Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems

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We construct a large number of coherent states satisfying the resolution of unity with a positive weight function, obtained through analytic solutions of the Stieltjes moment problem (coherent states on a plane) and the Hausdorff moment problem (coherent states on a disk). These solutions are obtained through the method of inverse Mellin transform. In addition, these coherent states induce a deformation of the metric that has been calculated analytically.

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I. INTRODUCTION

In this paper we shall expose a rather general method for constructing coherent states, as defined according to a minimal set of conditions, proposed by one of us [1,2]. For convenience, we shall focus on holomorphic coherent states, which up to normalization, are functions of a single complex variable z. The ensemble of states $|z\rangle$ labeled by the single complex number z is called a set of coherent state if (i) $|z\rangle$ is normalizable, (ii) $|z\rangle$ is continuous in the label z, i.e., $|z-z'|\to 0 \Rightarrow \| |z\rangle - |z'\rangle \| \to 0$, and (iii) the states $|z\rangle$, $z\in\mathbb{C}$, form a complete (in fact, an overcomplete) set and that allows a resolution of unity with the positive function $W(|z|^2)$ (completeness relation)

$$\int\int_{\mathbb{C}} d^2z|z\rangle W(|z|^2)\langle z| = I = \sum_{n=0}^{\infty} |n\rangle\langle n|, \qquad (1)$$

where, in Eq. (1) I is the unit operator and $|n\rangle$ is a set of orthonormal eigenfunctions of a Hermitian operator $\hat{H}$. As already noted, without loss of generality, in Eq. (1) the states $|z\rangle$ are normalized to one. In Eq. (1) the integration is restricted to the part of the complex plane where normalization converges, see Eq. (3) below. The general method of construction alluded to in the above consists of choosing a set of strictly positive parameters $\rho(n)$, $n=0,1,\ldots,M$, $M\leqslant\infty$, where $\rho(0)=1$, such that the normalized state $|z\rangle$ reads

$$|z\rangle = \mathcal{N}^{-1/2}(|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho(n)}}|n\rangle, \qquad (2)$$

where

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{\rho(n)} \qquad (3)$$

is the normalization, a convergent series in $|z|^2$ within the radius of convergence $|z|^2<R$, $0<R\leqslant\infty$, thus satisfying condition (i). While continuity in z is easily checked for the form of Eq. (2), the condition (1) presents a severe restriction on the choice of $\rho(n)$'s. In fact, only a relatively small number of distinct sets of $\rho(n)$'s is known, for which the function $W(|z|^2)$ can be extracted. As a result, the family of truly coherent states is small in number. The standard example leading to conventional coherent states is $\rho(n)=n!$ [2]. Recently progress has been made in finding a resolution of unity for selected choices of $\rho(n)$ [3–5]. The physical motivation behind the form of Eq. (2), is to propose a general linear combination of basis states $|n\rangle$, whose coefficients $\sim[\rho(n)]^{-1/2}$ are adapted to satisfy Eq. (1) and can be linked to a specific Hamiltonian $\hat{H}\neq\hat{H}_0$, where $\hat{H}_0$ is the linear harmonic oscillator. As we will show in the following, there exists only a very restricted set of families of $\rho(n)$ for which the above requirements can be satisfied.

The idea of building coherent states through an appropriate choice of $\rho(n)$ has been put forward in Refs. [6] and [7]. The states defined through Eq. (2) share for general $\rho(n)$ some universal features that we will enumerate now. For two different complex numbers z and z' the states $|z\rangle$ and $|z'\rangle$ are, in general, not orthogonal and their overlap is given by

$$\langle z|z'\rangle = \frac{\mathcal{N}(z^*z')}{[\mathcal{N}(|z|^2)\mathcal{N}(|z'|^2)]^{1/2}}, \qquad (4)$$

where we have extended the definition of the normalization, Eq. (3), to

$$\mathcal{N}(z^*z') = \sum_{n=0}^{\infty}\frac{(z^*z')^n}{\rho(n)}. \qquad (5)$$

Whereas, through the positivity of $\rho(n)$, $\mathcal{N}(|z|^2)$ is a strictly increasing function of its argument, the overlap $\langle z|z'\rangle$ is a complex function of its arguments.

The continuity in label z follows from the continuity of the overlap $\langle z|z'\rangle$ through

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$$\| |z\rangle - |z'\rangle \|^2 = 2(1 - \mathrm{Re}\langle z|z'\rangle), \tag{6}$$

and is easily satisfied in practice. The choice of the orthogonal set of $|n\rangle$'s is arbitrary. In the forthcoming paragraphs, we shall assume for simplicity that $|n\rangle$'s are eigenfunctions of the harmonic oscillator, $\hat{H}_0|n\rangle = n|n\rangle$, where $\hat{H}_0 = \hat{N} = \hat{a}^\dagger \hat{a}$ with $[\hat{a},\hat{a}^\dagger] = 1$. In the discussion (Sec. VII) we shall amply discuss other choices and their implication for the properties of coherent states. With the above choice, various expectation values of polynomial Hermitian operators are expressible through derivatives of $\mathcal{N}(|z|^2)$, such as

$$\langle z|(\hat{a}^\dagger)^r \hat{a}^r|z\rangle = \frac{|z|^{2r}}{\mathcal{N}(|z|^2)} \left(\frac{d}{d|z|^2}\right)^r \mathcal{N}(|z|^2), \quad r = 0,1,2,\ldots\ . \tag{7}$$

The generalization of Eq. (7) for non-Hermitian operators where $r \neq p$ reads

$$\langle z|(\hat{a}^\dagger)^p \hat{a}^r|z\rangle = \frac{(z^*)^p z^r}{\mathcal{N}(|z|^2)} \times \sum_{n=0}^{\infty} \left[\frac{(n+p)!(n+r)!}{\rho(n+p)\rho(n+r)}\right]^{1/2} \frac{|z|^{2n}}{n!},$$

$$r = 0,1,2,\ldots, \quad p = 0,1,2,\ldots\ . \tag{8}$$

From Eq. (2), one obtains the probability of finding the state $|n\rangle$ in the state $|z\rangle$. It is equal to $(x = |z|^2)$

$$p(n,x) = \frac{x^n}{\mathcal{N}(x)\rho(n)}, \tag{9}$$

which reduces to a Poisson distribution for the conventional coherent states $[\rho(n) = n!,\ \mathcal{N}(x) = e^x]$. A Poisson distribution is characterized by the fact that the variance of the number operator $\hat{N}$ is equal to its average. One aspect of the deviation from Poisson statistics can be measured with the Mandel parameter $Q_M(x)$ [8]

$$Q_M(x) = \frac{\langle z|\hat{N}^2|z\rangle - \langle z|\hat{N}|z\rangle^2 - \langle z|\hat{N}|z\rangle}{\langle z|\hat{N}|z\rangle}. \tag{10}$$

By using Eq. (7) to evaluate the averages in Eq. (10), one easily obtains

$$Q_M(x) = x\left(\frac{\mathcal{N}''(x)}{\mathcal{N}'(x)} - \frac{\mathcal{N}'(x)}{\mathcal{N}(x)}\right). \tag{11}$$

This relation implies that the statistical properties of the state $|z\rangle$ are solely dependent on the growth properties of the normalization function $\mathcal{N}(x)$. A state for which $Q_M(x) > 0$ (<0) is called super-(sub-)Poissonian.

The main objective of this paper is to formalize and extend this construction with the help of Mellin transform techniques together with their convolution properties. As will be explained below, it appears that this technique is ideally suited for greatly extending coherent state families. As a result, an extensive discussion of geometric properties of various coherent states can be now given, based on numerous examples of a qualitatively different nature.

The plan of the paper is as following: in Sec. II the relation between condition (1) and Stieltjes (for $R = \infty$) and Hausdorff (for $R < \infty$) moment problems will be established and explored. In Sec. III, we shall describe and develop the way to obtain the solutions of the moment problem through the method of inverse Mellin transform. In Sec. IV, the geometric properties of deformations of the complex plane, induced by the choice of coherent states themselves, are introduced and discussed. In Secs. V and VI, we shall present many examples of $\rho(n)$ for which the weight functions can be explicitly obtained, both for the Stieltjes ($R = \infty$, coherent states on a plane) and the (much less studied in the present context) Hausdorff ($R < \infty$, coherent states on a disk) moment problems. Section VII establishes a close link between the coherent states and physical potentials. Section VIII is devoted to a discussion and conclusions. In the appendix we carry out in detail examples of constructing the coherent states through the Mellin convolution.

II. RESOLUTION OF UNITY VERSUS MOMENT PROBLEM: STIELTJES AND HAUSDORFF

In this section, we shall establish the link between the completeness condition Eq. (1) and the classical moment problem. To this end, we substitute $z = r\,e^{i\theta}$ into the states $|z\rangle$ of Eq. (2) to obtain (D is a disk in the complex plane, centered at the origin, of radius R; if $R = \infty$, then $D = \mathbb{C}$)

$$\int\int_D d^2z |z\rangle W(|z|^2) \langle z| \tag{12}$$

$$= \sum_{n,n'=0}^{\infty} \left\{ \frac{1}{2\sqrt{\rho(n)\rho(n')}} \int_0^R \left[\frac{W(r^2)}{\mathcal{N}(r^2)}\right] r^{n+n'} d(r^2) \times \int_0^{2\pi} e^{i\theta(n-n')} d\theta \right\} |n\rangle\langle n'| \tag{13}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{\pi}{\rho(n)} \int_0^R x^n \left[\frac{W(x)}{\mathcal{N}(x)}\right] dx \right\} |n\rangle\langle n|, \quad (x = r^2), \tag{14}$$

from which the following infinite set of equations results

$$\pi \int_0^R x^n \left[\frac{W(x)}{\mathcal{N}(x)}\right] dx = \rho(n), \quad n = 0,1,\ldots, \quad 0 < R \leqslant \infty. \tag{15}$$

The quantities $\rho(n) > 0$ are then the power moments of the unknown function $\tilde{W}(x) = \pi W(x)/\mathcal{N}(x) > 0$ and the problem stated in Eq. (15) is the Stieltjes ($R = \infty$) or the Hausdorff ($R < \infty$) moment problem. These are classical mathematical problems on which an extensive and mathematically oriented literature exists [9–11]. As one approach, we could first verify the necessary and sufficient conditions for the existence of positive solutions, i.e., $\tilde{W}(x) > 0$. These conditions are known for both the Stieltjes and Hausdorff cases [9,10].

However, exploiting these conditions for general $\rho(n)$ is difficult if not impossible. A more practical approach would be simply to *construct* the required solutions by some auxiliary means. It is the essence of the present paper to demonstrate that these solutions can be obtained in many cases by the means of Mellin and inverse Mellin transforms. The links between the completeness of coherent states, the moment problem, and the Mellin transform have been the object of scattered remarks and some (partial and sometimes incomplete) calculations in the literature [12–14]. Here, we shall provide a more systematic exposition with an extensive use of the convolution properties of the inverse Mellin transform, an essential tool to extend the families of solutions of Eq. (15).

III. ESTABLISHING THE SOLUTIONS OF THE MOMENT PROBLEMS WITH THE INVERSE MELLIN TRANSFORM

We group here the main formulas of the Mellin and inverse Mellin transforms [15,16] as applied to the solution of the moment problem. The Mellin transform of a function $f(x)$, for complex s, is denoted by

$$f^*(s) := \int_0^\infty x^{s-1} f(x)\,dx \overset{\text{def}}{=} \mathcal{M}[f(x);s] \tag{16}$$

and its inverse then reads

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)\,x^{-s}\,ds \overset{\text{def}}{=} \mathcal{M}^{-1}[f^*(s);x]. \tag{17}$$

It turns out that for a vast class of functions $f^*(s)$ the value of $f(x)$ in Eq. (17) does not depend on c, which is equivalent to integration over the imaginary axis. A discussion of conditions that ensure the existence of $f^*(s)$ and its inverse $f(x)$ can be found in Ref. [16].

A linear shift of s in $f^*(s)$ produces ($b \in \mathbb{R};\ \ a,h>0$) the following transformation formulas

$$\mathcal{M}[x^b f(ax^h);s] = \frac{1}{h} a^{-(s+b)/h} f^*\left(\frac{s+b}{h}\right), \tag{18}$$

$$\mathcal{M}[x^b f(ax^{-h});s] = \frac{1}{h} a^{(s+b)/h} f^*\left(-\frac{s+b}{h}\right). \tag{19}$$

An essential property of the Mellin transformation is the so-called Mellin convolution expressed (for arbitrary a,b) as

$$x^a \int_0^\infty t^b f\left(\frac{x}{t}\right) g(t)\,dt = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} f^*(s+a)\,g^*(s+a+b+1)\,x^{-s}\,ds. \tag{20}$$

Observe that the formula for the Mellin convolutions, Eq. (20), contains the ratios of arguments of functions, as distinguished from the well-known Fourier convolutions where the difference of arguments of functions arise.

Equation (20) is also referred to as generalized Parseval relation. Conditions on the functions so that all the operations in Eq. (20) are well defined are discussed in the specialized treatises [17]. The expressions on the left-hand side (lhs) of Eq. (20) is called Mellin convolution of $f(x)$ and $g(x)$.

Let us rewrite the Stieltjes and Hausdorff moment problems as

$$\int_0^R x^n \widetilde{W}(x)\,dx = \rho(n), \quad n=0,1,2,\ldots, \tag{21}$$

where $\rho(0)=1$ in all subsequent cases. This condition means that the integral of $\widetilde{W}(x)$ is normalized to one. We extend the natural values of n to complex values s and rewrite it as

$$\int_0^\infty x^{s-1}\widetilde{W}(x)\,dx = \rho_\infty(s-1) \quad (R=\infty), \tag{22}$$

or

$$\int_0^\infty x^{s-1} H(R-x)\widetilde{W}(x)\,dx = \rho_R(s-1) \quad (R<\infty), \tag{23}$$

where $H(y)$ is the Heaviside function. According to Eq. (17), the solutions of Eq. (22) and Eq. (23) are given as

$$\widetilde{W}(x) = \mathcal{M}^{-1}[\rho_\infty(s-1);x], \tag{24}$$

$$H(R-x)\widetilde{W}(x) = \mathcal{M}^{-1}[\rho_R(s-1);x]. \tag{25}$$

If the choice of $\rho_\infty(n)$ and $\rho_R(n)$ is made so that the lhs of Eqs. (24) and (25) are positive functions, we have furnished a solution of the moment problem. For a general $\rho(n)$ it is evidently impossible to tell whether the inverse Mellin transforms in Eqs. (24) and (25) are positive functions. At this stage, we shall make use of quite extensive tables of Mellin and inverse Mellin transforms [18,21,22]. A word of explanation is in order to characterize these references. In Ref. [18] one finds inverse Mellin transforms $f(x)$ of $f^*(s)$ such that $f^*(s)$ are expressible as ratios of products of gamma functions. This constraint is not as strong as it would appear at first glance, since the resulting $f(x)$ include practically all the elementary functions, and a large number of special functions, including the general hypergeometric function. Reference [21] is an extended version of Ref. [18] that includes an all-embrassing terminology and notations of special functions in terms of Meijer's G function,

$$G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}\right).$$

In Ref. [22], one finds $f^*(s)$ including functions other than gamma functions and their respective inverse Mellin transforms $f(x)$, which include many different combinations of special functions. As will be explained below, in this paper we concentrate on generalizations of $\rho(n)=n!$ that naturally lead to gamma functions. Therefore, Refs. [18] and [21] are of main use for us. Our approach to find soluble moment problems consists of two stages: (i) look for $\rho(s)$ in the tables such that the variable s narrowed down to integers, $s-1=n$, permits all the values of $n=0,1,\ldots$; (ii) identify the corresponding $\widetilde{W}(x)$ and check for its positivity for all the $0\leqslant x\leqslant R$. If the checks (i) and (ii) are affirmative, the given $\rho(s)$ is suitable to construct a complete set of coherent states, provided the corresponding normalization has a nonvanishing radius of convergence.

We have scrutinized the tables in Refs. [18] and [21] according to the criteria (i) and (ii) and we have found approximately 30 instances, which we call generic, which suit themselves to our purposes. We shall not enumerate them all here but rather pick out several typical examples.

It is quite intriguing to notice that any countable set (including one) of such generic functions immediately extends itself into infinitely many positive solutions with the use of the convolution property Eq. (20). Suppose that we have found $f^*(s)$ and $g^*(s)$ such that $f(x)$ and $g(x)$ are positive. Then for a,b real, $f^*(s+a)g^*(s+a+b+1)$ is also a good candidate for the moments of a positive weight function as then $x^a\int_0^\infty t^b f(x/t)g(t)dt$ is again positive. Thus, solving one moment problem implicitly solves an infinity of moment problems.

In the following, in order to avoid a proliferation of indices, the symbols $\mathcal{N}(x)$, $W(x)$, etc., will refer to the normalization, the weight function, etc., of the state under current discussion.

We close this section by remarking that the positivity of the set of $\rho(n)$, $n=0,1,\ldots$ and the ensuing convergence of $\mathcal{N}(x)$ of Eq. (3) by no means guarantee the existence of a positive weight $W(x)$. We give here the illustration of such a situation by choosing $\rho(n)=(n+1)^2 n!$, $n=0,1,\ldots$, which yields

$$\mathcal{N}(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^2 n!}={}_2F_2(1,1;2,2;x) \tag{26}$$

$$=\frac{-\gamma-\ln(-x)-\mathrm{Ei}(1,-x)}{x}, \tag{27}$$

a convergent series.

In Eqs. (26) and (27) ${}_2F_2(a,b;c,d;x)$ is a higher-order hypergeometric function, $\mathrm{Ei}(1,z)$ is the first-order exponential integral [24] and γ is Euler's constant. For $x\geqslant 0$, $\mathcal{N}(x)$ is by construction a positive function. However, $\rho(n)=\int_0^\infty x^n\widetilde{W}(x)dx$, with $\widetilde{W}(x)=x\,e^{-x}(x-1)$, which is a nonpositive function.

Therefore, for $\rho(n)$ above,

$$|z\rangle=[{}_2F_2(1,1;2,2;|z|^2)]^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(n+1)^2 n!}}|n\rangle, \tag{28}$$

is a normalizable state, but it is not a coherent state.

IV. GEOMETRY OF THE COHERENT STATES

The map from z to $|z\rangle$ represented by Eq. (2) is a map from the space $\mathbb{C}$ of complex numbers (or a subset thereof if $R<\infty$) into a continuous subset of unit vectors in Hilbert space. As such, one may imagine that this map generates a two-dimensional (because $\mathbb{C}=\mathbb{R}\oplus\mathbb{R}$) surface ''sweeping'' through an infinite-dimensional (Hilbert) space. It is interesting to describe that two-dimensional surface by its *geometry*, which in explicit form, is represented by the induced two-dimensional Riemannian metric tensor implicit in the line element $d\sigma^2$. The metric in question is not that induced directly by the Hilbert space metric itself, but rather is one induced by the physical content of the Hilbert space in which vectors differing only in phase are identified. A suitable metric between any two Hilbert space vectors, say $|\psi\rangle$ and $|\phi\rangle$, is thus the *ray metric* defined by

$$d_{ray}(|\psi\rangle;|\phi\rangle):=\min_{0\leqslant\alpha<2\pi}\||\psi\rangle-e^{i\alpha}|\phi\rangle\|. \tag{29}$$

The infinitesimal form of this metric is given by the Fubini-Study metric, which restricted to coherent states explicitly takes the form

$$d\sigma^2:=\|d|z\rangle\|^2-|\langle z|d|z\rangle|^2. \tag{30}$$

Observe, in this line element, that any change of the form $d|z\rangle=\lambda|z\rangle$, $\lambda\in\mathbb{C}$, has *zero* distance, i.e., $d\sigma^2=0$. This fact is also useful in deriving an expression for $d\sigma^2$.

In forming $d|z\rangle$, therefore, we can ignore changes of the normalization and just adopt

$$d|z\rangle=\mathcal{N}^{-1/2}dz\sum_{n=0}^{\infty}\frac{nz^{n-1}}{\sqrt{\rho(n)}}|n\rangle. \tag{31}$$

It follows that

$$\|d|z\rangle\|^2=\mathcal{N}^{-1}(dz^*dz)\sum_{n=0}^{\infty}\frac{n^2|z|^{2(n-1)}}{\rho(n)}, \tag{32}$$

while

$$\langle z|d|z\rangle=\mathcal{N}^{-1}(z^*dz)\sum_{n=0}^{\infty}\frac{n|z|^{2(n-1)}}{\rho(n)}. \tag{33}$$

Hence, we learn that

$$d\sigma^2=\mathcal{N}^{-1}(dz^*dz)\left[\sum_{n=0}^{\infty}\frac{n^2|z|^{2(n-1)}}{\rho(n)}-\mathcal{N}^{-1}(z^*z)\times\left(\sum_{n=0}^{\infty}\frac{n|z|^{2(n-1)}}{\rho(n)}\right)^2\right]. \tag{34}$$

Because

$$\mathcal{N}(x):=\sum_{n=0}^{\infty}\frac{x^n}{\rho(n)}, \tag{35}$$

where $x:=|z|^2$, a moment's reflection shows that

$$d\sigma^2 := \omega(|z|^2) dz^* dz,$$

$$\omega(x) := \frac{1}{\mathcal{N}(x)}[x\mathcal{N}'(x)]' - x\left[\frac{1}{\mathcal{N}(x)}\mathcal{N}'(x)\right]^2 = \left[x\frac{\mathcal{N}'(x)}{\mathcal{N}(x)}\right]', \tag{36}$$

where $\mathcal{N}'(x) := d\mathcal{N}(x)/dx$. By construction, $\omega(x) > 0$, for all relevant x. In polar coordinates, $z := r\, e^{i\theta}$, it follows that

$$d\sigma^2 := \omega(r^2)(dr^2 + r^2 d\theta^2). \tag{37}$$

The result, therefore, is a *circularly symmetric, two-dimensional geometry*. If $\omega(r^2) := 1$, then $d\sigma^2$ describes a flat, two-dimensional surface. This situation occurs when $\rho(n) = n!$ and $\mathcal{N}(x) = e^x$. If $\mathcal{N}(x) \neq e^x$, then $\omega(r^2) \neq 1$ and the geometry is nonflat. We refer to $\omega(x)$ as the *metric factor*, and knowledge of $\omega(x)$ enables us, in some sense, to visualize the surface of coherent states embedded in the Hilbert space. More directly, we can also interpret $d\sigma^2$ as the geometry of the associated ‘‘classical phase space.’’

As usual, we can aid our visualization of such a surface by embedding it in a three-dimensional Euclidean or pseudo-Euclidean space. For the Euclidean case, we set

$$d\sigma^2 = du^2 + u^2 d\theta^2 + dz(u)^2, \tag{38}$$

where here $z = z(u)$ corresponds to the third dimension (and not a complex variable!), and set

$$\omega(r^2)(dr^2 + r^2 d\theta^2) = b(u)^2 du^2 + u^2 d\theta^2, \tag{39}$$

$$b(u)^2 := 1 + [dz(u)/du]^2. \tag{40}$$

To satisfy the last relation, we require that

$$u := \sqrt{\omega(r^2)}\, r, \tag{41}$$

$$\sqrt{\omega(r^2)}/(du/dr) := b(u) = \sqrt{1 + [dz(u)/du]^2}. \tag{42}$$

These two equations provide a set of parametric differential equations to find $z(u)$, and hence, a two-dimensional surface in the three-dimensional Euclidean space. For example, if $\omega(r^2) = 1$, then $u = r$, and thus, $b(u) = 1$, i.e., $dz(u)/du = 0$, or $z(u) = \text{const}$ describing a flat plane as expected. More generally, these equations provide the connection between the conformal form of the metric [left side of Eq. (39)] and the desired form [given in the right side of Eq. (39)], the latter form being more useful for visualization.

It may happen that Eq. (41) and Eq. (42) have no solution, in which case it may be that the embedding is pseudo-Euclidean rather than Euclidean. To cover this case, one initially sets

$$d\sigma^2 = \omega(r^2)(dr^2 + r^2 d\theta^2) = du^2 + u^2 d\theta^2 - dz(u)^2, \tag{43}$$

and therefore the required set of parametric differential equations becomes

$$u := \sqrt{\omega(r^2)}\, r, \tag{44}$$

$$\sqrt{\omega(r^2)}/(du/dr) := b(u) = \sqrt{1 - [dz(u)/du]^2}. \tag{45}$$

Again, a solution of these two equations enables one to obtain $z(u)$ from a knowledge of $\omega(r^2)$. In this case also the surface can be ‘‘visualized,’’ although not so naturally as in the Euclidean case. For all the new states introduced later we know explicit expressions for their normalizations. Consequently we are able, with Eq. (36), to investigate for the first time, the geometry of many different coherent states.

For simplicity in our presentation, we offer plots of $\omega(x)$ and make only qualitative remarks about the geometry of the coherent states. For an extensive discussion of the geometry of selected coherent states, see Ref. [23].

Figures illustrating metric factors are presented in the following sections in which specific examples are discussed.

V. COHERENT STATES ON THE PLANE

In this section we shall exploit a choice of $\rho(n)$'s such that $\mathcal{N}(x)$ is convergent everywhere on the positive axis, implying that $|z\rangle$ is well defined everywhere on the complex plane and the weight function is a solution of the Stieltjes moment problem. We shall follow now the prescription outlined in Sec. III and identify, partially with the help of tables in Refs. [18,21], such choices of $\rho(n)$ for which the weight function is positive. Our $\rho(n)$ will be always expressible through ratios of products of gamma functions. In this paragraph the $\rho(n)$'s satisfy $\lim_{n\to\infty}\rho(n) = \infty$. We will proceed roughly according to the increasing complexity of $\rho(n)$, choosing a few generic examples of positive weight functions.

(a) The first example consists in choosing $\rho(n) = (n+p)!/p!$, $p = 0,1,2,\ldots$, merely shifting n by p in the standard definition. The normalization is

$$\mathcal{N}(x) = {}_1F_1(1;1+p;x) = p\left(-\frac{1}{x} + x^{-p}e^x\Gamma(p) - (p-1)x^{-p}e^x\Gamma(p-1,x)\right), \tag{46}$$

where in Eq. (46), ${}_1F_1(a;b;x)$ is Kummer's confluent hypergeometric function and $\Gamma(\alpha,x)$ is the incomplete gamma function [24]. The weight $\widetilde{W}(x) = \pi W(x)/\mathcal{N}(x)$ is obtained from Eq. (18) by elementary means and is a term in the Poisson distribution:

$$\widetilde{W}(x) = \frac{1}{p!}x^p e^{-x}. \tag{47}$$

The metric factor is obtained from Eqs. (46) and (36) to be

$$\omega(p,x) = \frac{1}{(p+1)\,{}_1F_1(1;p+1;x)}\Bigg[{}_1F_1(2;p+2;x) + x\left(2\frac{{}_1F_1(3;p+3;x)}{p+2} - \frac{[{}_1F_1(2;p+2;x)]^2}{(p+1)\,{}_1F_1(1;p+1;x)}\right)\Bigg], \tag{48}$$

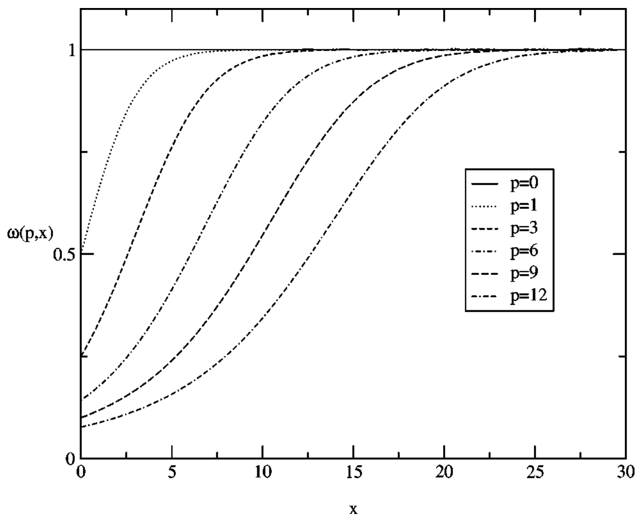

FIG. 1. The metric factors of Eq. (48) for different values of p. On this, as well as on all the subsequent figures, all the quantities plotted are dimensionless.

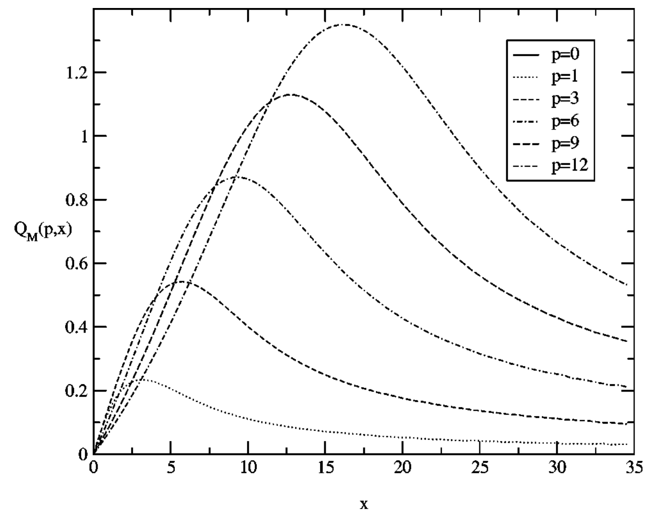

FIG. 2. The Mandel parameter of Eq. (51) for different values of p. For $p=0$, $Q_M(0,x)=0$.

which for every integer p evaluates in terms of elementary functions. We quote here only two expressions:

$$\omega(1,x)=\frac{1-e^{-x}(1+x)}{(e^{-x}-1)^2}, \tag{49}$$

$$\omega(2,x)=\frac{1+e^{-2x}-e^{-x}(2+x^2)}{(-1+e^{-x}(1+x))^2}. \tag{50}$$

Note that $\omega(p,0)=(p+1)^{-1}$.

The Mandel parameter $Q_M(p,x)$ is given by

$$Q_M(p,x)=x\left(\frac{2}{2+p}\,\frac{{}_1F_1(3;p+3;x)}{{}_1F_1(2;p+2;x)}-\frac{1}{1+p}\,\frac{{}_1F_1(2;p+2;x)}{{}_1F_1(1;p+1;x)}\right). \tag{51}$$

In Fig. (1) we have displayed the metric factors of Eq. (48) for $p=0,1,3,6,9,12$.

One observes that for a given p, the geometry represents a ''vase'' with the bottom at the value $(p+1)^{-1}$. For $x\to\infty$, it saturates at the value 1 for all p. In Fig. (2) we have presented the Mandel parameter that is positive for all $p\geqslant 1$ and x. The states are then super-Poissonian in nature.

(b) In this example we extend the exponential to its most natural generalization, the so-called Mittag-Leffler function [25] by choosing

$$\rho(n)=\frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)}, \quad \alpha,\beta>0. \tag{52}$$

Then

$$\mathcal{N}(x)=\Gamma(\beta)E_{\alpha,\beta}(x), \tag{53}$$

where $E_{\alpha,\beta}(x)$ are the Mittag-Leffler functions defined as $E_{\alpha,\beta}(x)=\Sigma_{n=0}^{\infty}(x^n)/\Gamma(\alpha n+\beta)$ [25]. (These functions have found recently important applications in dynamical problems [26,27]). This case was the subject of a separate study [4] where a rather detailed investigation of the corresponding moment problem was carried out. We quote the simplest solution of the moment problem that can be obtained with formula (18) above:

$$\widetilde{W}(x)=\pi\frac{W(x)}{\mathcal{N}(x)}=\frac{x^{(\beta-\alpha)/\alpha}e^{-x^{1/\alpha}}}{\alpha\,\Gamma(\beta)}. \tag{54}$$

We quote here for illustrative purposes two cases of $\mathcal{N}(x)$: for $\alpha=1$, β arbitrary $(\beta>0)$, $\mathcal{N}(x)=\Gamma(\beta)\times e^x x^{1-\beta}(1-[\Gamma(\beta-1,x)]/[\Gamma(\beta-1)])$; for $\alpha=1/2$, $\beta=1$, $\mathcal{N}(x)=e^{x^2}[1+\mathrm{erf}(x)]$, where $\mathrm{erf}(x)$ is the error function.

The two positive constants α,β in the expression for $\mathcal{N}(x)$ allow for a certain flexibility to investigate the behavior of physical quantitites. For instance, the case $\alpha=2$, $\beta=1$ gives $\mathcal{N}(x)=\cosh(\sqrt{x})$, the metric factor $\omega(x)=\frac{1}{4}([\tanh(\sqrt{x})]/(\sqrt{x})+1/[\cosh^2(\sqrt{x})])$ and the Mandel parameter $Q_M(x)=-\frac{1}{2}+(\sqrt{x})/[\sinh(2\sqrt{x})]$. Observe that $\omega(0)=1/2$ and that $Q_M(x)<0$, which indicates that the state is sub-Poissonian. The quantities $\omega(x)$ and $Q_M(x)$ are represented in Fig. (3). The case $\alpha=1/2$ and $\beta=2$ leads to

$$\mathcal{N}(x)=\frac{e^{x^2}[1+\mathrm{erf}(x)]-1}{x^2}-\frac{2}{x\sqrt{\pi}}. \tag{55}$$

The corresponding $\omega(x)$ and $Q_M(x)$ can be expressed through exponential and error functions but will not be quoted here. The drastic change compared with the $\alpha=2$ and $\beta=1$ case is observed.

(c) the next generic situation is obtained from a bit more complicated choice

$$\rho(n)=\frac{n!}{n+1}. \tag{56}$$

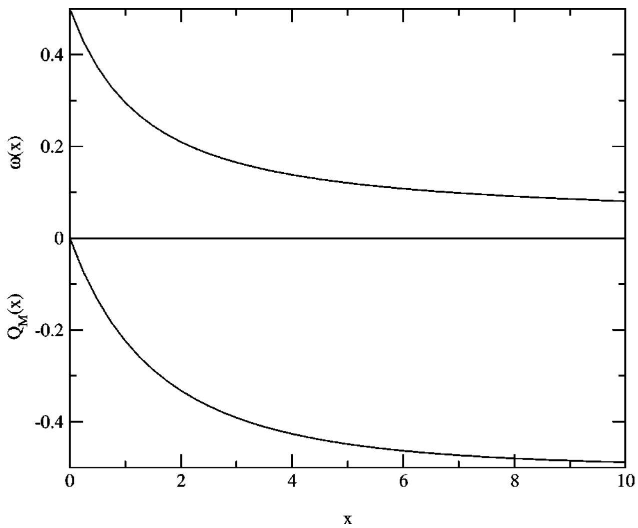

FIG. 3. The metric factor and the Mandel parameter for the moments of Eq. (52) for $\alpha=2$, $\beta=1$.

It leads to the normalization $\mathcal{N}(x)=e^{x}(1+x)$ and the positive weight function $\tilde{W}(x)=-\mathrm{Ei}(-x)$ where $\mathrm{Ei}(y)$ is the exponential integral [24] (see the formula 1(1) on p. 182 of Ref. [18] or the formula 8.4.11.1 on p. 642 of Ref. [21]). The weight $\tilde{W}(x)=-\mathrm{Ei}(-x)$ is plotted in Fig. (4). The metric factor is $\omega(x)=(x^2+2x+2)/(x+1)^2$ and $Q_M(x)=-x/[(x+1)(x+2)]$. Clearly, $\omega(x)\geqslant 1$ and $Q_M(x)<0$ indicating sub-Poissonian statistics.

(d) A generalization of (c) is to extend the factorial to a gamma function parametrized by $\alpha>0$. More precisely

$$\rho(n)=\frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)(n+1)} \tag{57}$$

originate from $\tilde{W}(x)=\Gamma(\alpha,x)/\Gamma(1+\alpha)$ (see formula 28(1) on p. 189 of Ref. [18] or formula 8.4.16.2 on p. 647 of Ref. [21]). It leads to

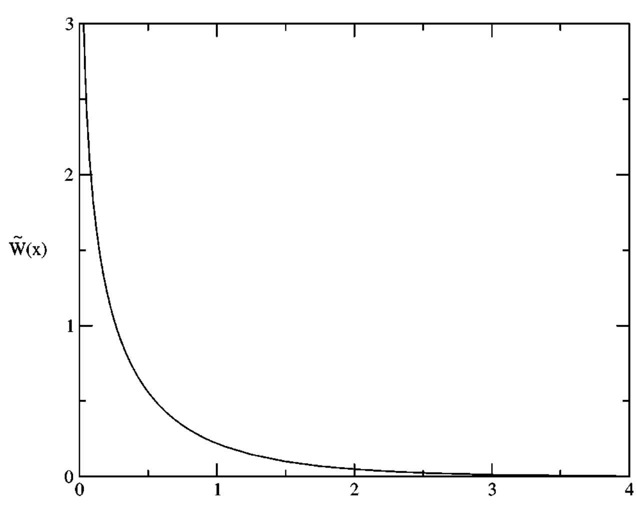

FIG. 4. The weight function $\tilde{W}(x)=-\mathrm{Ei}(-x)$ of example V (c).

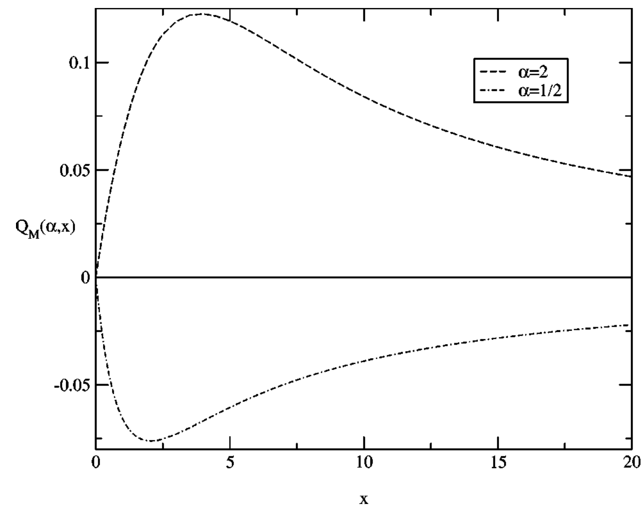

FIG. 5. The Mandel parameter of Eq. (60) for $\alpha=2$, and $\alpha=1/2$.

$$\mathcal{N}(x)={}_1F_1(2;1+\alpha;x). \tag{58}$$

The metric factor is

$$\omega(\alpha,x)=\frac{2}{(\alpha+1)\,{}_1F_1(2;\alpha+1;x)}\left[{}_1F_1(3;\alpha+2;x)\times\left(1-\frac{2x\,{}_1F_1(3;\alpha+2;x)}{(1+\alpha)\,{}_1F_1(2;\alpha+1;x)}\right)\right], \tag{59}$$

with $\omega(\alpha,0)=2/(\alpha+1)$. The function $\omega(\alpha,x)$ (as a function of x) is monotonically decreasing for $0<\alpha<1$, monotonically increasing for $\alpha>1$ and equals to 1 for $\alpha=1$. The Mandel parameter

$$Q_M(\alpha,x)=x\left(\frac{3}{\alpha+2}\,\frac{{}_1F_1(4;\alpha+3;x)}{{}_1F_1(3;\alpha+2;x)}-\frac{2}{\alpha+1}\,\frac{{}_1F_1(3;\alpha+2;x)}{{}_1F_1(2;\alpha+1;x)}\right), \tag{60}$$

is positive for $\alpha>1$ and negative for $0<\alpha<1$, as displayed in Fig. (5). It follows that the statistics of the coherent state with $\rho(n)$ given by Eq. (57) crosses over from sub- to super-Poissonian as α crosses $\alpha=1$.

(e) The choice $\rho(n)=(n!)^2$ is the particular case of states considered in the literature [13]. Their normalization is $\mathcal{N}(x)=I_0(2\sqrt{x})$, and the weight function is $\tilde{W}(x)=2K_0(2\sqrt{x})$ (see the Appendix for a simple demonstration of this result), where I_0 and K_0 are the modified Bessel functions of first and third kind, respectively. Here

$$\omega(x)=1-\left[\frac{I_1(2\sqrt{x})}{I_0(2\sqrt{x})}\right]^2, \tag{61}$$

$$Q_M(x)=\sqrt{x}\left(\frac{I_0(2\sqrt{x})}{I_1(2\sqrt{x})}-\frac{I_1(2\sqrt{x})}{I_0(2\sqrt{x})}\right)-1, \tag{62}$$

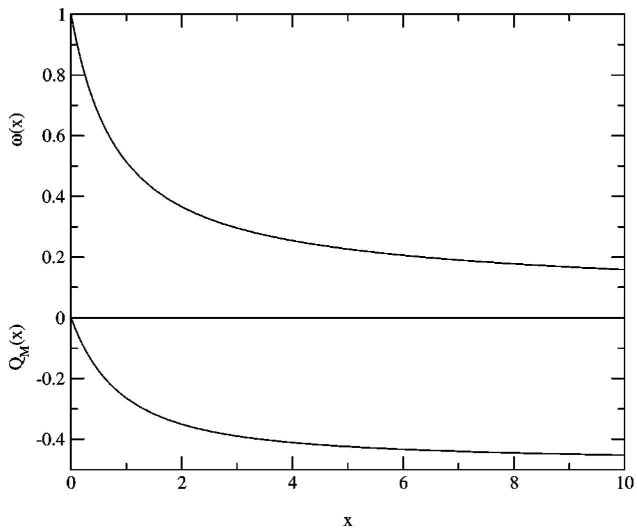

FIG. 6. The metric factor and the Mandel parameter of Eqs. (61) and (62).

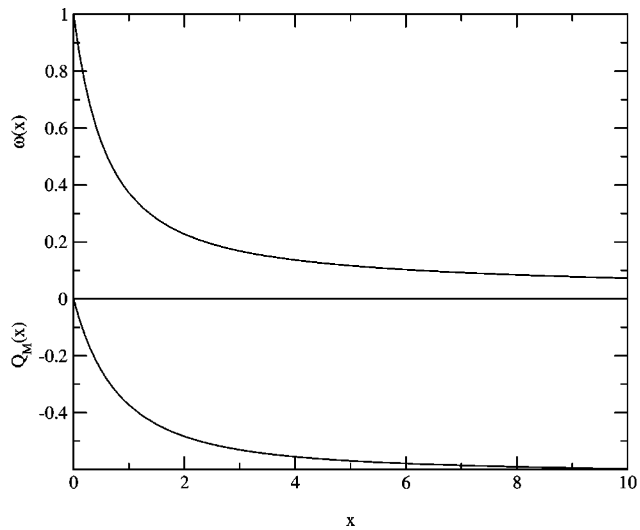

FIG. 7. The metric factor and the Mandel parameter of Eqs. (70) and (71).

with $Q_M(0)=0$, $Q_M(\infty)=-1/2$; $Q_M(x)$ is a strictly negative function, implying sub-Poissonian statistics. The plots of $\omega(x)$ and $Q_M(x)$ are displayed in Fig. (6).

(f) By setting $\rho(n)=(n!)^3$ we arrive at the normalization

$$\mathcal{N}(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n!)^3}\equiv {}_0F_2(1,1;x), \tag{63}$$

which is a generalization of the series ${}_0F_1(1;x)$ in terms of which all the Bessel functions are defined [including $I_0(2\sqrt{x})={}_0F_1(1;x)$ of the preceding example]. The weight function here, $\widetilde{W}(x)$, is a solution of the moment problem

$$\int_0^{\infty}x^n\widetilde{W}(x)dx=(n!)^3, \quad n=0,1,\ldots, \tag{64}$$

which, through Eq. (20) is the Mellin convolution of $2K_0(2\sqrt{y})$ with e^{-y}, or

$$\widetilde{W}(x)=2\int_0^{\infty}\frac{1}{t}e^{-x/t}K_0(2\sqrt{t})dt, \tag{65}$$

which is positive, but which cannot be expressed in simple terms. A possible way to obtain a closed form expression for Eq. (65) is to use the Meijer G-function [21] that is defined as a certain Mellin transform. When applied to Eq. (64) the weight is expressed through a known series representation of

$G^{3,0}_{0,3}\left(x\middle|\begin{matrix}\cdot\\ 0,\quad 0,\quad 0\end{matrix}\right)$:

$$\widetilde{W}(x)=G^{3,0}_{0,3}\left(x\middle|\begin{matrix}\cdot\\ 0,\quad 0,\quad 0\end{matrix}\right) \tag{66}$$

$$=\tfrac{1}{2}[\ln(x)]^2\,{}_0F_2(1,1;-x)+[\ln(x)]T_1(x)+T_2(x), \tag{67}$$

where $T_{1,2}(x)$ are rapidly converging power series defined by

$$T_1(x)=-3\sum_{n=0}^{\infty}\frac{\psi(n+1)}{(n!)^3}(-x)^n, \tag{68}$$

$$T_2(x)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\pi^2+9\psi^2(n+1)-3\psi^{(1)}(n+1)}{(n!)^3}(-x)^n, \tag{69}$$

where $\psi(z)$ and $\psi^{(1)}(z)$ are, respectively, digamma and polygamma functions of order one [24]. From Eq. (63) the analytic formulas for $\omega(x)$ and $Q_M(x)$ follow:

$$\omega(x)=\frac{1}{{}_0F_2(1,1;x)}\left[{}_0F_2(2,2;x)+x\left(\frac{1}{4}{}_0F_2(3,3;x)-\frac{{}_0F_2(2,2;x)}{{}_0F_2(1,1;x)}\right)\right], \tag{70}$$

$$Q_M(x)=x\left(\frac{1}{4}\,\frac{{}_0F_2(3,3;x)}{{}_0F_2(2,2;x)}-\frac{{}_0F_2(2,2;x)}{{}_0F_2(1,1;x)}\right). \tag{71}$$

They are plotted in Fig. (7). Note in passing that $\mathcal{N}(x)$ of Eq. (63) can be related to the so-called hyper-Bessel function [18–20] of ''type'' I_0. In this perspective, the relation $I_0\leftrightarrow K_0$ between the normalization and weight in example (e) can be extended to the current example if we tentatively identify the hyper-Bessel function of ''type'' K_0 with $\widetilde{W}(x)$ of Eq. (67). By changing the moments from $(n!)^2$ to $(n!)^3$ the singularity of the weight function at $x=0$ becomes stronger. This is illustrated on Fig. (8) where we have plotted $2K_0(2\sqrt{x})$ and $\widetilde{W}(x)$ of Eq. (67) on the same graph.

(g) An extension of example (e) in the form

$$\rho(n)=\frac{n!\,\Gamma(n+4/3)}{\Gamma(4/3)}, \tag{72}$$

leads to the normalization $\mathcal{N}(x)=\Gamma(4/3)[I_{1/3}(2\sqrt{x})]/(x^{1/6})$. From the formula 8.4.29.1 on p. 676 of Ref. [21] one con-

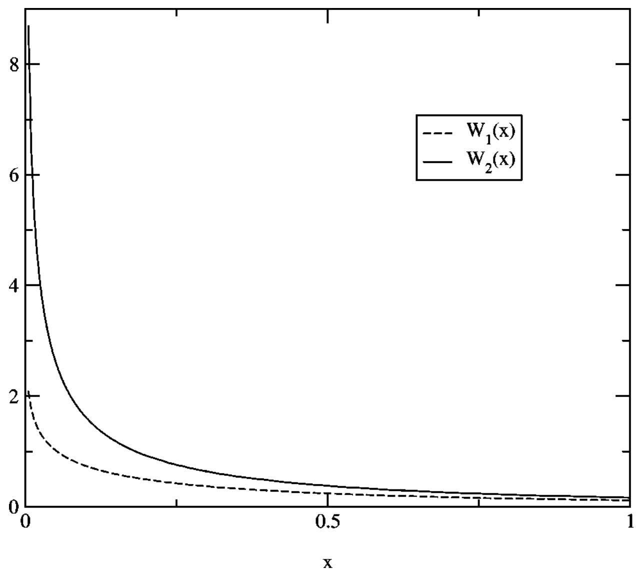

FIG. 8. Comparison of weight function $W_1(x)=2K_0(2\sqrt{x})$ and of $W_2(x)=\widetilde{W}(x)$ of Eq. (67).

cludes that the corresponding positive $\widetilde{W}(x)$ is given by $\mathrm{Ai}[(9x)^{1/3}]/\Gamma(4/3)$, where $\mathrm{Ai}(y)$ is the Airy function [24]. The metric factor is

$$\omega(x)=3+\left(\frac{1}{3\sqrt{x}}-3\right)\frac{I_{2/3}(2\sqrt{x})}{I_{1/3}(2\sqrt{x})}, \tag{73}$$

a monotonically decreasing function of x and $Q_M(x)$ turns out to be negative. They are illustrated on Fig. (9).

(h) The last example involves the moments

$$\rho(n)=\frac{(n!)^3\Gamma(3/2)}{\Gamma(n+3/2)} \tag{74}$$

leading to

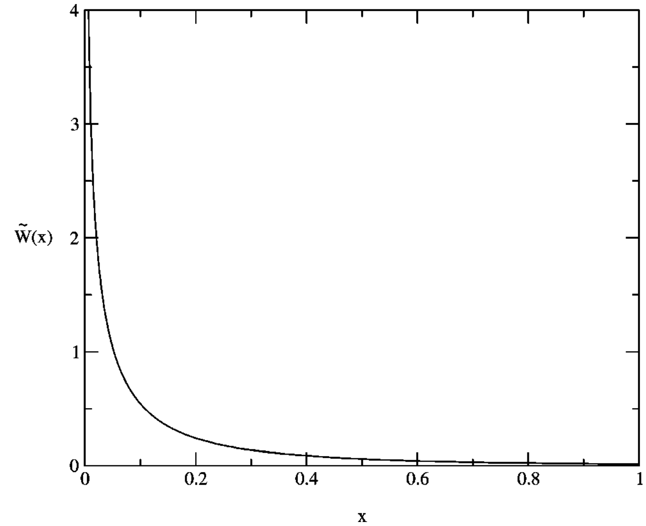

FIG. 10. The weight function $\widetilde{W}(x)$ of Eq. (76).

$$\mathcal{N}(x)={}_1F_2\left(\frac{3}{2};1,1;x\right)=[I_0(\sqrt{x})]^2+2\sqrt{x}I_0(\sqrt{x})I_1(\sqrt{x}) \tag{75}$$

and the explicitly positive weight function (see formula 37(1) on p. 205 of Ref. [18] or formula 8.4.23.27 on p. 668 of Ref. [21]):

$$\widetilde{W}(x)=[K_0(\sqrt{x})]^2, \tag{76}$$

which is plotted in Fig. (10). The metric factor and the Mandel parameter are rather complicated expressions involving $I_0(\sqrt{x})$ and $I_1(\sqrt{x})$. We display them in Fig. (11). Observe that $\omega(x)$ is monotonically decreasing from $\omega(0)=3/2$, whereas $Q_M(x)$ is negative everywhere, with $\lim_{x\to\infty}Q_M(x)=-1/2$.

As mentioned above, the preceding $\rho(n)$'s, from (a)–(h), can still be convoluted (with themselves or bilaterally) to

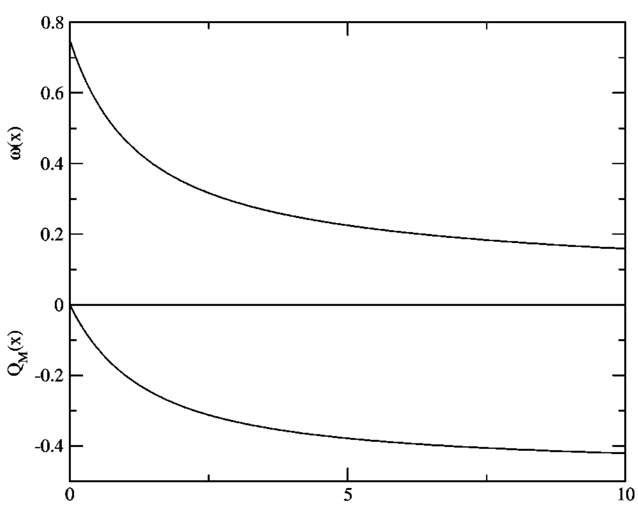

FIG. 9. The metric factor of Eq. (73) and the Mandel parameter for the moments of Eq. (72).

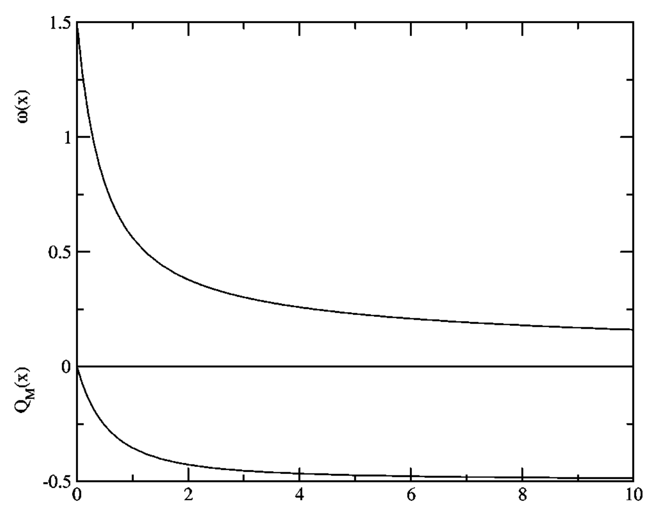

FIG. 11. The metric factor and the Mandel parameter for the moments of Eq. (74).

produce again generally new positive weights. In the appendix we have presented examples of such calculations.

VI. COHERENT STATES ON THE DISK

We turn now to a situation where the normalization given by the series Eq. (3) is only convergent for $x<R<\infty$. Then the domain of definition of the coherent states is a disk on the complex plane centered at $x=0$ with radius R. A trivial change of variables transforms it to a unit disk, $R=1$, which we shall adopt for all the subsequent examples. The resolution of unity condition, Eq. (1) takes the form of

$$\int\int_{|z|^2<1} d^2z|z\rangle W(|z|^2)\langle z|=I=\sum_{n=0}^{\infty}|n\rangle\langle n|, \tag{77}$$

$$\pi\int_0^1 x^n\left[\frac{W(x)}{\mathcal{N}(x)}\right]dx=\rho(n), \quad n=0,1,\ldots, \tag{78}$$

which is a classical Hausdorff moment problem [9] for a sought for positive function $\pi W(x)/\mathcal{N}(x)$. In contrast to the Stieltjes problem, Eq. (22), it is relatively easy here to furnish the solutions of Eq. (78) with a simple form of the moments. In this section, all the $\rho(n)$'s satisfy $\lim_{n\to\infty}\rho(n)=0$. We give here first two illustrative examples and then go over to more complicated moments and the applications of the inverse Mellin transform method applied to Eq. (78), quite analogous to the preceding paragraph.

(a) First, choosing $\tilde{W}(x)=2x$, we obtain $\rho(n)=2/(n+2)$, which leads to the normalization

$$\mathcal{N}(x)=\frac{1}{2}\sum_{n=0}^{\infty}(n+2)x^n=\frac{1}{2}\frac{2-x}{(1-x)^2}, \quad (0\leqslant x<1), \tag{79}$$

and through Eq. (36), to the metric factor

$$\omega(x)=2\frac{3-2x}{(2-x)^2(1-x)^2}, \tag{80}$$

both of which are plotted in Fig. (12). The Mandel parameter is

$$Q_M(x)=\frac{x(x^2-6x+7)}{(x-2)(x-3)(1-x)}. \tag{81}$$

(b) We complicate the moments a little by (arbitrarily) choosing $\rho(n)=6/[(n+2)(n+3)]$, which can be shown to originate from $\tilde{W}(x)=6\,x(1-x)$. The corresponding normalization and metric factor are given, respectively, by

$$\mathcal{N}(x)=\frac{1}{3}\frac{x^2-3x+3}{(1-x)^3}, \tag{82}$$

$$\omega(x)=3\frac{3x^2-8x+6}{(x^2-3x+3)^2(x-1)^2}, \tag{83}$$

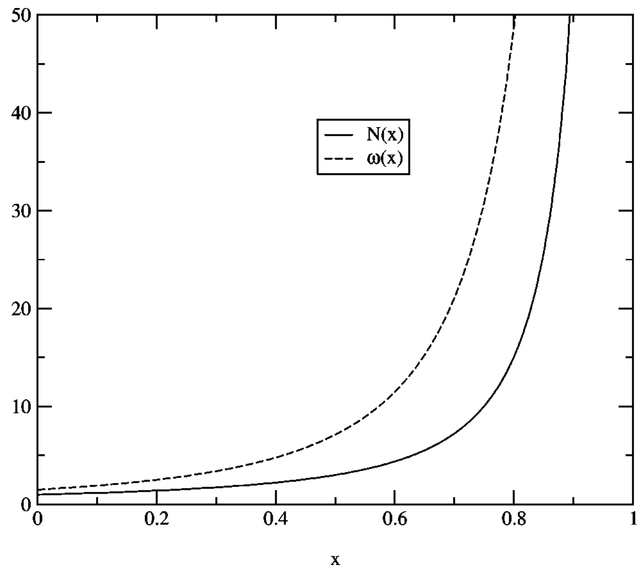

FIG. 12. Normalization and metric factor of example VI (a), Eqs. (79) and (80).

both of which are plotted in Fig. (13). The Mandel parameter is

$$Q_M(x)=\frac{x(x^4-8x^3+28x^2-42x+24)}{(x^2-3x+3)(x^2-4x+6)(1-x)}. \tag{84}$$

We now extend our procedure of construction of coherent states by choosing the moments in form of $(\Gamma\Gamma)/(\Gamma\Gamma)$, with Γ denoting some gamma function. This choice is not as arbitrary as it may seem to be, since, by examining the tables in Refs. [18,21], one finds at least four generic cases of positive weight function.

(c) Consider formula 3(1), p. 174 of Ref. [18] or formula 8.4.40.2 on p. 692 of Ref. [21], which reads

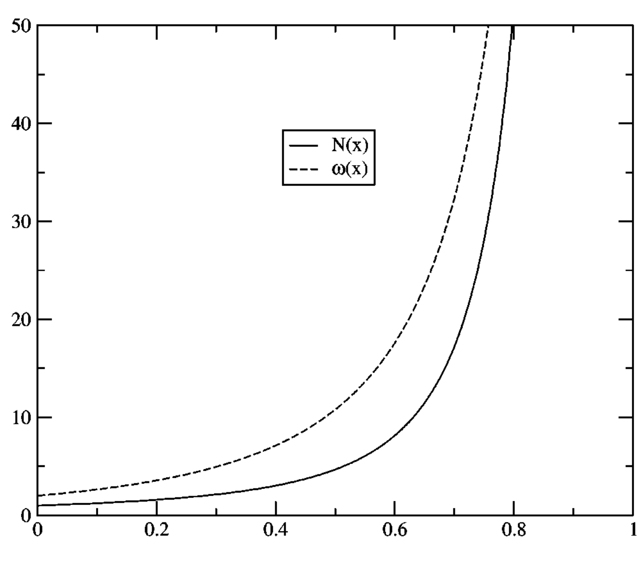

FIG. 13. Normalization and metric factor of example VI (b), Eqs. (82) and (83).

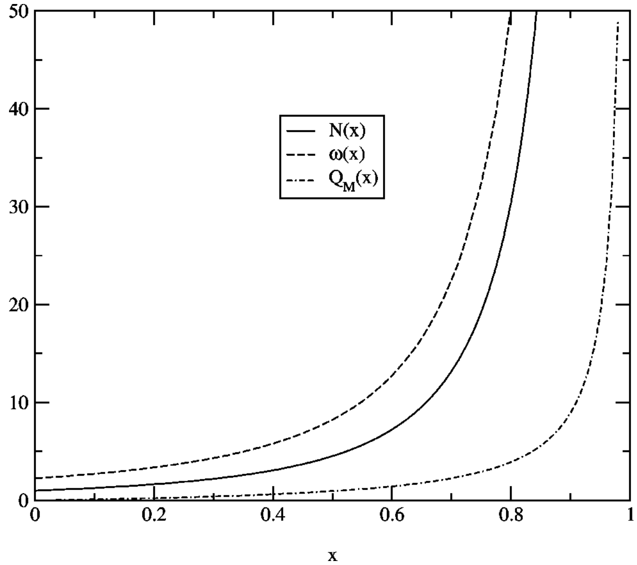

FIG. 14. Normalization, metric factor, and Mandel parameter of example VI (c).

$$\int_0^1 \frac{1}{2}\mathbf{K}(\sqrt{1-x})x^n dx = \frac{\pi}{4}\,\frac{(n!)^2}{\Gamma^2\left(n+\frac{3}{2}\right)} = \rho(n). \tag{85}$$

For this example, it follows that the normalization is given by

$$\mathcal{N}(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} x^n \frac{\Gamma^2\left(n+\frac{3}{2}\right)}{(n!)^2} = {}_2F_1\left(\frac{3}{2},\frac{3}{2};1;x\right) \tag{86}$$

$$= \frac{2}{\pi(x-1)}\left(\mathbf{K}(\sqrt{x}) + \frac{2}{x-1}\mathbf{E}(\sqrt{x})\right), \tag{87}$$

where, in Eqs. (85) and (87), $\mathbf{K}(k)$ is the complete elliptic integral of the first kind, and $\mathbf{E}(k)$ is the complete elliptic integral of the second kind [24]. The expressions for $\omega(x)$ and $Q_M(x)$ are quite involved and will not be reproduced here. The functions $\mathcal{N}(x)$, $\omega(x)$, and $Q_M(x)$ are plotted in Fig. (14).

(d) A further example is provided by formula 23(1) on p. 180 of Ref. [18] or formula 8.4.40.39 of Ref. [21], which reads

$$\int_0^1 \frac{3}{4}\mathbf{E}(\sqrt{1-x})x^n\,dx = \frac{3\pi}{8}\,\frac{n!(n+1)!}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{5}{2}\right)} = \rho(n), \tag{88}$$

giving the normalization

$$\mathcal{N}(x) = \frac{8}{3\pi}\sum_{n=0}^{\infty} x^n \frac{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{5}{2}\right)}{n!(n+1)!} = {}_2F_1\left(\frac{3}{2},\frac{5}{2};2;x\right) \tag{89}$$

$$= \frac{4}{3\pi}\,\frac{1}{x(x-1)}\left[\mathbf{K}(\sqrt{x}) + \frac{x+1}{x-1}\mathbf{E}(\sqrt{x})\right]. \tag{90}$$

The weight function $(3/4)\mathbf{E}(\sqrt{1-x})$ is a perfectly regular and nonsingular function on $[0,1]$. The metric factor and the Mandel parameter have a very similar behavior to the previous case and will not be considered here.

The other two examples involve as weight functions a rather general form of the Gauss hypergeometric function ${}_2F_1(a,b;c;x)$.

(e) We rewrite the formula 11(1) on p. 288 of Ref. [18] or formula 8.4.49.22 on p. 720 of Ref. [21] as

$$\begin{aligned}\int_0^1 x^n\bigg[&\frac{\Gamma(1+c-a)\Gamma(1+c-b)}{\Gamma(c)\Gamma(1+c-a-b)}\\ &\times(1-x)^{c-1}{}_2F_1(a,b;c;1-x)\bigg]dx\\ &= \frac{\Gamma(1+c-a)\Gamma(1+c-b)}{\Gamma(1+c-a-b)}\\ &\times\frac{\Gamma(n+1)\Gamma(n+1+c-a-b)}{\Gamma(n+1+c-a)\Gamma(n+1+c-b)}\\ &= \rho(n),\end{aligned} \tag{91}$$

with the restrictions $a+b-c<1$ and $c>0$. It turns out that for many choices of a, b, and c the weight function [the expression in square brackets in Eq. (91)] is positive.

We illustrate Eq. (91) here with the values $a=1/2$, $b=1/2$, and $c=3/2$. For this choice, one can show that $\tilde{W}(x) = 4/\pi \arcsin(\sqrt{1-x})$ [which is plotted on Fig. (15) with the weight functions of example (d) in Sec. VI and example (c) in Sec. VI] and Eq. (91) reduces to

$$\frac{4}{\pi}\int_0^1 x^n \arcsin(\sqrt{1-x})dx = \frac{2}{\sqrt{\pi}}\,\frac{\Gamma(n+3/2)}{(n+1)^2 n!} = \rho(n), \tag{92}$$

with the normalization,

$$\mathcal{N}(x) = {}_2F_1\left(2,2;\frac{3}{2};x\right) = \frac{1}{4(1-x)^2}\left[3 + \frac{(1+2x)\arcsin(\sqrt{x})}{\sqrt{x(1-x)}}\right], \tag{93}$$

the metric factor

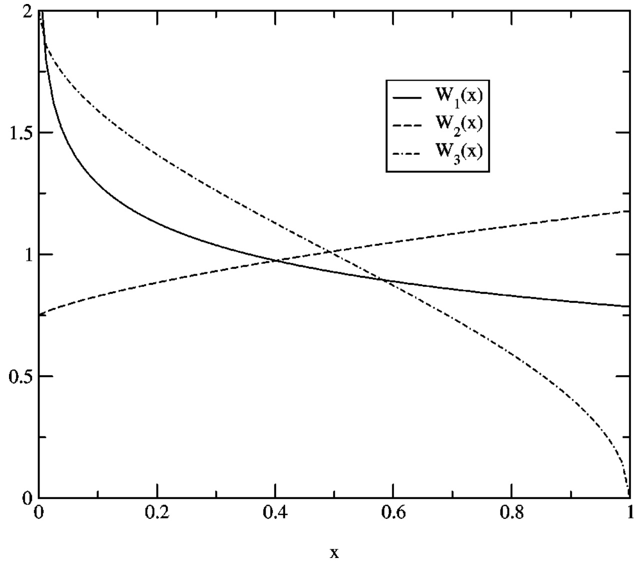

FIG. 15. Plots of weight functions: $W_1(x)$: weight function of example VI (c), $W_2(x)$: weight function of example VI (d), $W_3(x)$: weight function of example VI (e).

$$\omega(x)=\frac{8}{{}_2F_1\left(2,2;\frac{3}{2};x\right)}\left[\frac{1}{3}\,{}_2F_1\left(3,3;\frac{5}{2};x\right)-\frac{8}{9}x\frac{{}_2F_1\left(3,3;\frac{5}{2};x\right)}{{}_2F_1\left(2,2;\frac{3}{2};x\right)}+\frac{6}{5}x\,{}_2F_1\left(4,4;\frac{7}{2};x\right)\right],\tag{94}$$

and the Mandel parameter

$$Q_M(x)=2x\left(\frac{9}{5}\frac{{}_2F_1\left(4,4;\frac{7}{2};x\right)}{{}_2F_1\left(3,3;\frac{5}{2};x\right)}-\frac{4}{3}\frac{{}_2F_1\left(3,3;\frac{5}{2};x\right)}{{}_2F_1\left(2,2;\frac{3}{2};x\right)}\right).\tag{95}$$

These expressions are plotted in Fig. (16). The functions in Eqs. (94) and (95) can still be expressed by formulas of type (93) but these will not be reproduced here.

(f) The last example in this paragraph involves formula 12(4) on p. 289 of Ref. [18] or formula 8.4.49.25 on p. 721 of Ref. [21]:

$$\int_0^1 x^n\left[\gamma(a,b,c)(1-x)^{c-1}{}_2F_1\left(a,b;c;1-\frac{1}{x}\right)\right]dx =\Gamma(c)\,\gamma(a,b,c)\frac{\Gamma(n+1+a)\Gamma(n+1+b)}{\Gamma(n+1+c)\Gamma(n+1+a+b)}=\rho(n),\tag{96}$$

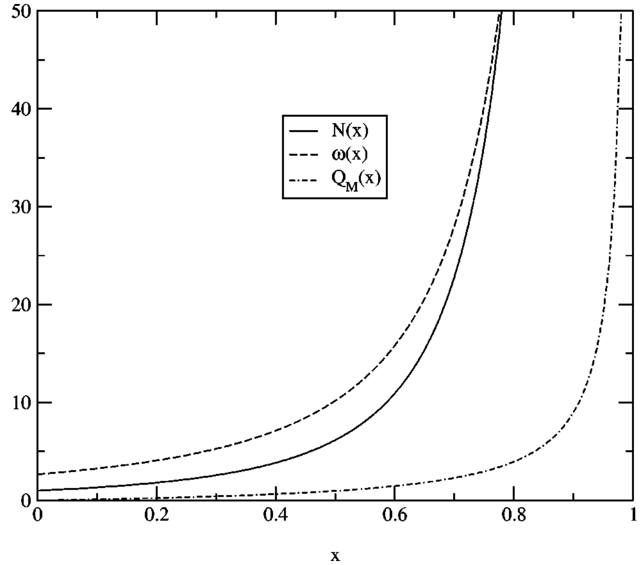

FIG. 16. Normalization, metric factor, and the Mandel parameter of example VI (e), Eqs. (93), (94), and (95).

with the restrictions $c>0$, $a,b>-1$, and where $\gamma(a,b,c):=[\Gamma(1+c)\Gamma(1+a+b)]/[\Gamma(c)\Gamma(1+a)\Gamma(1+b)]$. We choose now the set $a=1$, $b=2$, $c=3/2$ for which the moments are $\rho(n)=3\Gamma(\frac{5}{2})(n+1)!/[(n+3)\Gamma(n+\frac{5}{2})]$ and the weight function [in square bracket in lhs of Eq. (96)] is positive and reads

$$\tilde{W}(x)=\frac{9}{4}x\left[\sqrt{1-x}+x\,\text{arcsinh}\left(\sqrt{\frac{1-x}{x}}\right)\right],\tag{97}$$

while the corresponding normalization is

$$\mathcal{N}(x)=\frac{1}{9}\left(\frac{4-x}{x(1-x)^{5/2}}-\frac{4}{x}\right).\tag{98}$$

These quantities as well as the corresponding metric factor and the Mandel parameter are depicted in Figs. (17) and (18).

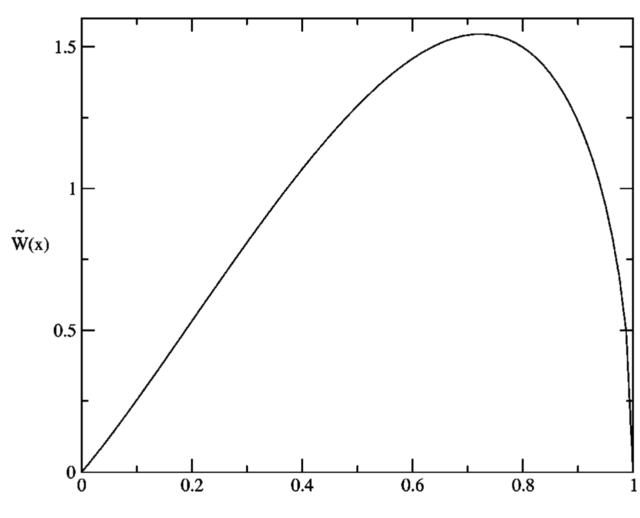

FIG. 17. Weight function of example VI (f), Eq. (97).

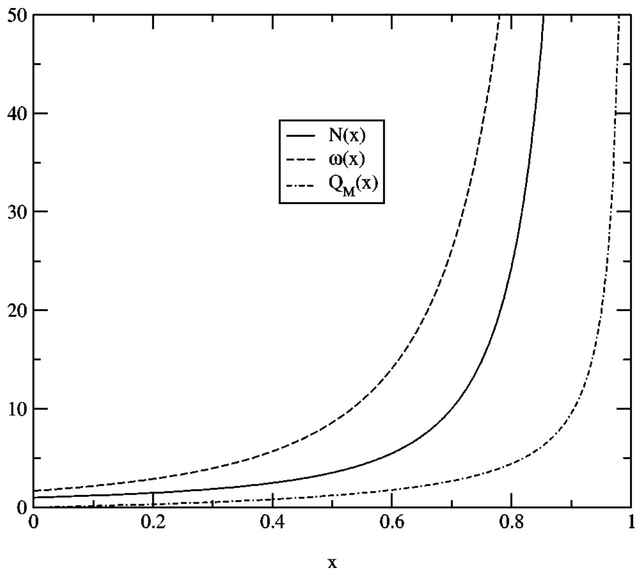

FIG. 18. Normalization, Eq. (98), metric factor, and the Mandel parameter for example VI (f).

This terminates our list of examples of coherent states on the unit disk for which the weight functions are positive. This list is by no means exhaustive, since by Mellin convoluting any one of these examples, either with itself or with any other of them, we will still obtain a new set of $\rho(n)$ with a positive weight function and desired convergence properties of the normalization. Furthermore, as explained in the preceding paragraph, a similar construction procedure can be designed using for $\rho(n)$ functions other than ratios of products of gamma functions using our procedure but this time based mostly on Ref. [22].

In the following, we shall point out that the foregoing examples are sufficient to recognize a class of physical potentials for which the constructed coherent states are asymptotically relevant.

VII. COHERENT STATES AND PHYSICAL POTENTIALS

The large variety of quantum wave functions introduced and discussed in the preceding paragraphs needs to be confronted with more specific physical situations usually formulated with a one-particle nonrelativistic Hamiltonian $\mathcal{H}(p,q)$ with p momentum and q coordinate, in the form $\mathcal{H}(p,q)=p^2/2m+V(q)$ where $V(q)$ is the potential. We shall limit ourselves in this work to one-dimensional Hamiltonian problems, and try to make contact between the coherent states and physical potentials. One way to do it is to consider a recent formulation of coherent states for systems with discrete and continuous spectrum [30]. In this paper we shall only take into account Hamiltonians with discrete spectra. In order to be able to associate the coherent states with Hamiltonian problems, some important modifications in the definition of coherent states discussed in the introduction should be made. We shall list them here. First of all, the idea of parametrizing the state $|z\rangle$ in terms of a single complex number z is extended by replacing z by two *independent* real numbers J and γ, such that $J\geqslant 0$ and $-\infty<\gamma<\infty$. Then the actual meaning of the orthogonal set $|n\rangle$ is extended to a general Hermitian operator $\mathcal{H}$, such that

$$\mathcal{H}|n\rangle=\omega e_n|n\rangle, \tag{99}$$

where the eigenvalues e_n satisfy

$$0=e_0<e_1<e_2,\ \dots\ . \tag{100}$$

Note that the usual boson operators do not enter these definitions, as in general $e_n\neq n$. The modified definition of the general coherent state becomes:

$$|J,\gamma\rangle:=\mathcal{N}^{-1/2}(J)\sum_{n=0}^{\infty}\frac{J^{n/2}\exp(-i\,e_n\gamma)}{\sqrt{\rho(n)}}|n\rangle \tag{101}$$

and it satisfies an extended set of conditions for a state to be coherent state [30]:

(1) $|J,\gamma\rangle$ is normalizable, i.e., the radius of convergence R of $\mathcal{N}(J)=\Sigma_{n=0}^{\infty}J^n/\rho(n)$ is not zero.

(2) $|J,\gamma\rangle$ is continuous in *two* labels J and γ, i.e., $(J',\gamma')\to(J,\gamma)\Rightarrow|J',\gamma'\rangle\to|J,\gamma\rangle$.

(3) The states $|J,\gamma\rangle$ satisfy the resolution of unity, with $W(J,\gamma)>0$ such that

$$\int_{-\infty}^{\infty}d\gamma\int_0^{\infty}dJ|J,\gamma\rangle W(J,\gamma)\langle J,\gamma|=I=\sum_{n=0}^{\infty}|n\rangle\langle n|. \tag{102}$$

(4) Temporal stability for a specific Hamiltonian $\mathcal{H}$.

$$e^{-i\,\mathcal{H}t}|J,\gamma\rangle=|J,\gamma+\omega\,t\rangle,\quad \omega=\text{const.} \tag{103}$$

(5) Action identity: $\langle J,\gamma|\mathcal{H}|J,\gamma\rangle=\omega\,J$.

The properties (4) and (5), which supplement the fundamental characteristics elaborated upon in the introduction, originate from the adaptation of $|J,\gamma\rangle$ to a specific Hamiltonian $\mathcal{H}$ of a truly interacting system, away from the harmonic oscillator, the free boson system. The property (5) forces the quantities $\rho(n)$ to be a unique function of e_n's, namely,

$$\rho(n)=\prod_{k=1}^{n}e_k,\quad \rho(0)=1. \tag{104}$$

Note, that these quite far-reaching extensions of the initial definition (2) to arrive at (101), do not affect much the property (3)—the resolution of unity: if we assume the product form for $W(J,\gamma)=W(J)U(\gamma)$ such that

$$\int\ \dots\ U(\gamma)d\gamma=\lim_{\Gamma\to\infty}\frac{1}{2\,\Gamma}\int_{-\Gamma}^{\Gamma}\dots d\gamma, \tag{105}$$

then resolution of unity for Eq. (102) boils down to the equations

$$\int_0^R J^n\left[\frac{W(J)}{\mathcal{N}(J)}\right]dJ=\rho(n),\quad n=0,1,2,\dots, \tag{106}$$

which are the previous Eq. (15). It means that the states defined by Eq. (101) have the same completeness relations as the states of Eq. (2). In addition, they are specifically adapted to a Hamiltonian with the spectrum of Eq. (100).

We shall now try to extract some information about possible Hamiltonians, by first deriving their spectra with Eq. (104) and then attempting to reconstruct the potentials $V(q)$, thereby obtaining the estimations for $\mathcal{H}$. It should be borne in mind that these steps are only approximate, and for at least two reasons: first and foremost, the reconstruction of the Hamiltonian from its known spectrum is never unique. This has been explicitly demonstrated for instance by Abraham and Moses [31] who have found highly anharmonic potentials, possessing the spectrum of the harmonic oscillator. The second reason is, that the only feasible method to estimate $V(q)$ from e_n's is to use the quasiclassical quantization conditions of Bohr-Sommerfeld

$$\frac{1}{\hbar}\int_a^b p(q)dq=\pi\left(n+\frac{1}{2}\right), \quad n=0,1,2,\ldots, \tag{107}$$

or

$$=\frac{1}{\hbar}\int_a^b\sqrt{2m(e_n-V(q))}dq=\pi\left(n+\frac{1}{2}\right), \quad n=0,1,2,\ldots, \tag{108}$$

where a and b are the turning points of the potential $V(q)$, defined by $p(a)=p(b)=0$. We will consider only the symmetric power-law potentials $V(q)=V_0|q|^\sigma$.

The usual disclaimer at this point is to restrict the validity of Eq. (107) to $n\gg 1$, although many examples are known for $V(q)$ for which Eq. (107) reproduces exactly the whole spectrum e_n, $n=0,1,\ldots$ [32]. We will now reconsider the whole set of coherent states from Secs. V and VI in the light of the definition, Eq. (101), using the $\rho(n)$'s from the preceding examples. According to Eq. (104), the spectrum of the (unknown) Hamiltonian is given by

$$e_n=\frac{\rho(n)}{\rho(n-1)}, \quad n=1,2,\ldots, \tag{109}$$

$$e_0=0. \tag{110}$$

The n dependence of the (quasiclassical) spectrum of the potential $V(q)=V_0|q|^\sigma$, $(V_0,\sigma>0)$ is given from the Eq. (108) $(\hbar=1)$ by

$$e_n=\left[\frac{\pi}{2}\frac{(n+1/2)}{\sqrt{2m}C(\sigma)}\right]^{2\sigma/(2+\sigma)}|V_0|^{2/(2+\sigma)} \underset{n\to\infty}{\sim} n^{2\sigma/(2+\sigma)}, \tag{111}$$

where $C(\sigma)=\int_0^1\sqrt{1-x^\sigma}dx=(\sqrt{\pi}/(2+\sigma))\Gamma(1/\sigma)/\Gamma(1/2+1/\sigma)$.

Let us calculate by Eq. (109) the e_n's for the $\rho(n)$ of examples (a)–(h) of Sec. V; the result is: (a) $n+p$; (b) $\Gamma(n/2+2)/\Gamma(n/2+3/2)$; (c) $n^2/(n+1)$; (d) $n(n+\alpha)/(n+1)$; (e) n^2; (f) n^3; (g) $n(n+1/3)$; and (h) $n^3/(n+1/2)$. All these expressions represent an unbounded spectrum. In the limit $n\to\infty$, the cases (a), (c), and (d) correspond to $\sigma=2$, or the quadratic potentials; the cases (e), (g), and (h) can be interpreted as $\sigma=\infty$, which can be represented as an infinite square-well potential. The case (f) cannot be fitted to Eq. (111). The case (b) for $n\to\infty$ gives $e_n\sim n^{1/2}$ indicating $\sigma=2/3$. Taking literally Eq. (52) with general $\alpha,\beta>0$ we obtain

$$e_n=\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta-\alpha)} \underset{n\to\infty}{\longrightarrow} n^\alpha, \tag{112}$$

which is independent of β. This leads to

$$\sigma=\frac{2\alpha}{2-\alpha}, \quad 0<\alpha<2, \tag{113}$$

which includes any positive exponent σ as long as $0<\alpha<2$. Here, the Mittag-Leffler functions prove to be a particularly flexible tool to identify any power-law potentials.

For the coherent states on the unit disk of Sec. VI the situation is a little different. In fact, the states of Sec.VI are appropriate for attractive potentials of inverse power-law type, which typically exhibit bounded spectrum, i.e., the Coulomb problem, $\sigma=-1$.

We derive the appropriate estimate for e_n from Eq. (108) with $V(q)=-|V_0|q^\sigma$ $(-2<\sigma<0)$; it reads

$$e_n=-\left[\frac{\pi}{2}\frac{(n+1/2)}{\sqrt{2m}D(\sigma)}\right]^{2\sigma/(2+\sigma)}|V_0|^{2/(2+\sigma)} \underset{n\to\infty}{\sim} -n^{2\sigma/(2+\sigma)}. \tag{114}$$

The integral $D(\sigma)=\int_0^1\sqrt{x^\sigma-1}dx$ is known exactly only for selected values of σ: $D(-1/2)=\pi/4$, $D(-1)=\pi/2$, $D(-3/2)=2\sqrt{\pi}\Gamma(7/6)/\Gamma(5/3)$, etc. Together with the condition $e_0=0$, we arrive at the following representation of the bound spectrum $(n\gg 1)$:

$$e_n\sim 1-n^{2\sigma/(2+\sigma)}, \quad -2<\sigma<0. \tag{115}$$

The expression (115) will now be confronted with the e_n's obtained with Eq. (109) using the set of $\rho(n)$'s from the examples of Sec. VI. After simple transformations we arrive at the following n dependence:

$$\text{(a)}\quad e_n=1-\frac{1}{n+2} \underset{n\to\infty}{\longrightarrow} 1-\frac{1}{n}, \tag{116}$$

$$\text{(b)}\quad e_n=1-\frac{2}{n+3} \underset{n\to\infty}{\longrightarrow} 1-\frac{2}{n}, \tag{117}$$

$$\text{(c)}\quad e_n=1-\frac{4n+1}{(2n+1)^2} \underset{n\to\infty}{\longrightarrow} 1-\frac{1}{n}, \tag{118}$$

$$\text{(d)}\quad e_n=1-\frac{4n+3}{4\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)} \underset{n\to\infty}{\longrightarrow} 1-\frac{1}{n}, \tag{119}$$

$$\text{(e)}\quad e_n = 1 - \frac{\frac{3}{2}n+1}{(n+1)^2} \xrightarrow[n\to\infty]{} 1-\frac{3}{2n}, \tag{120}$$

$$\text{(f)}\quad e_n = 1 - \frac{\frac{3}{2}n+\frac{5}{2}}{(n+3)(n+3/2)} \xrightarrow[n\to\infty]{} 1-\frac{3}{2n}. \tag{121}$$

We see that a common pattern emerges as the behavior $e_n \sim 1 - c/n$ $(c = \text{const} > 0)$ applies to all these examples. According to the estimate Eq. (115) the corresponding exponent is $\sigma = -2/3$, thus the Hamiltonian $\mathcal{H}(p,q)$ for which the states VI (a)–(f) are asymptotically relevant is

$$\mathcal{H}(p,q) = \frac{p^2}{2m} - \frac{|V_0|}{|q|^{2/3}}. \tag{122}$$

As far as we know the eigenfunctions $|n\rangle$ of the discrete part of spectrum of Eq. (122) are not known so that the state $|J,\gamma\rangle$ of Eq. (101) cannot be completely determined.

We shall now attempt to describe qualitatively the coherent state and its weight function that is asymptotically relevant to a situation somewhere in between two known cases, $\sigma = -2/3$ [case of Eq. (122)] and the Coulomb case $V(q) \sim |q|^{-1}$, for which the weight function is known exactly [30]

$$W(x) = \frac{1}{2}[1 + \delta(x - 1^-)], \tag{123}$$

where $\delta(y)$ is the Dirac delta function. We choose $\sigma = -3/4$, which through Eq. (115) yields $e_n = 1 - n^{-6/5}$. Now we go back to Eq. (104), which indicates that $\rho(n) = \Pi_{k=2}^n e_k = \Pi_{k=2}^n (1 - k^{-6/5})$. As it appears that this product cannot be evaluated in a closed form, we shall make a rough approximation to its behavior. We first note that

$$\ln \rho(n) = \sum_{k=2}^{n} \ln(1 - k^{-6/5}) = -\sum_{k=2}^{n} [k^{-6/5} + (1/2)k^{-12/5} + (1/3)k^{-18/5} + \cdots] . \tag{124}$$

For all x, $0 < x \leq 2^{-6/5}$, and with $c \equiv -2^{6/5}\ln(1-2^{-6/5}) (=1.31277)$, we next observe that

$$-cx \leq \ln(1-x) \leq -x. \tag{125}$$

Thus, we can assert that

$$-c\sum_{k=2}^{n} k^{-6/5} \leq \sum_{k=2}^{n} \ln(1-k^{-6/5}) \leq -\sum_{k=2}^{n} k^{-6/5}. \tag{126}$$

Since

$$\int_2^{n+1} x^{-6/5}dx \leq \sum_{k=2}^{n} k^{-6/5} \leq \int_1^n x^{-6/5}dx, \tag{127}$$

it follows that

$$-5c[1-n^{-1/5}] \leq \sum_{k=2}^{n} \ln(1-k^{-6/5}) \leq -5[2^{-1/5} - (n+1)^{-1/5}]. \tag{128}$$

Doing better than this would require using more terms in the series on the right side of Eq. (124), and with additional effort this could be done. However, for convenience, we stop at this point and take from this estimate the rough approximation that

$$\rho(n) \approx e^{c'[(n+1)^{-1/5}-1]} \tag{129}$$

for large n for some constant $c' \approx 5$. To proceed further, we need to change our problem somewhat, and being guided by Eq. (129), we adopt

$$\rho(n) := \frac{[1 + c'(n+1)^{-1/5}]}{1+c'}, \tag{130}$$

as our definition of $\rho(n)$ for all n, $0 \leq n < \infty$. This final form defines a normalized expression for $\rho(n)$ that exhibits the desired characteristic behavior $n^{-1/5}$ isolated above. Having done so, we may then seek a weight function $\widetilde{W}(x)$ so that

$$\int_0^1 x^n \widetilde{W}(x)dx = \frac{[1 + c'(n+1)^{-1/5}]}{1+c'}, \tag{131}$$

which again can be done using the inverse Mellin transform; compare the formula 4.37 on p. 38 of Ref. [22]. The solution of Eq. (131) is then

$$\widetilde{W}(x) = \frac{1}{\Gamma\left(\frac{1}{5}\right)} \frac{c'}{1+c'} \frac{1}{\left[\ln\left(\frac{1}{x}\right)\right]^{4/5}} + \frac{1}{1+c'}\delta(x-1^-), \tag{132}$$

which, as explained, may be considered as an approximate form of the weight function corresponding to $\sigma = -3/4$. In Fig. (19) we plot the first term of Eq. (132) for $c' = 5$ together with the weight of example (c) in Sec. VI, $1/2\mathbf{K}(\sqrt{1-x})$. We discern a clear pattern as σ goes through $-2/3, -3/4$ and -1: for $\sigma = -2/3$, the weight is singular at $x = 0$, for $\sigma = -3/4$ it is singular at $x = 1$, and for $\sigma = -1$ it is also singular at $x = 1$.

Still more cases can be encompassed by choosing in $\rho(n)$ a general exponent $r > 0$, such that $\rho(n) = \Pi_{k=2}^n (1 - k^{-r})$. This implies $\rho(\infty) = \lim_{n\to\infty}\rho(n) = 0$ if $r \leq 1$ and $0 < \rho(\infty) < 1$ if $r > 1$. Since from Eq. (114) $r = -2\sigma/(2+\sigma)$ then for $-2 < \sigma < -2/3$, $\rho(\infty) \neq 0$, whereas for $-2/3 \leq \sigma < 0$, $\rho(\infty) = 0$. Both of these circumstances define Hausdorff moment problems; the case of Eq. (124) corresponds to the former case, while all the examples of Sec. VI correspond to the latter case.

VIII. CONCLUSION

There are two main goals that we have tried to develop in the present paper. For the first goal, in which attention is

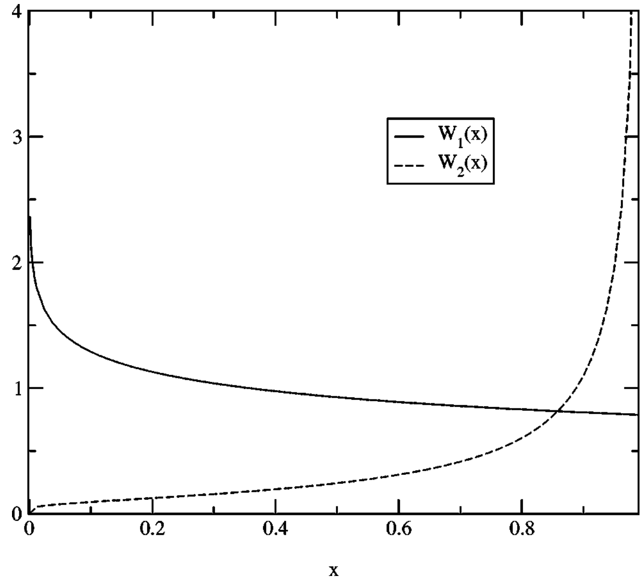

FIG. 19. Comparison of weight functions of example VI (c) $[W_1(x)=(1/2)\mathbf{K}(\sqrt{1-x})]$ with that of Eq. (132) $[W_2(x)]$.

confined to holomorphic coherent states (coherent states that up to normalization are functions of a single complex variable), our purpose has been to display examples of the infinite variety of possible coherent states. In so doing we have given attention to many examples that can be characterized by various special functions, and have generally been able to provide closed expressions for the normalization factor and many expectation values such as the Mandel parameter, and what we have called the metric factor. Study of the Mandel parameter has provided many examples of coherent states with either sub- or super-Poissonian behavior. The metric factor, on the other hand, has given us a direct handle on the geometry of the coherent states. It is noteworthy that for the class of coherent states we have studied, different geometries are associated with different sets of coherent states in a one-to-one fashion.

A principal tool in developing our various coherent state families has been the use of Mellin and inverse Mellin transforms associated with both Stieltjes and Hausdorff moment problems. Fortunately, tables of such transforms are sufficiently rich to enable us to extract many examples of non-negative weights in these transforms that can serve as probability distributions. Convolution formulas implicitly extend these examples to an unlimited supply of acceptable weight functions. In our studies we have offered examples of weight functions defined over the entire non-negative real numbers as well as over only a finite interval of the non-negative real numbers including zero. These two categories are associated with the different moment problems and correspond to coherent states defined over the entire complex plane or over a disk of finite radius centered at the origin, respectively.

In this regard, it is interesting to recall that certain Stieltjes moment problems have a nonunique solution [9,10], that is, different, non-negative probability densities can lead to identical moments. The uniqueness of solutions can be determined with the (sufficient) Carleman condition [10]: if $S=\Sigma_{n=0}^{\infty}[\rho(n)]^{-1/2n}$ diverges then the solution is unique. This is for instance the case of examples (a) and (b) in Sec. V and for $\alpha<2$, $\beta>0$, in examples (c) and (a) in Sec. V. In contrast for example (b) with $\alpha>2,\beta>0$ the solution is definitively nonunique [4].

From the point of view of the construction of coherent states this situation would imply, in particular, that such sets of coherent states would possess distinct, but still non-negative, weight functions in their resolution of unity. What, if anything, would be the physical consequences of such a situation remains a problem for the future. Some thoughts have been devoted to this question recently [33].

The solutions of the Hausdorff moment problems are always unique, which implies that the coherent states pertinent to Hamiltonians with bounded spectra of Sec. VI have a unique resolution of unity.

We mention here a recent study of so-called molecular coherent states [34] constructed with a finite set of arbitrarily chosen numbers $\rho(n)$. In this case, the resolution of unity can only be achieved if the $\rho(n)$'s satisfy certain specific relations. In a recent work [35], a related construction has been undertaken.

The second goal of this paper has been an effort to relate our various coherent states to possible physical systems. In so doing we were guided by a recent discussion of coherent states [30] that generally does not involve holomorphic coherent states. Most importantly, we have taken from that study a means of relating the sequence of moments $\{\rho(n)\}$ with the sequence of rescaled energy eigenvalues $\{e_n\}$. The large n behavior of the energy eigenvalues can be related to a potential function by the Bohr-Sommerfeld quantization rule, and by that route, we can give a qualitative association of our coherent states with selected quantum-mechanical problems. This association serves to add a degree of physics to our discussion.

Our states of Eq. (101) offer specific advantages as compared with previous constructions of Nieto and Simmons [36,37], of coherent states for general potentials. Nieto and Simmons work is strictly semiclassical in character in regard to its association with the presumed physical problem. There is no control over what approximate kind of description Nieto's work has for a given physical system. Last but not least, there is no general proof that Nieto's states even span the required Hilbert space, let alone whether they can be used to form the usual kind of resolution of unity.

One clear result of our discussion has been to offer a wealth of specific and concrete examples of coherent states. These sets of coherent states stand ready to join their brethren in the service of providing explicit representations that can be applied to the study of various problems.

ACKNOWLEDGMENTS

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APPENDIX: APPLICATIONS OF THE MELLIN CONVOLUTION

In this appendix we shall carry out in detail two cases of Eq. (20) applied to specific forms of $\rho(n)$. The objective is to obtain the weight function $\tilde{W}_{12}(x)$ of the state described by $\rho_{12}(n)=\rho_1(n)\cdot\rho_2(n)$ if we know the individual positive weights $\tilde{W}_1(x)$ and $\tilde{W}_2(x)$ originating from $\rho_1(n)$ and $\rho_2(n)$, respectively.

(1) The simplest nontrivial example is to obtain the weight $\tilde{W}_{12}(x)$ by Mellin ''convoluting'' two conventional coherent states, i.e., $\rho_{12}(n)=\rho_1(n)\rho_2(n)\equiv\rho_1^2(n)=(n!)^2$. We use Euler's representation of the gamma function, $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx$, which implies $\mathcal{M}^{-1}[\Gamma(s);x]=e^{-x}$. This in turn, using Eq. (20) with $a=0$ and $b=-1$ gives

$$\mathcal{M}^{-1}[\Gamma^2(s);x]=\int_0^\infty t^{-1}\exp\left[-\left(\frac{x}{t}+t\right)\right]dt. \quad \text{(A1)}$$

The Eq. (A1) will now be compared with Sommerfeld's representation of the modified Bessel function (see formula 8.432.6 on p. 969 of Ref. [28]):

$$K_\nu(x)=\frac{1}{2}\left(\frac{x}{2}\right)^\nu\int_0^\infty e^{-(t+x^2/4t)}\frac{1}{t^{\nu+1}}dt, \quad \text{(A2)}$$

immediately giving $\tilde{W}_{12}(x)=\mathcal{M}^{-1}[\Gamma^2(s);x]=2K_0(2\sqrt{x})>0$. Note that although the individual $\tilde{W}_{1,2}(x)=e^{-x}$ are regular functions, their Mellin convolution $\tilde{W}_{12}(x)$ is singular at $x=0$. It is evidently an integrable singularity.

(2) The second example involves $\rho_1(n)=n!/(n+1)$ [see Eq. (56)] with $\tilde{W}_1(x)=-\mathrm{Ei}(-x)>0$ and $\rho_2(n)=2/\sqrt{\pi}\Gamma(n+3/2)$, which by means of Eq. (18) originates from $\tilde{W}_2(x)=2/(\sqrt{\pi})x^{1/2}e^{-x}$. We are going to obtain the function $\tilde{W}_{12}(x)$ whose nth moment $\rho_{12}(n)$ is equal to

$$\rho_{12}(n)=\int_0^\infty x^n\tilde{W}_{12}(x)dx=\frac{2}{\sqrt{\pi}}\frac{n!\Gamma(n+3/2)}{n+1}. \quad \text{(A3)}$$

We use again Eq. (20) and obtain directly

$$\tilde{W}_{12}(x)=\frac{2}{\sqrt{\pi}}\int_0^\infty\sqrt{\frac{x}{t}}e^{-x/t}[-\mathrm{Ei}(-t)]\frac{dt}{t}, \quad \text{(A4)}$$

which upon utilizing the formula 2.5.4.3 on p. 72 of Ref. [29] results in

$$\tilde{W}_{12}(x)=-4\,\mathrm{Ei}(-2\sqrt{x}), \quad \text{(A5)}$$

which is again a positive function. The normalization of the state $|z\rangle$ defined by the $\rho_{12}(n)$'s above is equal to

$$\mathcal{N}(x)=\frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{x^n(n+1)}{n!\Gamma(n+3/2)} \quad \text{(A6)}$$

$$=\frac{1}{2\sqrt{x}}\left(\frac{\sinh(2\sqrt{x})}{2}+\sqrt{x}\cosh(2\sqrt{x})\right). \quad \text{(A7)}$$

We have thus obtained a coherent state satisfying the required fundamental properties. The reader may well imagine other Mellin convolutions that can be either obtained analytically or, more generally, numerically.

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