_{C,2}$ *clicks*'' (we only consider clicks in the first one of the two detectors here). Yet it should be stressed that the derived inequalities are in no way dependent on this particular interpretation. Any other, in particular, one evolving spin-state measurements, would do just as well. Let us specify the angles at $\phi_{l,1}=0$ and $\phi_{l,2}=\pi/2$ for all particles labeled by $l=A,B,C$. Then, Eqs. (2)–(5) are among the 1329 equalities (out of 53 856) which violate Boole's condition of possible experience. The corresponding factors are 2:9/8, 3:25/8, 0:1/2, 0:1/2, respectively. Figure 1 represents a numerical study of the case $\phi_{l,1}=0$ and $0\leqslant\phi_{l,2}\leqslant\pi$ (the drawing is π periodic) for all particles labeled by $l=A,B,C$. All inequalities of the form $x\geqslant y$ have been rewritten as functions $f(x,y)=y-x$, such that the zero baseline indicates the borderline between the conditions of possible classical experience and the quantum violation thereof.

Notice that the inequalities can also be written in a form containing only coincidence probabilities of three events. For instance, Eq. (5) yields

$$\begin{aligned}0\geqslant&-P(A_1B_1C_1)-2P(A_2B_1C_1)-3P(A_1B_2C_1)\\&-3P(A_1B_1C_2)-P(A_2B_2C_1)-P(A_2B_1C_2)\\&-P(A_1B_2C_2)+4P(A_2B_2C_2),\end{aligned}\tag{6}$$

which is maximally violated by 1:0.55 for $\phi_{l,1}=0$ and $\phi_{l,2}\approx 1.45$. We find that it is not possible to obtain a violation of Boole-Bell type inequalities if only single-particle and three-particle coincidences are taken into account. This occurs only if the two-particle coincidences are also added.

We shall next consider the case of two particles, labeled by A,B, and three properties per particle, denoted by 1,2,3, respectively. The set of 15 propositions involve all three-particle events and is given by $\{A_1, A_2, A_3, B_1, B_2, B_3, A_1B_1, A_1B_2, A_1B_3, A_2B_1, A_2B_2, A_2B_3, A_3B_1, A_3B_2, A_3B_3\}$.

The resulting correlation polytope is 15-dimensional and has 684 faces, corresponding to 684 Boole-Bell type inequalities. For a complete listing of all Boole-Bell type inequalities, see Ref. [26]. Again, many of these inequalities are trivial; e.g., $P(A_2)\geqslant P(A_1B_3)\geqslant 0$. Many inequalities are familiar ones, such as the inequalities associated with the Bell-Wigner polytope ($\{A_1,A_2,A_3,A_1A_2,A_1A_3,A_2A_3\}$); i.e.,

$$\begin{aligned}1\geqslant&+P(A_2)+P(B_3)+P(A_1B_1)-P(A_1B_3)\\&-P(A_2B_1)-P(A_2B_3)\\\geqslant&+P(A_1)+P(A_2)+P(A_3)-P(A_1A_2)\\&-P(A_1A_3)-P(A_2A_3),\end{aligned}\tag{7}$$

if one identifies $A_i\equiv B_i$, $i=1,2,3$ [recall that $P(A_1A_1)=P(A_1)$]. The following Boole-Bell inequalities are less known.

$$\begin{aligned}3\geqslant&2P(A_1)+P(A_2)+P(B_2)+2P(B_3)-P(A_1B_1)\\&-P(A_1B_2)-P(A_1B_3)+P(A_2B_1)-P(A_2B_2)\\&-P(A_2B_3)+P(A_3B_2)-P(A_3B_3),\end{aligned}\tag{8}$$

$$\begin{aligned}1\geqslant&-P(A_1)+P(A_2)-P(B_2)+P(B_3)+P(A_1B_1)\\&+P(A_1B_2)-P(A_2B_1)+P(A_2B_2)-P(A_2B_3)\\&+P(A_3B_1)-P(A_3B_2)-P(A_3B_3),\end{aligned}\tag{9}$$

$$\begin{aligned}1\geqslant&P(A_2)-P(A_3)-2P(B_1)+P(B_3)+P(A_1B_1)\\&+P(A_1B_2)-P(A_1B_3)+P(A_2B_1)-P(A_2B_2)\\&-P(A_2B_3)+P(A_3B_1)+P(A_3B_3),\end{aligned}\tag{10}$$

$$\begin{aligned}2\geqslant&P(A_2)+P(A_3)+P(B_1)+P(B_3)+P(A_1B_1)\\&-P(A_1B_2)-P(A_1B_3)-P(A_2B_1)+P(A_2B_2)\\&-P(A_2B_3)-P(A_3B_1)-P(A_3B_2),\end{aligned}\tag{11}$$

$$\begin{aligned}0\geqslant&-P(A_1)-P(A_2)-P(B_1)-P(B_2)-P(A_1B_1)\\&+P(A_1B_2)+P(A_1B_3)+P(A_2B_1)+P(A_2B_3)\\&+P(A_3B_1)+P(A_3B_2)-P(A_3B_3),\end{aligned}\tag{12}$$

$$\begin{aligned}0\geqslant&-P(A_1)-P(B_3)+P(A_1B_2)+P(A_1B_3)\\&-P(A_2B_2)+P(A_2B_3).\end{aligned}\tag{13}$$

Let us specify our experiment now by choosing the common spin-state measurements of two spin 1/2 particles prepared in a singlet state. Thereby, every elementary proposition A_x can be stated as, ''the spin of particle A in the

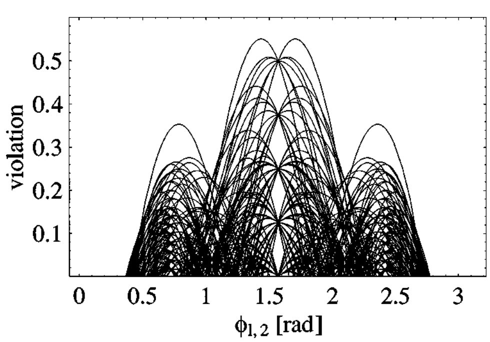

FIG. 1. Evaluation of the quantum expressions corresponding to all Boole-Bell type inequalities for $\phi_{l,1}=0$ and $0\leqslant\phi_{l,2}\leqslant\pi$ for all particles labeled by $l=A,B,C$. Any value above the zero baseline indicates violation of the conditions of possible experience.

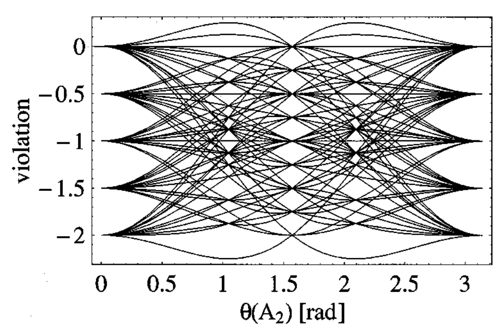

FIG. 2. Evaluation of the quantum expressions corresponding to all 648 Boole-Bell type inequalities for $\theta(A_1=B_1)=0$, $0\leqslant\theta(A_2=B_2)=2\pi-\theta(A_3=B_3)\leqslant\pi$. (The periodicity is π.) Any value above the zero baseline indicates violation of the conditions of possible experience.

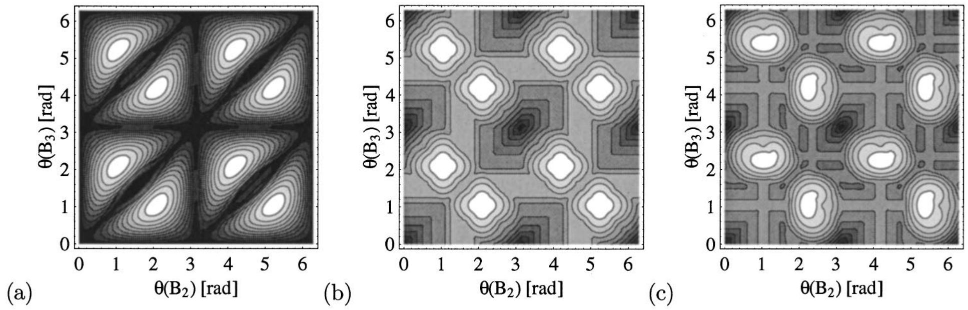

FIG. 3. Contour plot of the violations of classical conditions of possible experience for three scenarios using different angles: (a) $\theta(A_1=B_1)=0,\ 0\leqslant\theta(A_2=B_2),\theta(A_3=B_3)\leqslant 2\pi$, (b) $\theta(A_1=B_1)=0,\ \theta(A_2)=\pi/3,\ \theta(A_3)=2\pi/3,\ 0\leqslant\theta(B_2),\theta(B_3)\leqslant 2\pi$, (c) $\theta(A_1=B_1)=0,\ \theta(A_2)=\pi/2,\ \theta(A_3)=2\pi/3,\ 0\leqslant\theta(B_2),\theta(B_3)\leqslant 2\pi$. A nonblack region indicates the violation of conditions of possible experience by the quantum values.

direction x is up.'' It is well known that, for the singlet state of spin 1/2 particles, the probability to find the particles both either in spin-''up'' or both in spin-''down'' states is given by $P^{\uparrow\uparrow}(\theta)=P^{\downarrow\downarrow}(\theta)=(1/2)\sin^2(\theta/2)$, where θ is the angle between the measurement directions. Likewise, the probabilities for different spin states is given by $P^{\uparrow\downarrow}(\theta)=P^{\downarrow\uparrow}(\theta)=(1/2)\cos^2(\theta/2)$. In searching for possible violations of the inequalities, one may choose a symmetric configuration such as $\theta(A_1=B_1)=0$, $\theta(A_2=B_2)=2\pi/3$, $\theta(A_3=B_3)=4\pi/3$, in which case one obtains for the parallel case ($\uparrow\uparrow$ or $\downarrow\downarrow$) a violation of $0:1/4$ for (12) and of $0:1/8$ for (13). Figure 2 is a plot of the combined evaluation of quantum expressions for all the 684 equations corresponding to inequalities. The zero baseline indicates a threshold for a violation of Boole-Bell type inequalities. For the opposite case ($\uparrow\downarrow$ or $\downarrow\uparrow$), the violation of (7) is $1:9/8$ and of (11) is $2:5/4$. In the less symmetric configuration $\theta(A_1)=0$, $\theta(B_1)=-\pi/4$, $\theta(A_2)=\pi/2$, $\theta(B_2)=\pi/4$, $\theta(A_3)=2\pi/3$, $\theta(B_3)=\pi/3$, more inequalities violate the Bell inequalities, although to a lesser degree. In Figure 3 the violations of classical conditions of possible experience are plotted for a variety of angles.

Besides its conceptual clarity as a royal road to the understanding and constructive generation of Boole-Bell type inequalities, the importance of the correlation polytopes method lies in the fact that, unlike previous methods, these inequalities can be guaranteed to yield maximal bounds for consistent conditions of possible classical experience.

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