Evanescent waves in a time-of-arrival measurement model

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(Received 12 December 2000; published 31 May 2001)

The analysis of the model quantum clocks proposed by Aharonov *et al.* [Phys. Rev. A **57**, 4130 (1998); e-print quant-ph/9709031] requires considering evanescent components in the transient regime, previously not examined, and consideration of several aspects of the asymptotic regime. We also clarify the meaning of the operational time of arrival distribution that had been investigated. The accuracy limitation due to the back reaction of the clock is not affected.

DOI: 10.1103/PhysRevA.64.014101 PACS number(s): 03.65.Ta, 03.65.−w

The concept of time has a very problematic status in the development of quantum theory. This has led several researchers to investigate and propose quantum clocks, i.e., quantum devices that in some way will capture and reflect a given aspect of time. A recent such set of proposals for quantum clocks, in the specific case of measurement of times of arrival, has been put forward by Aharonov *et al.* [1]. We shall concentrate here on the first quantum clock proposed in that paper, which was introduced to demonstrate that the back reaction of the clock leads to an accuracy limitation. This clock is coupled to the otherwise free system whose times of arrival one wishes to measure in such a way that the total Hamiltonian (system plus clock) is

$$H=\frac{1}{2m}\mathbf{P}_x^2+\theta(-\mathbf{x})\mathbf{P}_y. \tag{1}$$

The system variable is x, while the clock corresponds to the time variable y. This is in fact a cyclic variable, which entails the conservation of the energy $\mathbf{P}_y$. Let us then consider $\mathbf{P}_y$ restricted to a given value p, in the classical case.

The motion of the particle (the system) is not fully free: moving from left to right, it runs into a step potential at $x=0$, either downwards if $p>0$, or upwards for negative p. For $p>0$, the classical y variable encodes how much time elapses from when the particle is released and the clock is started, to the instant the particle crosses $x=0$. For negative barrier height p, the classical particle would be reflected if its energy were not big enough to overcome the step, and the clock variable y would keep running after the particle is reflected. This suggests using predominantly positive p values in the analysis, even though quantum mechanically there is reflection even for downward steps.

In the quantum case, restricting ourselves to an eigenspace of $\mathbf{P}_y$ with eigenvalue p, either positive or negative, the (generalized) eigenstates of the (restricted) Hamiltonian are (i) scattering states, with degeneracy 2; and (ii) ''evanescent'' states, whose eigenvalue is not degenerate. On choosing an adequate orthogonal basis of these scattering and evanescent states, one can simply write the time evolution of any given state. However, for states with support in the x space restricted to one side ($x<0$, in particular), there is a compact alternative expression in terms of an integral over a path in complex momentum space that provides us with the whole space-time dependence of the state [2,3].

More explicitly, consider an initial wave function in (x,y) space, assumed to be factorized, $\psi(x,y,0)=\psi_1(x)\psi_2(y)$, such that $\psi_1(x)$ has no support on positive x, and compute its Fourier components, that is to say,

$$\psi(x,y,0)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}dp\, e^{ikx/\hbar}e^{ipy/\hbar}g(k)f(p).$$

It is then the case that, for any time t, the state evolved with the total Hamiltonian H, defined in Eq. (1), can be written as

$$\psi(x,y,t)=\int_{-\infty}^{+\infty}dp\int_{\Gamma(p)}dk\, f(p)g(k)\phi_{kp}(x,y,t), \tag{2}$$

where

$$\phi_{kp}(x,y,t)=\frac{1}{2\pi\hbar}\times\begin{cases}\left(e^{ikx/\hbar}+\dfrac{k-q}{k+q}e^{-ikx/\hbar}\right)e^{ipy/\hbar}e^{-i(p+k^2/2m)t/\hbar}, & x\leqslant 0\\[2ex] \left(\dfrac{2k}{k+q}\right)e^{iqx/\hbar}e^{ipy/\hbar}e^{-i(p+k^2/2m)t/\hbar}, & x\geqslant 0,\end{cases} \tag{3}$$

$q=\sqrt{k^2+2mp}$, defined with a branch cut in the k plane, which, for positive p, goes from $-i\sqrt{2mp}$ to $i\sqrt{2mp}$, and $\Gamma(p)$ is a path in the complex k plane from real negative to real positive infinity that goes *above* the branch cut.

Thus, in order to write the general solution for Schrödinger's equation with the Hamiltonian H and initial support at $x<0$, it is imperative to take into account the contribution of the branch cut in Eq. (2), which was omitted in [1]. It should be noted that supposing that $g(k)$ is zero along the branch cuts would in turn bring in the problem that $g(k)$ would be forced to be either zero everywhere or nonanalytic; however, if the initial wave function is normalizable and has initial support at $x<0$, its Fourier transform $g(k)$ must be analytic in the upper half plane. It follows that we cannot consistently assume both initial localization and that $g(k)$ is zero along the branch cuts, and the contribution of the branch cuts is of necessity present. Quite another issue is whether the contribution of the branch cut can be neglected with regard to the physics that we want to describe. If the state were initially peaked at high energies, it would present negligible overlap with evanescent states, and their contribution to the posterior evolution of the state would remain ignorable. However, we should point out that generically the evanescent component cannot be ignored, even in the asymptotic limit $t\to\infty$. As a first example, to be supplemented later graphically, consider the total probability that a particle, which starts at $t=0$ from the left-hand side, $x<0$, has to be found at positive x for large times, after colliding with a downward step barrier of height p. This (transmission) probability tends to

$$P_T=\int_0^\infty dq\left|\frac{2q}{q+k}g(k)\right|^2, \tag{4}$$

where q and k are related as before, and the evanescent component provides the lower part of the integration interval, from 0 to $\sqrt{2mp}$ (this expression can be obtained by using a variant of Riemann-Lebesgue's lemma, in a way similar to the computation performed by Allcock in the free case [4]). Note that there is a contribution to Eq. (4) from positive and imaginary k, but not from negative values of k.

In order to graphically illustrate the relevance of evanescent waves, let us first examine Fig. 1, where we depict the probability density for positive x at a given instant, as well as the modulus square of the contribution to the wave function of evanescent components [corresponding to imaginary k along the branch cut in Eq. (3)], and the modulus square of the contribution of scattering (real k) components, in the case of a step potential (fixed p); that is to say, we are computing the result of a collision of a wave function with a downward step potential. The initial state is a truncated sine function, i.e., $\psi_0(x)=[2/(b-a)]^{1/2}\sin[\pi(x-a)/(b-a)][\theta(x-a)-\theta(x-b)]$, where $\theta(x)$ is the step function and $0>b>a$. It is apparent that evanescent waves are numerically important, and even more so is the interference term between the evanescent and scattering components.

From classical considerations, Aharonov *et al.* were led to suggest as an operational candidate for the distribution of times of arrival of the x particle the following expression:

$$\rho_c(y,t)=\int_0^\infty dx\,|\psi(x,y,t)|^2. \tag{5}$$

We would suggest that this quantity be better understood as an operational distribution of *dwell* times *conditional* on the particle being found at positive x. The corresponding *unconditional* operational distribution would be given by

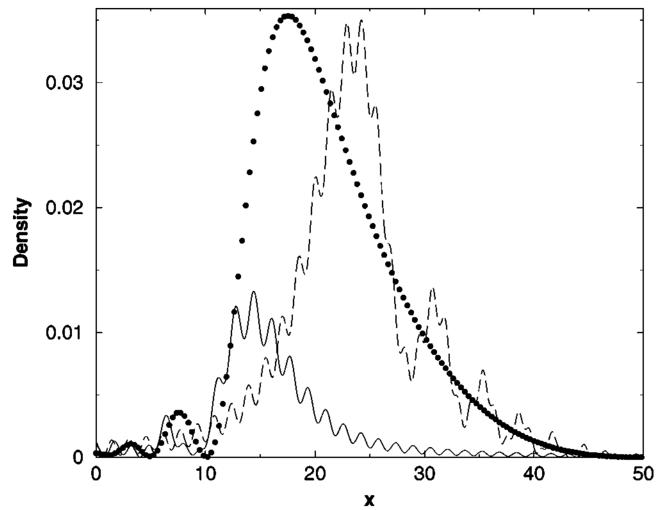

FIG. 1. Collision with a downward step potential: The continuous line portrays the modulus squared of the contribution of evanescent waves to the wave function; the dashed line is assigned to the modulus squared for scattering waves; the dotted line corresponds to the total probability density. All quantities are displayed in atomic units. A time $t=10$ a.u. has elapsed. Step height $p=2$ a.u. The initial state is a truncated sine (see text), with $a=-2.01$ a.u. and $b=-0.01$ a.u.

$$\rho_u(y,t)=\int_{-\infty}^{+\infty} dx\,|\psi(x,y,t)|^2. \tag{6}$$

The definition of the unconditional distribution makes its normalization to unity apparent. However, if the evanescent components were not included, there would be a probability deficit. Carrying back this argument to the conditional distribution (5), which need not be normalized, one sees that leaving out the evanescent components leads to a probability deficit. This is apparent in Fig. 2.

Another effect of considering or not considering the evanescent components is that the peak of arrivals is slightly shifted towards higher values of y. This can be understood as due to the fact that the evanescent components are, in a way, ''slower'' than those going over the step, thus remaining longer in $x<0$. This also brings down the tail in negative y coordinates.

The additional conceptual distinction between arrival and dwell time distributions is due to the fact that, even in a classical picture, particles that arrived at $x=0$ but did not cross this point, being reflected, would force the clock dial y to keep on moving. Similarly, particles that had crossed $x=0$ and were later reflected back would have the corresponding dial stop for a while and then resume its ticking. In this way both ρ_c and ρ_u are (conditional and unconditional) distributions of dwell times of the particle in the left half line.

Another way of understanding the need of reservations with respect to the interpretation of ρ_c as a time-of-arrival distribution comes about because of Allcock's analysis of the possibility of ideal distribution of times of arrival [4]. He identified the final ($t\to\infty$) probability of finding the particle

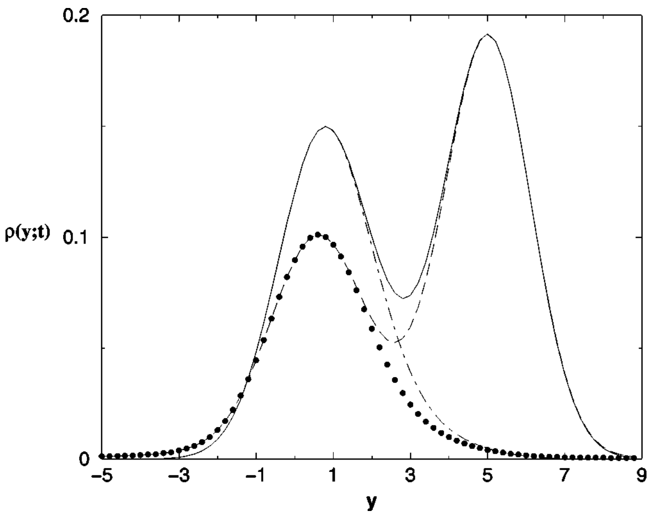

FIG. 2. The conditional distribution (5) without evanescent components is represented by a dotted line; with evanescent components by the dashed-dotted line. The dashed line corresponds to the unconditional distribution (6) without evanescent components; the continuous line depicts the full unconditional distribution, with the evanescent components taken into account. The initial state is a truncated sine wave for the particle, as before, in x space, and a minimum-uncertainty-product Gaussian for the clock, with central p being 2 a.u., $\langle y\rangle=0$, and $\Delta y=1.1$ a.u.

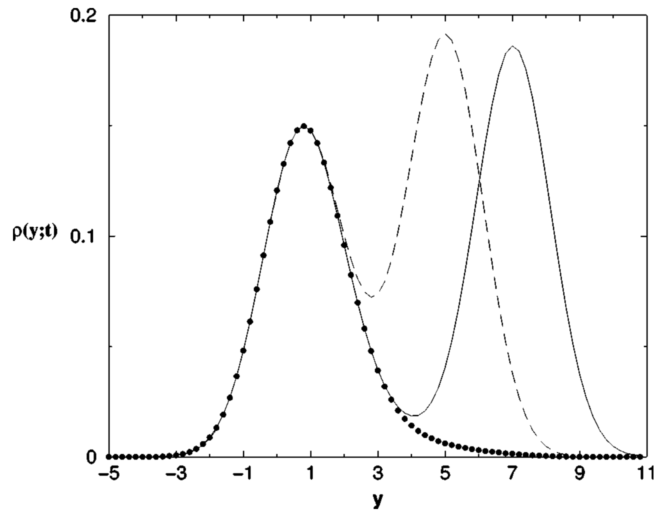

FIG. 3. The full conditional distributions at times $t=5$ a.u. and $t=7$ a.u., for the same initial wave function as in Fig. 2, are shown with the dotted line. The full unconditional distribution at $t=5$ a.u. is depicted with a dashed line, while the continuous line represents the full unconditional distribution at $t=7$ a.u.

in the half-space of positive x with the final probability of having arrived at $x=0$, for a particle initially confined at negative x. He concluded from this identification that an ideal time-of-arrival probability distribution could not exist; however, it is by now well known that an ideal time-of-arrival distribution does indeed exist for the free particle, namely, Kijowski's distribution [5,6]. As has been discussed elsewhere [7,8], the flaw in Allcock's argument was to ignore that negative momenta components (in the free particle case and in the case at hand too) would contribute to the arrival probability in a transient regime, thus invalidating the stated identification. Similarly, $\rho_c(t\to\infty)$ only contains asymptotic information on dwell times, but not detailed information on the transients, so it cannot generically be interpreted as a distribution of times of arrival.

The major difference between the conditional and unconditional distributions, when enough time has passed for the situation to be considered asymptotic, is the presence in the unconditional distribution of a second peak, associated with those components of the wave function that are still in $x<0$ and will in fact remain there. This second peak is present no matter whether we do or do not take into account the evanescent components, and, in fact, does not get any contribution from them, as can be seen in Fig. 2. The evolution of the second peak, moving forward in y space as time goes by, is represented in Fig. 3.

At any rate, it should be noticed that the numerical change of the conditional distribution when the evanescent components are taken into account, in the asymptotic regime, is mostly a scale change, thus rendering the difference irrelevant for the analysis carried out in Ref. [1] with regard to uncertainties in the position of the clock dial.

In this respect, it is interesting to note that the lower bound suggested in Ref. [1] for the product of the average energy of the particle and the uncertainty in the time of arrival has been obtained in a completely different context, i.e., for Kijowski's distribution, without recourse to any coupling with clock variables [9]. The interpretation, however, is different: in this latter case the lower bound is derived as an uncertainty relation, whereas Aharonov *et al.* suggest that there are intrinsic limitations in the accurate measurement of time.

We acknowledge support from the Ministerio de Educación y Cultura (AEN99-0315), the University of the Basque Country (Grant No. UPV 063.310-EB187/98), and the Basque Government (PI-1999-28). A.D. Baute acknowledges financial support from the Ministerio de Educación y Cultura.

[1] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, and W. G. Unruh, Phys. Rev. A **57**, 4130 (1998), e-print quant-ph/9709031.

[2] C. L. Hammer, T. A. Weber, and V. S. Zidell, Am. J. Phys. **45**, 933 (1977).

[3] A. D. Baute, I. L. Egusquiza, and J. G. Muga, Int. J. Theor. Phys. (to be published); e-print quant-ph/0007079.

[4] G. R. Allcock, Ann. Phys. (N.Y.) **53**, 253 (1969).

[5] J. Kijowski, Rep. Math. Phys. **6**, 361 (1974).

[6] I. L. Egusquiza and J. G. Muga, Phys. Rev. A **61**, 012104

(2000); e-print quant-ph/9905023; see also related erratum, *ibid.* **61**, 059901(E) (2000).

[7] J. G. Muga and C. R. Leavens, Phys. Rep. **338**, 353 (2000).

[8] A. D. Baute, I. L. Egusquiza, and J. G. Muga, e-print quant-ph/0007066, J. Phys. A (to be published).

[9] A. D. Baute, R. Sala-Mayato, J. P. Palao, J. G. Muga, and I. L. Egusquiza, Phys. Rev. A **61**, 022118 (2000); e-print quant-ph/9904055.