Resonant photon creation in a three-dimensional oscillating cavity

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(Received 8 December 2000; published 5 June 2001)

We analyze the problem of photon creation inside a perfectly conducting, rectangular, three-dimensional cavity with one oscillating wall. For some particular values of the frequency of the oscillations the system is resonant. We solve the field equation using multiple scale analysis and show that the total number of photons inside the cavity grows exponentially in time. This is also the case for slightly off-resonance situations. Although the spectrum of a cavity is in general nonequidistant, we show that the modes of the electromagnetic field can be coupled, and that the rate of photon creation strongly depends on this coupling. We also analyze the thermal enhancement of the photon creation.

DOI: 10.1103/PhysRevA.64.013808 PACS number(s): 42.50.Lc, 42.50.Dv, 12.20.−m

I. INTRODUCTION

The existence of an attractive force between two perfectly conducting plates was predicted by Casimir in 1948 [1]. It has been measured with accurate precision in recent years by Lamoreaux [2] and by Mohideen and Roy [3]. These experiments confirm the existence of vacuum field fluctuations in the framework of field quantization with static boundaries, and increase the interest in the case of dynamical boundaries as well.

The dynamical effect consists in the generation of photons due to the instability of the vacuum state of the electromagnetic field in the presence of time-dependent boundaries. In the literature it is referred to as the dynamical Casimir effect [4] or motion-induced radiation [5]. Up to now no concrete experiment has been carried out to confirm this photon generation, but an experimental verification is not out of reach. From the theoretical point of view it is widely accepted that the most favorable configuration in order to observe the phenomenon is a vibrating cavity in which it is possible to produce resonant effects between the mechanical and field oscillations.

Many previous papers have focused their attention on the field quantization within a one-dimensional cavity with one or two walls performing small amplitude oscillations, at twice the eigenfrequency of some unperturbed electromagnetic mode. For these cavities there exists a strong intermode interaction, which is a consequence of the equidistant character of the frequency spectrum. The main features of the one-dimensional model are that photons are created in all electromagnetic modes (due to mode-mode coupling), and that the total energy inside the cavity grows exponentially at the expense of the energy given to the system to keep the wall moving. A simple approach is to make a perturbative expansion in terms of the amplitude of oscillations, as was done in [6]. However, this perturbative treatment breaks down after a short period of time due to the appearence of secular terms. In [7] the renormalization group technique was used in order to obtain a solution valid for a period of time longer than that of the pertubative case. There it was shown that the energy spectrum develops a nontrivial structure formed by peaks traveling at the speed of light and bouncing against the walls (in agreement with other authors [8]). Reference [9] makes use of the fact that two different time scales characterize the problem; the usual one, related to the wall's oscillation period, and a ''slow'' one which accounts for the cumulative resonance effect. It is then possible to isolate the resonant part after averaging over the fast oscillations the initial equations for the electromagnetic field modes.

There is some work in the literature dealing with higher-dimensional cavities. In [10] the radiation emitted in each polarization of the electromagnetic field was computed perturbatively, when two parallel, plane, and perfectly conducting plates oscillate along the direction perpendicular to their surfaces. Such geometry constitutes the simplest example of an open three-dimensional cavity. In [11] the authors obtained the distribution of the created photons for the case of parametric resonance inside a three-dimensional cavity. In both cases [10,11] a perturbative method was applied so the results are valid in the short time limit. In [9] a nonperturbative analysis was presented, generalizing the method of averaging over fast oscillations to higher-dimensional cavities. However, the intermode coupling was neglected, reducing the problem to that of one single parametric oscillator.

It is of particular interest to find out how the finite temperature affects the photon production. This was studied in [12] with a nonperturbative method (see also [13]). A remarkable enhancement of the pure vacuum effect was found, but again neglecting the coupling between modes.

In this paper we present a detailed analysis of the photon generation inside a three-dimensional resonant cavity. We also discuss the finite temperature case, showing the enhancement of the effect with respect to $T=0$. We apply a multiple scale analysis (MSA) which provides us with a

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1050-2947/2001/64(1)/013808(10)/$20.00

simple technique equivalent to summing the most secular terms to all orders in the perturbative treatment. In this way we can get a solution valid for a period of time longer than that of the perturbative case. We pay particular attention to the resonant coupling between different modes.

The paper is organized as follows. In Sec. II we obtain the time evolution of the quantized field by expanding it over the *instantaneous basis*. For simplicity we deal with a scalar bosonic field. Following the steps given in [14] we arrive at an infinite set of coupled differential equations for the coefficients of the expansion. We also explain there how to compute the number of motion-induced photons for the zero-temperature case. In Sec. III we describe and apply the MSA to our problem. We find the coupling conditions between different modes that can be satisfied depending on the cavity's spectrum. In Sec. IV we present a general analysis of the coupling conditions, and discuss some examples. In particular we find that the fundamental mode of a cubic cavity is coupled to another mode in the parametric resonance case, giving as result that the number of photons with two different frequencies increases exponentially in time. However, the production rate for the fundamental mode is only one-half of that expected if we neglected the coupling, as previous works did. At the end of this section we study slightly off-resonance situations. In Sec. V we obtain an expression for the number of photons in each mode assuming that the field was initially in thermal equilibrium. Section VI contains our final remarks and comments on the generalization to the more realistic case of an electromagnetic field.

II. SCALAR FIELD QUANTIZATION WITH MOVING BOUNDARIES

We consider a rectangular cavity formed by perfectly reflecting walls with dimensions L_x, L_y, and L_z. The wall placed at $x=L_x$ is at rest for $t<0$ and begins to move following a given trajectory $L_x(t)$ at $t=0$. Note that we assume this trajectory as prescribed for the problem (not a dynamical variable) and that it works as a time-dependent boundary condition for the field. The field $\phi(\mathbf{x},t)$ satisfies the wave equation $\Box\phi=0$ in $3+1$ dimensions, and the boundary conditions $\phi|_{\rm walls}=0$ for all times. The Fourier expansion of the field for an arbitrary moment of time, in terms of creation and annihilation operators, can be written as

$$\phi(\mathbf{x},t)=\sum_{\mathbf{n}} \hat{a}_{\mathbf{n}}^{\rm in} u_{\mathbf{n}}(\mathbf{x},t)+\text{H.c.}, \tag{1}$$

where the mode functions $u_{\mathbf{n}}(\mathbf{x},t)$ form a complete orthonormal[1] set of solutions of the wave equation with vanishing boundary conditions.

When $t\leqslant 0$ (static cavity) each field mode is determined by three positive integers n_x, n_y and n_z. That is,

$$u_{\mathbf{n}}(\mathbf{x},t<0)=\frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi}{L_y}y\right)\times\sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z\pi}{L_z}z\right)e^{i\omega_{\mathbf{k}}t}, \tag{2}$$

$$\omega_{\mathbf{n}}=\pi\sqrt{\left(\frac{n_x}{L_x}\right)^2+\left(\frac{n_y}{L_y}\right)^2+\left(\frac{n_z}{L_z}\right)^2}, \tag{3}$$

with the shorthand notation $\mathbf{n}=(n_x,n_y,n_z)$.[2]

When $t>0$ the boundary condition on the moving wall becomes $\phi(x=L_x(t),y,z,t)=0$. In order to satisfy it we expand the mode functions in Eq. (1) with respect to an instantaneous basis

$$u_{\mathbf{n}}(\mathbf{x},t>0)=\sum_{\mathbf{k}} Q_{\mathbf{k}}^{(\mathbf{n})}(t)\sqrt{\frac{2}{L_x(t)}}\sin\left(\frac{k_x\pi}{L_x(t)}x\right)\times\sqrt{\frac{2}{L_y}}\sin\left(\frac{k_y\pi}{L_y}y\right)\sqrt{\frac{2}{L_z}}\sin\left(\frac{k_z\pi}{L_z}z\right) \tag{4}$$

$$=\sum_{\mathbf{k}} Q_{\mathbf{k}}^{(\mathbf{n})}(t)\varphi_{\mathbf{k}}(\mathbf{x},L_x(t)), \tag{5}$$

with the initial conditions

$$Q_{\mathbf{k}}^{(\mathbf{n})}(0)=\frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\delta_{\mathbf{k},\mathbf{n}},\quad \dot{Q}_{\mathbf{k}}^{(\mathbf{n})}(0)=-i\sqrt{\frac{\omega_{\mathbf{n}}}{2}}\delta_{\mathbf{k},\mathbf{n}}. \tag{6}$$

In this way we ensure that, as long as $L_x(t)$ and $\dot{L}_x(t)$ are continuous at $t=0$, each field mode and its time derivate are also continuous functions. The expansion in Eq. (5) for the field modes must be a solution of the wave equation. Taking into account that the $\varphi_{\mathbf{k}}$'s form a complete and orthonormal set and that they depend on t only through $L_x(t)$, we obtain a set of coupled equations for $Q_{\mathbf{k}}^{\mathbf{n}}(t)$: [14]

$$\ddot{Q}_{\mathbf{k}}^{(\mathbf{n})}+\omega_{\mathbf{k}}^2(t)Q_{\mathbf{k}}^{(\mathbf{n})}=2\lambda(t)\sum_{\mathbf{j}} g_{\mathbf{kj}}\dot{Q}_{\mathbf{j}}^{(\mathbf{n})}+\dot{\lambda}(t)\sum_{\mathbf{j}} g_{\mathbf{kj}}Q_{\mathbf{j}}^{(\mathbf{n})}+\lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} g_{\mathbf{lk}}g_{\mathbf{lj}}Q_{\mathbf{j}}^{(\mathbf{n})}, \tag{7}$$

where

$$\omega_{\mathbf{k}}(t)=\pi\sqrt{\left(\frac{k_x}{L_x(t)}\right)^2+\left(\frac{k_y}{L_y}\right)^2+\left(\frac{k_z}{L_z}\right)^2},\quad \lambda(t)=\frac{\dot{L}_x(t)}{L_x(t)}. \tag{8}$$

The coefficients $g_{\mathbf{kj}}$ are defined by

$$g_{\mathbf{kj}}=L_x(t)\int_0^{L_x(t)} dx\frac{\partial\varphi_{\mathbf{k}}}{\partial L_x}\varphi_{\mathbf{j}}, \tag{9}$$

[1]The inner product is the usual for the Klein-Gordon equation, namely, $(\psi,\xi)=-i\int_{\rm cavity}d^3x[\psi\,\dot{\xi}^\star-\dot{\psi}\,\xi^\star]$.

[2]We are using units where $\hbar=c=1$.

and read

$$g_{\mathbf{kj}}=-g_{\mathbf{jk}}=\begin{cases}(-1)^{k_x+j_x}\dfrac{2k_xj_x}{j_x^2-k_x^2}\delta_{k_yj_y}\delta_{k_zj_z} & \text{if } k_x\neq j_x\\ 0 & \text{if } k_x=j_x.\end{cases} \tag{10}$$

Furthermore, in deriving Eq. (7) we have used that $\Sigma_{\mathbf{l}} g_{\mathbf{kl}} g_{\mathbf{jl}} = L_x^2 \int dx (\partial\varphi_{\mathbf{k}}/\partial L_x)(\partial\varphi_{\mathbf{j}}/\partial L_x)$, which follows from the completeness relation of the $\varphi_{\mathbf{k}}$'s.

The annihilation and creation operators $\hat{a}_{\mathbf{k}}^{\text{in}}$ and $\hat{a}_{\mathbf{k}}^{\dagger\,\text{in}}$ correspond to the particle notion in the ''in'' region ($t<0$). If the wall stops for $t>t_{\text{final}}$, we can define a new set of operators $\hat{a}_{\mathbf{k}}^{\text{out}}$ and $\hat{a}_{\mathbf{k}}^{\dagger\,\text{out}}$, associated with the particle notion in the ''out'' region ($t>t_{\text{final}}$). These two sets of operators are connected by means of the Bogoliubov transformation

$$\hat{a}_{\mathbf{k}}^{\text{out}}=\sum_{\mathbf{n}}(\hat{a}_{\mathbf{n}}^{\text{in}}\alpha_{\mathbf{nk}}+\hat{a}_{\mathbf{n}}^{\dagger\,\text{in}}\beta_{\mathbf{nk}}^{\star}). \tag{11}$$

The coefficients $\alpha_{\mathbf{nk}}$ and $\beta_{\mathbf{nk}}$ can be obtained as follows. When the wall returns to its initial position the right hand side in Eq. (7) vanishes and the solution reads

$$Q_{\mathbf{k}}^{(\mathbf{n})}(t>t_{\text{final}})=A_{\mathbf{k}}^{(\mathbf{n})}e^{i\omega_{\mathbf{k}}t}+B_{\mathbf{k}}^{(\mathbf{n})}e^{-i\omega_{\mathbf{k}}t}, \tag{12}$$

with $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$ being some constant coefficients to be determined by the continuity conditions at $t=t_{\text{final}}$. Inserting Eq. (12) into Eqs. (1) and (5) we obtain an expansion of ϕ in terms of $\hat{a}_{\mathbf{k}}^{\text{in}}$ and $\hat{a}_{\mathbf{k}}^{\dagger\,\text{in}}$ for $t>t_{\text{final}}$. Comparing this with the equivalent expansion in terms of $\hat{a}_{\mathbf{k}}^{\text{out}}$ and $\hat{a}_{\mathbf{k}}^{\dagger\,\text{out}}$ it is easy to see that

$$\alpha_{\mathbf{nk}}=\sqrt{2\omega_{\mathbf{k}}}B_{\mathbf{k}}^{(\mathbf{n})},\quad \beta_{\mathbf{nk}}=\sqrt{2\omega_{\mathbf{k}}}A_{\mathbf{k}}^{(\mathbf{n})}. \tag{13}$$

The amount of photons created in the mode $\mathbf{k}$ is the average value of the number operator $\hat{a}_{\mathbf{k}}^{\dagger\,\text{out}}\hat{a}_{\mathbf{k}}^{\text{out}}$ with respect to the initial vacuum state (defined through $\hat{a}_{\mathbf{k}}^{\text{in}}|0_{\text{in}}\rangle=0$). With the help of Eq. (11) and Eq. (13) we get

$$\langle\mathcal{N}_{\mathbf{k}}\rangle=\langle 0_{\text{in}}|\hat{a}_{\mathbf{k}}^{\dagger\,\text{out}}\hat{a}_{\mathbf{k}}^{\text{out}}|0_{\text{in}}\rangle=\sum_{\mathbf{n}}2\omega_{\mathbf{k}}|A_{\mathbf{k}}^{(\mathbf{n})}|^2. \tag{14}$$

III. MULTIPLE SCALE ANALYSIS

Up to this point the equations are valid for an arbitrary motion of the wall [we only assume $L(0)=L_0$ and $\dot{L}(0)=0$ because the wall is at rest for $t<0$]. We are interested in the number of photons created inside the cavity, so it is natural to look for harmonic oscillations of the wall that could enhance that number by means of resonance effects for some specific external frequencies. So we study the trajectory

$$L(t)=L_0[1+\epsilon\sin(\Omega t)+\epsilon f(t)], \tag{15}$$

where $f(t)$ is some decaying function that allows us to meet the continuity conditions at $t=0$ [for example, $f(t)=-\Omega t\, e^{-\alpha t}$]. For small amplitudes of oscillations ($\epsilon\ll 1$), the equations for the modes Eq. (7) take the form

$$\begin{aligned}\ddot{Q}_{\mathbf{k}}^{(\mathbf{n})}+\omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(\mathbf{n})}=&\,2\epsilon\left(\frac{\pi k_x}{L_x}\right)^2\sin(\Omega t)Q_{\mathbf{k}}^{(\mathbf{n})}\\ &-\epsilon\Omega^2\sin(\Omega t)\sum_{\mathbf{j}} g_{\mathbf{kj}}Q_{\mathbf{j}}^{(\mathbf{n})}\\ &+2\epsilon\Omega\cos(\Omega t)\sum_{\mathbf{j}} g_{\mathbf{kj}}\dot{Q}_{\mathbf{j}}^{(\mathbf{n})}+\epsilon O(f)\\ &+O(\epsilon^2),\end{aligned} \tag{16}$$

where $O(f)$ denotes terms proportional to f, $\dot{f}$, and $\ddot{f}$.

It is known that a naive perturbative solution of these equations in powers of the displacement ϵ breaks down after a short amount of time, of order $(\epsilon\Omega)^{-1}$. This happens for those particular values of the external frequency Ω such that there is a resonant coupling with the eigenfrequencies of the static cavity. In this situation, to find a solution valid for longer times (of order $\epsilon^{-2}\Omega^{-1}$) we use the multiple scale analysis technique [15]. We introduce a second time scale $\tau=\epsilon t$ and expand $Q_{\mathbf{k}}^{(\mathbf{n})}$ as follows (we shall content ourselves with first order MSA):

$$Q_{\mathbf{k}}^{(\mathbf{n})}(t)=Q_{\mathbf{k}}^{(\mathbf{n})(0)}(t,\tau)+\epsilon Q_{\mathbf{k}}^{(\mathbf{n})(1)}(t,\tau)+O(\epsilon^2). \tag{17}$$

The derivatives with respect to the time scale t read

$$\begin{aligned}\dot{Q}_{\mathbf{k}}^{(\mathbf{n})}&=\partial_t Q_{\mathbf{k}}^{(\mathbf{n})(0)}+\epsilon[\partial_\tau Q_{\mathbf{k}}^{(\mathbf{n})(0)}+\partial_t Q_{\mathbf{k}}^{(\mathbf{n})(1)}],\\ \ddot{Q}_{\mathbf{k}}^{(\mathbf{n})}&=\partial_t^2 Q_{\mathbf{k}}^{(\mathbf{n})(0)}+\epsilon[2\partial_{\tau t}^2 Q_{\mathbf{k}}^{(\mathbf{n})(0)}+\partial_t^2 Q_{\mathbf{k}}^{(\mathbf{n})(1)}].\end{aligned} \tag{18}$$

The initial conditions are

$$\begin{aligned}Q_{\mathbf{k}}^{(\mathbf{n})(0)}(0)&=\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\delta_{\mathbf{n},\mathbf{k}},\\ \dot{Q}_{\mathbf{k}}^{(\mathbf{n})(0)}(0)&=-i\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\delta_{\mathbf{n},\mathbf{k}}.\end{aligned} \tag{19}$$

To zeroth order in ϵ we get the equation of a harmonic oscillator

$$Q_{\mathbf{k}}^{(\mathbf{n})(0)}=A_{\mathbf{k}}^{(\mathbf{n})}(\tau)e^{i\omega_{\mathbf{k}}t}+B_{\mathbf{k}}^{(\mathbf{n})}(\tau)e^{-i\omega_{\mathbf{k}}t}, \tag{20}$$

and using the initial conditions it follows that

$$A_{\mathbf{k}}^{(\mathbf{n})}(\tau=0)=0, \tag{21}$$

$$B_{\mathbf{k}}^{(\mathbf{n})}(\tau=0)=\frac{1}{\sqrt{2w_{\mathbf{k}}}}\delta_{\mathbf{n},\mathbf{k}}. \tag{22}$$

To first order in ϵ we obtain[3]

$$\partial_t^2 Q_{\mathbf{k}}^{(\mathbf{n})(1)} + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(\mathbf{n})(1)} = -2\partial_{\tau t}^2 Q_{\mathbf{k}}^{(\mathbf{n})(0)} + 2\left(\frac{\pi k_x}{L_x}\right)^2 \sin(\Omega t) Q_{\mathbf{k}}^{(\mathbf{n})(0)} - \Omega^2 \sin(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}} g_{\mathbf{kj}} Q_{\mathbf{j}}^{(\mathbf{n})(0)} + 2\Omega\cos(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}} g_{\mathbf{kj}}\partial_t Q_{\mathbf{j}}^{(\mathbf{n})(0)} + O(f). \tag{23}$$

The basic idea of MSA is to impose the condition that any term on the right-hand side of the previous equation with a time dependency of the form $e^{\pm i\omega_{\mathbf{k}}t}$ must vanish. If not, these terms would be in resonance with the left-hand-side term and secularities would appear. The terms contained in $O(f)$ are not relevant because they are exponentially suppressed, and do not produce secularities. After imposing the requirement that no term $e^{+i\omega_{\mathbf{k}}t}$ appear, we get

$$\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau} = -\frac{\pi^2 k_x^2}{2\omega_{\mathbf{k}} L_x^2} B_{\mathbf{k}}^{(\mathbf{n})}\delta(2\omega_{\mathbf{k}}-\Omega) + \sum_{\mathbf{j}}\left(-\omega_{\mathbf{j}}+\frac{\Omega}{2}\right)\times\delta(-\omega_{\mathbf{k}}-\omega_{\mathbf{j}}+\Omega)\frac{\Omega}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} B_{\mathbf{j}}^{(\mathbf{n})} + \sum_{\mathbf{j}}\left[\left(\omega_{\mathbf{j}}+\frac{\Omega}{2}\right)\times\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{j}}-\Omega) + \left(\omega_{\mathbf{j}}-\frac{\Omega}{2}\right)\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{j}}+\Omega)\right]\times\frac{\Omega}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} A_{\mathbf{j}}^{(\mathbf{n})}. \tag{24}$$

In a similar fashion, the fact that no secularities should arise from the $e^{-i\omega_{\mathbf{k}}t}$ term leads to

$$\frac{dB_{\mathbf{k}}^{(\mathbf{n})}}{d\tau} = -\frac{\pi^2 k_x^2}{2\omega_{\mathbf{k}} L_x^2} A_{\mathbf{k}}^{(\mathbf{n})}\delta(2\omega_{\mathbf{k}}-\Omega) + \sum_{\mathbf{j}}\left(-\omega_{\mathbf{j}}+\frac{\Omega}{2}\right)\times\delta(-\omega_{\mathbf{k}}-\omega_{\mathbf{j}}+\Omega)\frac{\Omega}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} A_{\mathbf{j}}^{(\mathbf{n})} + \sum_{\mathbf{j}}\left[\left(\omega_{\mathbf{j}}+\frac{\Omega}{2}\right)\times\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{j}}-\Omega) + \left(\omega_{\mathbf{j}}-\frac{\Omega}{2}\right)\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{j}}+\Omega)\right]\times\frac{\Omega}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} B_{\mathbf{j}}^{(\mathbf{n})}. \tag{25}$$

[3]It is not straightforward to compute the next order corrections using MSA. The introduction of new time scales like $\tau_1=\epsilon t, \tau_2=\epsilon^2 t$, etc., is in general not sufficient to determine the second order solution unambiguously [15]. The renormalization group method [16] (which is equivalent to MSA to first order in ϵ) seems to be more appropiate to systematically improve the result. In any case, the next order corrections will be very small for $\epsilon^2 t \leqslant \Omega^{-1}$.

The previous set of two equations are nontrivial (i.e., lead to resonant behavior) if $\Omega=2\omega_{\mathbf{k}}$ (resonant condition). Moreover, there is intermode coupling between modes $\mathbf{j}$ and $\mathbf{k}$ if any of the following conditions is satisfied:

$$\Omega=\omega_{\mathbf{k}}+\omega_{\mathbf{j}}, \tag{26}$$

$$\Omega=\omega_{\mathbf{k}}-\omega_{\mathbf{j}}, \tag{27}$$

$$\Omega=\omega_{\mathbf{j}}-\omega_{\mathbf{k}}. \tag{28}$$

There is an alternative, equivalent way of deriving the equations of motion [9]. For $\epsilon\ll 1$, it is natural to assume that the solution of Eq. (16) is of the form

$$Q_{\mathbf{k}}^{(\mathbf{n})}(t) = A_{\mathbf{k}}^{(\mathbf{n})}(t)e^{i\omega_{\mathbf{k}}t} + B_{\mathbf{k}}^{(\mathbf{n})}(t)e^{-i\omega_{\mathbf{k}}t}, \tag{29}$$

where the functions $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$ are slowly varying. In order to obtain differential equations for them, we insert this ansatz into Eq. (16) and neglect second derivatives of $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$. After multiplying the equation by $e^{\pm i\omega_{\mathbf{k}}t}$ we average over the fast oscillations. The resulting equations coincide with Eqs. (24) and (25).

We derived the equations for $3+1$ dimensions. It is very easy to obtain the corresponding ones in $1+1$ and $2+1$ dimensions. In all cases the resonant conditions are given by Eqs. (26)–(28) above. The main difference between the $1+1$ case and higher dimensions is that in $1+1$ the eigenfrequencies ω_k are proportional to integers. The spectrum is equidistant and therefore an infinite set of modes may be coupled. For example, when the external frequency is $\Omega=2\omega_1$, the mode k is coupled with the modes $k\pm 2$. This has been extensively studied in the literature [7,9,11,17]. In what follows we will be concerned with cavities with non equidistant spectra.

IV. RESONANT PHOTON CREATION

In this section we shall solve the coupled Eqs. (24) and (25). We will see that there are different kinds of solution depending both on the wall's frequency and on the spectrum of the static cavity. Note that the spectrum is related to the cavity's dimensions through Eq. (3). In Sec. IV A we will present a general analysis of the resonant conditions and the solutions. We will show some particular examples in Sec. IV B. In Sec. IV C we will analyze the case in which the external frequency is slightly off-resonance.

A. General analysis

Let us consider the ''parametric resonance case,'' in which the frequency of the wall is twice the frequency of some unperturbed mode, say $\Omega=2\omega_{\mathbf{k}}$. Under this condition we expect that the number of created photons in the mode $\mathbf{k}$ will grow exponentially in time due to resonance effects. In order to find $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$ from Eq. (24) and Eq. (25) we have to analyze whether the coupling conditions $|\omega_{\mathbf{k}}\pm\omega_{\mathbf{j}}|=\Omega$ can be satisfied or not. If we set $\Omega=2\omega_{\mathbf{k}}$, the resonant mode $\mathbf{k}$ will be coupled to some other mode $\mathbf{j}$ only if $\omega_{\mathbf{j}}$

$-\omega_{\mathbf{k}}=\Omega=2\omega_{\mathbf{k}}$. Clearly, the latter relation will be satisfied depending on the spectrum of the particular cavity under consideration.

First, let us assume that this condition is not fullfilled. In this case, the equations for $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$ reduce to

$$\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=\frac{-1}{2\omega_{\mathbf{k}}}\left(\frac{\pi k_x}{L_x}\right)^2 B_{\mathbf{k}}^{(\mathbf{n})}, \tag{30}$$

$$\frac{dB_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=\frac{-1}{2\omega_{\mathbf{k}}}\left(\frac{\pi k_x}{L_x}\right)^2 A_{\mathbf{k}}^{(\mathbf{n})}. \tag{31}$$

The solution that satisfies the initial conditions (21) and (22) reads

$$B_{\mathbf{k}}^{(\mathbf{n})}=\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\delta_{\mathbf{k},\mathbf{n}}\cosh(\gamma k_x\tau), \tag{32}$$

$$A_{\mathbf{k}}^{(\mathbf{n})}=-\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\delta_{\mathbf{k},\mathbf{n}}\sinh(\gamma k_x\tau), \tag{33}$$

where $\gamma=(k_x/\Omega)(\pi/L_x)^2$. With the help of Eq. (14) we obtain

$$\langle\mathcal{N}_{\mathbf{k}}\rangle=\sinh^2(\gamma k_x\tau_f), \tag{34}$$

where $\tau_f=\epsilon t_f$. In this uncoupled resonance case the average number of created photons in the mode **k** increases exponentially in time with a rate given by $2\gamma k_x$. The same result has been obtained in previous papers (see Ref. [9] and Ref. [12]). There it was assumed that the coupling condition $\omega_{\mathbf{j}}=3\omega_{\mathbf{k}}$ cannot be fullfilled for two- and three-dimensional cavities, essentially due to the nonequidistant character of the spectrum. As we shall see, this is not always true. In what follows we will solve Eq. (24) and Eq. (25) with coupled modes, and we will show some explicit examples.

Let us now assume the existence of one mode, say **j**, in the infinite sum in Eq. (24) and Eq. (25), that satisfies $\omega_{\mathbf{j}}=3\omega_{\mathbf{k}}$. We obtain for $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$

$$\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=\gamma[-k_x B_{\mathbf{k}}^{(\mathbf{n})}+(-1)^{j_x+k_x}j_x A_{\mathbf{j}}^{(\mathbf{n})}], \tag{35}$$

$$\frac{dB_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=\gamma[-k_x A_{\mathbf{k}}^{(\mathbf{n})}+(-1)^{j_x+k_x}j_x B_{\mathbf{j}}^{(\mathbf{n})}], \tag{36}$$

where we have used the fact that the relation $\omega_{\mathbf{j}}=3\omega_{\mathbf{k}}$ is equivalent to

$$j_x^2=9\,k_x^2+8\left[\left(\frac{L_x}{L_y}k_y\right)^2+\left(\frac{L_x}{L_z}k_z\right)^2\right] \tag{37}$$

because the coupling coefficient $g_{\mathbf{kj}}$ is proportional to $\delta_{k_y j_y}\delta_{k_z j_z}$. The next step is to obtain the equations for $A_{\mathbf{j}}^{(\mathbf{n})}$ and $B_{\mathbf{j}}^{(\mathbf{n})}$. The mode **j** is coupled to modes **s** that satisfy $2\omega_{\mathbf{k}}=|\omega_{\mathbf{j}}\pm\omega_{\mathbf{s}}|$. Since $\omega_{\mathbf{j}}=3\,\omega_{\mathbf{k}}$ this relation is satisfied for $\omega_{\mathbf{s}}=\omega_{\mathbf{k}}$ (as expected) and for $\omega_{\mathbf{s}}=5\,\omega_{\mathbf{k}}$. We assume that the spectrum under consideration does not satisfy the latter. In this case, the equations read

$$\frac{dA_{\mathbf{j}}^{(\mathbf{n})}}{d\tau}=-\frac{(-1)^{j_x+k_x}\gamma j_x}{3}A_{\mathbf{k}}^{(\mathbf{n})}, \tag{38}$$

$$\frac{dB_{\mathbf{j}}^{(\mathbf{n})}}{d\tau}=-\frac{(-1)^{j_x+k_x}\gamma j_x}{3}B_{\mathbf{k}}^{(\mathbf{n})}. \tag{39}$$

In order to find the solution to the above equations we write the system in matrix form:

$$\frac{d\vec{v}}{d\tau}=\mathcal{M}\,\vec{v}, \tag{40}$$

where

$$\vec{v}(\tau)=\begin{pmatrix} B_{\mathbf{k}}^{(\mathbf{n})}(\tau)\\ A_{\mathbf{k}}^{(\mathbf{n})}(\tau)\\ B_{\mathbf{j}}^{(\mathbf{n})}(\tau)\\ A_{\mathbf{j}}^{(\mathbf{n})}(\tau)\end{pmatrix},\quad \mathcal{M}=\gamma\begin{pmatrix} 0 & -k_x & (-1)^{k_x+j_x}j_x & 0\\ -k_x & 0 & 0 & (-1)^{k_x+j_x}j_x\\ -(-1)^{k_x+j_x}j_x/3 & 0 & 0 & 0\\ 0 & -(-1)^{k_x+j_x}j_x/3 & 0 & 0\end{pmatrix}, \tag{41}$$

and the initial condition reads

$$\vec{v}(0)=\begin{pmatrix} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\delta_{\mathbf{n},\mathbf{k}}\\ 0\\ \frac{1}{\sqrt{2\omega_{\mathbf{j}}}}\delta_{\mathbf{n},\mathbf{j}}\\ 0\end{pmatrix}. \tag{42}$$

The solution is easily obtained after diagonalizing $\mathcal{M}$. The eigenvalues are given by

$$\lambda=\pm\frac{\gamma k_x}{2}\pm\frac{i\gamma}{6}\sqrt{|9k_x^2-12j_x^2|}, \tag{43}$$

where we have used Eq. (37). The solution can be formally written as

$$\vec{v}(\tau)=\mathcal{C}\,e^{\mathcal{D}\tau}\mathcal{C}^{-1}\vec{v}(0), \tag{44}$$

where $\mathcal{D}$ is the eigenvalue diagonal matrix and $\mathcal{C}$ is the corresponding eigenvector matrix. This means that $A_{\mathbf{k}}^{(\mathbf{n})}$ and $A_{\mathbf{j}}^{(\mathbf{n})}$ are linear combinations of exponential functions of the eigenvalues in Eq. (43) times τ. The exponential growth of $A_{\mathbf{k}}^{(\mathbf{n})}$ and $A_{\mathbf{j}}^{(\mathbf{n})}$ is determined by the eigenvalues with positive real part. Looking at Eq. (14) we conclude that the number of created photons, in both the mode $\mathbf{k}$ and the mode $\mathbf{j}$, will increase exponentially in time with a rate given by γk_x.

This is our main result. In the resonance parametric case the resonant mode may be coupled to some other mode. In this case the number of created photons in both modes grows exponentially in time with the same rate, which is exactly one-half of the rate expected for the resonant mode when the coupling is neglected.

We have derived Eq. (40) assuming that only two modes are coupled. If the spectrum contains one mode $\mathbf{s}$ such that $\omega_{\mathbf{s}}=5\omega_{\mathbf{k}}$, besides $\omega_{\mathbf{j}}=3\omega_{\mathbf{k}}$, we get three coupled modes. The resulting equation will be similar to Eq. (40) but with a 6×6 matrix to diagonalize. The number of photons in each mode ($\mathbf{s}$, $\mathbf{j}$, and $\mathbf{k}$) will grow exponentially in time. Due to the nonequidistant character of the spectrum, it is not common to have three modes coupled. For a cubic cavity (i.e., $L_x=L_y=L_z=L$), the first three modes coupled are $\mathbf{k}=(11,16,13)$, $\mathbf{j}=(67,16,13)$, and $\mathbf{s}=(115,16,13)$, the frequency of the lowest mode being an order of magnitude larger than the fundamental frequency of the cavity. This case is therefore of less interest.

Let us now discuss briefly what happens with the remaning cases in which the MSA is nontrivial and differs from the naive perturbation approach. We first study two nonresonant modes $\mathbf{s}$ and $\mathbf{p}$ (i.e., $\omega_{\mathbf{s}}\neq\Omega/2\neq\omega_{\mathbf{p}}$) satisfying the condition (26), $\omega_{\mathbf{s}}+\omega_{\mathbf{p}}=\Omega$.[4] After some algebra on Eqs. (24) and (25) we get for $A_{\mathbf{s}}^{(\mathbf{n})}$ and $B_{\mathbf{p}}^{(\mathbf{n})}$ the following:

$$\frac{dA_{\mathbf{s}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{s}}}\left(\frac{\pi}{L_x}\right)^2(-1)^{s_x+p_x}s_xp_xB_{\mathbf{p}}^{(\mathbf{n})}, \tag{45}$$

$$\frac{dB_{\mathbf{p}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{p}}}\left(\frac{\pi}{L_x}\right)^2(-1)^{s_x+p_x}s_xp_xA_{\mathbf{s}}^{(\mathbf{n})}, \tag{46}$$

and the same equations hold for $B_{\mathbf{s}}^{(\mathbf{n})}$ and $A_{\mathbf{p}}^{(\mathbf{n})}$. The solutions are straightforwardly obtained, giving for the average value of the number operator

$$\langle\mathcal{N}_{\mathbf{s}}\rangle=\langle\mathcal{N}_{\mathbf{p}}\rangle=\sinh^2\left[\left(\frac{\pi}{L_x}\right)^2\frac{s_xp_x}{2\sqrt{\omega_{\mathbf{s}}\omega_{\mathbf{p}}}}\tau_f\right]. \tag{47}$$

Note that if we set $s_x=p_x(\mathbf{s}=\mathbf{p})$ we recover the parametric resonance case. This example shows the possibility of obtaining exponential growth of photons in modes that are not in resonance with the external frequency.

If the spectrum contains some sequence of equidistant frequencies $\omega_{\mathbf{p}_i}$ separated by Ω, the corresponding modes will be coupled through the conditions $\Omega=\omega_{\mathbf{p}_{i+1}}-\omega_{\mathbf{p}_i}$ and $\Omega=\omega_{\mathbf{p}_i}-\omega_{\mathbf{p}_{i-1}}$. One can show that, as long as the modes $\mathbf{p}_i$ are not coupled to modes outside the sequence, the number of created photons in each of these modes will be an oscillatory function of time.

[4]Note that we are not necessarily within the parametric resonance case. The external frequency Ω could be or not be twice the frequency of some other unperturbed mode.

B. Examples

The first and more important example is the cubic cavity. In order to obtain parametric resonance we fix Ω as twice the lowest cavity frequency,

$$\Omega=2\omega_{(1,1,1)}=\frac{2\pi\sqrt{3}}{L}. \tag{48}$$

For this example we will assume that $L=1$ cm. The fundamental mode $\mathbf{k}=(1,1,1)$ will be coupled to $\mathbf{j}=(5,1,1)$ because $\omega_{(5,1,1)}=3\omega_{(1,1,1)}$. Only these two modes are coupled, since there does not exist in the spectrum any mode $\mathbf{s}$ satisfaying $\omega_{\mathbf{s}}=5\omega_{(1,1,1)}$. The exponential growth for the modes $\mathbf{k}$ and $\mathbf{j}$ will be one-half of that expected by previous authors [12]. Now we can write explicitly Eq. (44) for this particular case. The result is

$$\begin{aligned}B_{\mathbf{k}}^{(\mathbf{n})}(\tau)=&\frac{\delta_{\mathbf{n},\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}[\cos(2.56\,\tau)\cosh(0.45\tau)\\&+0.176\sin(2.57\tau)\sinh(0.45\tau)]\\&+\frac{\delta_{\mathbf{n},\mathbf{j}}}{\sqrt{2\omega_{\mathbf{j}}}}[1.76\sin(2.57\tau)\cosh(0.45\tau)],\end{aligned} \tag{49}$$

$$\begin{aligned}A_{\mathbf{k}}^{(\mathbf{n})}(\tau)=&\frac{\delta_{\mathbf{n},\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}[\cos(2.56\tau)\sinh(0.45\tau)\\&+0.176\sin(2.57\tau)\cosh(0.45\tau)]\\&+\frac{\delta_{\mathbf{n},\mathbf{j}}}{\sqrt{2\omega_{\mathbf{j}}}}[1.76\sin(2.57\tau)\sinh(0.45\tau)],\end{aligned} \tag{50}$$

$$\begin{aligned}B_{\mathbf{j}}^{(\mathbf{n})}(\tau)=&\frac{\delta_{\mathbf{n},\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}[-0.586\sin(2.56\tau)\cosh(0.45\tau)]\\&+\frac{\delta_{\mathbf{n},\mathbf{j}}}{\sqrt{2\omega_{\mathbf{j}}}}[\cos(2.56\tau)\cosh(0.45\tau)\\&-0.176\sin(2.56\tau)\sinh(0.45\tau)],\end{aligned} \tag{51}$$

$$\begin{aligned}A_{\mathbf{j}}^{(\mathbf{n})}(\tau)=&\frac{\delta_{\mathbf{n},\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}[-0.586\sin(2.56\tau)\sinh(0.45\tau)]\\&+\frac{\delta_{\mathbf{n},\mathbf{j}}}{\sqrt{2\omega_{\mathbf{j}}}}[\cos(2.56\tau)\sinh(0.45\tau)\\&-0.176\sin(2.56\tau)\cosh(0.45\tau)].\end{aligned} \tag{52}$$

An important remark is that this solution satisfies the unitary condition for the Bogoliubov transformation (11),

$$\sum_{\mathbf{n}} |B_{\mathbf{k}}^{(\mathbf{n})}|^2 - |A_{\mathbf{k}}^{(\mathbf{n})}|^2 = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}. \tag{53}$$

We can compute the number of created photons in each mode inserting by Eqs. (50) and (52) into Eq. (14). The result is

$$\begin{aligned}\langle \mathcal{N}_{\mathbf{k}} \rangle = &\cos^2(2.56\tau_f)\sinh^2(0.45\tau_f)\\ &+1.06\sin^2(2.56\tau_f)\cosh^2(0.45\tau_f)\\ &+0.088\sin(5.12\tau_f)\sinh(0.9\tau_f),\end{aligned} \tag{54}$$

$$\begin{aligned}\langle \mathcal{N}_{\mathbf{j}} \rangle = &\cos^2(2.56\tau_f)\sinh^2(0.45\tau_f)\\ &+1.06\sin^2(2.56\tau_f)\cosh^2(0.45\tau_f)\\ &-0.088\sin(5.12\tau_f)\sinh(0.9\tau_f).\end{aligned} \tag{55}$$

When $\tau_f \gg 1$ these expressions are approximated by

$$\langle \mathcal{N}_{\mathbf{k}} \rangle \approx \langle \mathcal{N}_{\mathbf{j}} \rangle \approx e^{0.9\tau_f}. \tag{56}$$

In a previous paper [9] the authors considered two-dimensional cavities, which means that one of the cavity's dimensions is much smaller than the others (say $L_z \ll L_x, L_y$). We can easily recover this limit by omitting the z dimension. In what follows we will discuss this case, for increasing external frequencies.

Let us first assume that $L_x = L_y$. If $\Omega = 2\omega_{(1,1)}$, then the fundamental mode $\mathbf{k} = (1,1)$ does not couple to any other mode and it grows exponentially in time. The next resonant frequencies are $\Omega = 2\omega_{(1,2)}$ and $\Omega = 2\omega_{(2,2)}$. In both cases the resonant mode is not coupled. If $\Omega = 2\omega_{(1,3)}$, the mode (1,3) will be coupled to the mode (9,3) and both will grow exponentially in time. However, the mode (3,1) also satisfies the parametric resonance condition and, being uncoupled to other modes, it will grow faster than the previous ones. For the same frequency $\Omega = 2\omega_{(1,3)}$, we have found by inspection three equidistant modes, $\omega_{(13,39)} = 13\pi\sqrt{10}/L$ $\omega_{(27,39)} = 15\pi\sqrt{10}/L$, and $\omega_{(37,39)} = 17\pi\sqrt{10}/L$. The number of created photons in each mode will oscillate in time.

Now we choose $L_x = 3L_y$. This choice causes the fundamental mode to be coupled in parametric resonance. If we set $\Omega = 2\omega_{(1,1)}$, the mode (9,1) satisfies $\omega_{(9,1)} = 3\omega_{(1,1)}$, so both modes will grow exponentially.

C. Off resonance

In this subsection we study what happens when the external frequency $\tilde{\Omega}$ is slightly off resonance, i.e., $\tilde{\Omega} = \Omega + h$, where Ω is a resonant frequency and $h \ll \Omega$. We assume that $h = \epsilon\alpha$, where $\alpha = O(\Omega)$. We show how to apply MSA to this case. Off-resonance motions have already been considered in the literature [18] using a different approach, and it was shown that there are threshold conditions on h for exponential photon creation.

For small amplitudes of oscillations, the equations for the modes are still Eq. (16) with Ω replaced by the external frequency $\tilde{\Omega}$. Since $h \ll \Omega$, all factors of the form $e^{\pm iht}$ may be regarded as slow oscillations, so that the MSA conditions to get rid of secularities are exactly the same as in the resonant case, Eqs. (26), (27) and (28). However, the equations for the modes (24), (25) do get modified. Their off-resonant version reads

$$\begin{aligned}\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau} = &-\frac{\pi^2 k_x^2}{2\omega_{\mathbf{k}} L_x^2} e^{i\alpha\tau} B_{\mathbf{k}}^{(\mathbf{n})} \delta(2\omega_{\mathbf{k}} - \Omega) + \sum_{\mathbf{j}} \left(-\omega_{\mathbf{j}} + \frac{\Omega+h}{2}\right) \delta(-\omega_{\mathbf{k}} - \omega_{\mathbf{j}} + \Omega) \frac{\Omega+h}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} e^{i\alpha\tau} B_{\mathbf{j}}^{(\mathbf{n})}\\ &+\sum_{\mathbf{j}} \left[e^{i\alpha\tau}\left(\omega_{\mathbf{j}} + \frac{\Omega+h}{2}\right) \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{j}} - \Omega) + e^{-i\alpha\tau}\left(\omega_{\mathbf{j}} - \frac{\Omega+h}{2}\right) \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{j}} + \Omega)\right] \frac{\Omega+h}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} A_{\mathbf{j}}^{(\mathbf{n})},\end{aligned} \tag{57}$$

and

$$\begin{aligned}\frac{dB_{\mathbf{k}}^{(\mathbf{n})}}{d\tau} = &-\frac{\pi^2 k_x^2}{2\omega_{\mathbf{k}} L_x^2} e^{-i\alpha\tau} A_{\mathbf{k}}^{(\mathbf{n})} \delta(2\omega_{\mathbf{k}} - \Omega) + \sum_{\mathbf{j}} \left(-\omega_{\mathbf{j}} + \frac{\Omega+h}{2}\right) \delta(-\omega_{\mathbf{k}} - \omega_{\mathbf{j}} + \Omega) \frac{\Omega+h}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} e^{-i\alpha\tau} A_{\mathbf{j}}^{(\mathbf{n})}\\ &+\sum_{\mathbf{j}} \left[e^{-i\alpha\tau}\left(\omega_{\mathbf{j}} + \frac{\Omega+h}{2}\right) \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{j}} - \Omega) + e^{i\alpha\tau}\left(\omega_{\mathbf{j}} - \frac{\Omega+h}{2}\right) \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{j}} + \Omega)\right] \frac{\Omega+h}{2\omega_{\mathbf{k}}} g_{\mathbf{kj}} B_{\mathbf{j}}^{(\mathbf{n})}.\end{aligned} \tag{58}$$

Let us solve these equations in the quasi-parametric-resonant case, that is, for $\tilde{\Omega} - h = \Omega = 2\omega_{\mathbf{k}}$. In the case when there are no coupled modes, we get two coupled first order differential equations for $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$. After the change of variables

$$\begin{aligned}A_{\mathbf{k}}^{(\mathbf{n})} &= e^{i\alpha\tau/2} a_{\mathbf{k}}^{(\mathbf{n})},\\ B_{\mathbf{k}}^{(\mathbf{n})} &= e^{-i\alpha\tau/2} b_{\mathbf{k}}^{(\mathbf{n})},\end{aligned} \tag{59}$$

the equations take the form

$$\frac{da_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-i\frac{\alpha}{2}a_{\mathbf{k}}^{(\mathbf{n})}-\gamma k_x b_{\mathbf{k}}^{(\mathbf{n})}, \tag{60}$$

$$\frac{db_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=i\frac{\alpha}{2}b_{\mathbf{k}}^{(\mathbf{n})}-\gamma k_x a_{\mathbf{k}}^{(\mathbf{n})}. \tag{61}$$

There will be growing exponential solutions if an eigenvalue of the corresponding matrix

$$\begin{pmatrix} -\frac{i\alpha}{2} & -\gamma k_x \\ -\gamma k_x & \frac{i\alpha}{2} \end{pmatrix} \tag{62}$$

has a positive real part. The eigenvalues are given by

$$\lambda=\pm\sqrt{\gamma^2 k_x^2-\frac{\alpha^2}{4}} \tag{63}$$

and lead to the following threshold for resonant behavior:

$$|h|<2\epsilon\gamma k_x \Leftrightarrow \frac{|h|}{\Omega}<\frac{\epsilon}{2}\,\frac{(k_x/L_x)^2}{(k_x/L_x)^2+(k_y/L_y)^2+(k_z/L_z)^2}. \tag{64}$$

For this uncoupled, quasiresonant situation, the resonant mode satisfies a Mathieu equation [see Eq. (16)]. The threshold we obtained in Eq. (64) coincides with the one obtained for the Mathieu equation using a different method [19].

As a second example, let us consider the case of two coupled modes, say modes $\mathbf{k}$ and $\mathbf{j}$, for which $\omega_{\mathbf{j}}=3\omega_{\mathbf{k}}$. After the change of variables (59), and defining

$$A_{\mathbf{j}}^{(\mathbf{n})}=e^{3i\alpha\tau/2}a_{\mathbf{j}}^{(\mathbf{n})},$$

$$B_{\mathbf{j}}^{(\mathbf{n})}=e^{-3i\alpha\tau/2}b_{\mathbf{j}}^{(\mathbf{n})}, \tag{65}$$

the off-resonance form of Eqs. (35), (36), (38), and (39) is

$$\frac{da_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-i\frac{\alpha}{2}a_{\mathbf{k}}^{(\mathbf{n})}+\gamma[-k_x b_{\mathbf{k}}^{(\mathbf{n})}+(-1)^{j_x+k_x}j_x a_{\mathbf{j}}^{(\mathbf{n})}], \tag{66}$$

$$\frac{db_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=i\frac{\alpha}{2}b_{\mathbf{k}}^{(\mathbf{n})}+\gamma[-k_x a_{\mathbf{k}}^{(\mathbf{n})}+(-1)^{j_x+k_x}j_x b_{\mathbf{j}}^{(\mathbf{n})}], \tag{67}$$

$$\frac{da_{\mathbf{j}}^{(\mathbf{n})}}{d\tau}=-3i\frac{\alpha}{2}a_{\mathbf{j}}^{(\mathbf{n})}-(-1)^{j_x+k_x}\frac{\gamma j_x}{3}a_{\mathbf{k}}^{(\mathbf{n})}, \tag{68}$$

$$\frac{db_{\mathbf{j}}^{(\mathbf{n})}}{d\tau}=3i\frac{\alpha}{2}b_{\mathbf{j}}^{(\mathbf{n})}-(-1)^{j_x+k_x}\frac{\gamma j_x}{3}b_{\mathbf{k}}^{(\mathbf{n})}, \tag{69}$$

where we have neglected terms proportional to $h/\omega_{\mathbf{k}}=O(\epsilon)$ in the right-hand side since they would introduce only small corrections to the eigenvalues and thresholds. Just as in Sec. IV A, we can rewrite these equations in matrix form, and find the solution by diagonalizing such a matrix. The corresponding eigenvalues are

$$\lambda=\pm\frac{1}{\sqrt{2}}\sqrt{-U\pm\sqrt{V}}, \tag{70}$$

where $U=5\alpha^2/2+\gamma^2/3[2j_x^2-3k_x^2]$ is a positive number [see Eq. (37)], and

$$V=4\alpha^4+4\alpha^2\gamma^2\left(k_x^2+\frac{4}{3}j_x^2\right)+\gamma^4 k_x^2\left(k_x^2-\frac{4}{3}j_x^2\right). \tag{71}$$

For $\alpha=0$, V is negative, so there are two eigenvalues with positive real part, which corresponds to exponential growth. As $|\alpha|$ grows, V will eventually become positive, all eigenvalues are purely imaginary for $\sqrt{V}<U$, and no exponential growth is obtained. The condition for exponential growth is $V<0$, which again sets a threshold for the off-resonance frequency difference h,

$$|h|<\epsilon\gamma\sqrt{-d_1+\sqrt{d_1^2+d_2}}, \tag{72}$$

with

$$d_1=\left(k_x^2+\frac{4}{3}j_x^2\right)>0, \tag{73}$$

$$d_2=k_x^2\left(\frac{4}{3}j_x^2-k_x^2\right)>0. \tag{74}$$

For the example of the cubic cavity discussed in the Sec. IV B the threshold is $|h|<0.68\epsilon\gamma$, that is, $|h|/\Omega<0.06\epsilon$.

V. DYNAMICAL CASIMIR EFFECT AT FINITE TEMPERATURE

Up to this point we have assumed that the field was in the in-vacuum state $T=0$ [see, for example, Eq. (14)]. It is well known that the temperature contribution can dominate the pure vacuum effect when computing the *static* Casimir force [20]. Thus, it is expected that temperature plays an important role in the dynamical Casimir effect as well. In what follows we shall derive an expression for the number of created photons inside the cavity equivalent to Eq. (14), but now assuming an initial state in equilibrium at finite temperature. After that, we apply the result obtained to the resonant vibrating cavity.

We adopt the scheme of scalar field quantization developed in Sec. II. For $t<0$ the wall is at rest, so we assume the system to be at thermal equilibrium at finite temperature $T=1/\beta$. The state of the system is described by a statistical operator ρ which does not evolve in time. We expand this operator in Fock states of the field at $t<0$,

$$\rho=\frac{1}{Z}\sum_{n_{\mathbf{k}_1}\geq 0}\sum_{n_{\mathbf{k}_2}\geq 0}\cdots\exp-\left(\beta\sum_i\left(n_{\mathbf{k}_i}+\frac{1}{2}\right)E_{\mathbf{k}_i}\right)\times|n_{\mathbf{k}_1}n_{\mathbf{k}_2}\cdots\rangle\langle n_{\mathbf{k}_1}n_{\mathbf{k}_2}\cdots|, \tag{75}$$

where $E_{\mathbf{k}_i}=\omega_{\mathbf{k}_i}$ and the normalization factor is given by

$$Z=\sum_{n_{\mathbf{k}_1}\geq 0}\sum_{n_{\mathbf{k}_2}\geq 0}\cdots\exp-\left(\beta\sum_i\left(n_{\mathbf{k}_i}+\frac{1}{2}\right)E_{\mathbf{k}_i}\right). \quad (76)$$

At $t=0$ the wall begins to oscillate and the system will no longer remain at thermal equilibrium. Following the steps given in Sec. II we assume that the wall stops at $t=t_{\mathrm{final}}$. In that case, the number of photons in the $\mathbf{k}$ mode is the average value of $\hat{a}_{\mathbf{k}}^{\dagger\,\mathrm{out}}\hat{a}_{\mathbf{k}}^{\mathrm{out}}$ with respect to the initial state ρ,

$$\langle\mathcal{N}_{\mathbf{k}}\rangle_\rho=\mathrm{Tr}(\rho\hat{a}_{\mathbf{k}}^{\dagger\,\mathrm{out}}\hat{a}_{\mathbf{k}}^{\mathrm{out}}). \quad (77)$$

Using Eq. (11) we can write this expression in terms of $\hat{a}^{\mathrm{in}}$ and $\hat{a}^{\dagger\,\mathrm{in}}$

$$\begin{aligned}\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{k}}^{\dagger\,\mathrm{out}}\hat{a}_{\mathbf{k}}^{\mathrm{out}})=\sum_{\mathbf{n}\,\mathbf{j}}\;&\beta_{\mathbf{n}\,\mathbf{k}}\alpha_{\mathbf{j}\,\mathbf{k}}\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\mathrm{in}}\hat{a}_{\mathbf{j}}^{\mathrm{in}})\\&+\beta_{\mathbf{n}\,\mathbf{k}}\beta^{\star}_{\mathbf{j}\,\mathbf{k}}\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\mathrm{in}}\hat{a}_{\mathbf{j}}^{\dagger\,\mathrm{in}})\\&+\alpha^{\star}_{\mathbf{n}\,\mathbf{k}}\alpha^{\star}_{\mathbf{j}\,\mathbf{k}}\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\dagger\,\mathrm{in}}\hat{a}_{\mathbf{j}}^{\mathrm{in}})\\&+\alpha^{\star}_{\mathbf{n}\,\mathbf{k}}\beta^{\star}_{\mathbf{j}\,\mathbf{k}}\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\dagger\,\mathrm{in}}\hat{a}_{\mathbf{j}}^{\dagger\,\mathrm{in}}). \quad (78)\end{aligned}$$

With the help of Eqs. (75) and (76) it is a straightforward calculation to find that

$$\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\mathrm{in}}\hat{a}_{\mathbf{j}}^{\mathrm{in}})=0=\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\dagger\,\mathrm{in}}\hat{a}_{\mathbf{j}}^{\dagger\,\mathrm{in}}), \quad (79)$$

$$\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\dagger\,\mathrm{in}}\hat{a}_{\mathbf{j}}^{\mathrm{in}})=\frac{1}{e^{\beta E_{\mathbf{n}}}-1}\,\delta_{\mathbf{n}\,\mathbf{j}}=\mathrm{Tr}(\rho\,\hat{a}_{\mathbf{n}}^{\mathrm{in}}\hat{a}_{\mathbf{j}}^{\dagger\,\mathrm{in}})-1. \quad (80)$$

Inserting this into Eqs. (78) and (77) we arrive at

$$\langle\mathcal{N}_{\mathbf{k}}\rangle_\rho=\sum_{\mathbf{n}}|\beta_{\mathbf{n}\,\mathbf{k}}|^2+\sum_{\mathbf{n}}(|\beta_{\mathbf{n}\,\mathbf{k}}|^2+|\alpha_{\mathbf{n}\,\mathbf{k}}|^2)\frac{1}{e^{\beta E_{\mathbf{n}}}-1}. \quad (81)$$

Finally, using Eqs. (13) and (14) we get

$$\langle\mathcal{N}_{\mathbf{k}}\rangle_\rho=\langle\mathcal{N}_{\mathbf{k}}\rangle_{T=0}+\sum_{\mathbf{n}}(|B_{\mathbf{k}}^{(\mathbf{n})}|^2+|A_{\mathbf{k}}^{(\mathbf{n})}|^2)\frac{2\omega_{\mathbf{k}}}{e^{\beta E_{\mathbf{n}}}-1}. \quad (82)$$

Let us now apply the results obtained in Sec. IV for the vibrating cavity to the case in which the field is initially at thermal equilibrium. In the parametric resonance case without coupling the Bogoliubov coefficients are diagonal [see Eqs. (32) and (33)], so Eq. (82) can be reduced to

$$\langle\mathcal{N}_{\mathbf{k}}\rangle_\rho=\langle\mathcal{N}_{\mathbf{k}}\rangle_{T=0}\left(1+2\frac{1}{e^{\beta E_{\mathbf{k}}}-1}\right)+\frac{1}{e^{\beta E_{\mathbf{k}}}-1}, \quad (83)$$

with $\langle\mathcal{N}_{\mathbf{k}}\rangle_{T=0}$ given by Eq. (34). We see that the effect of the temperature is to enhance the amount of created photons in the pure vacuum case by a thermal distribution factor. Note that the second term in expression (83) corresponds to the average number of photons, in the mode $\mathbf{k}$, present in the cavity when $t<0$. The same result was obtained in Ref. [12] for the fundamental mode of a cubic cavity using a different approach. However, we have seen that in this case the Bogoliubov coefficients are not diagonal, due to the coupling between the fundamental mode $\mathbf{k}=(1,1,1)$ and the mode $\mathbf{j}=(5,1,1)$. With the help of Eqs. (49), (50), (51), and (52) it is easy to obtain the number of photons present in the fundamental mode after the wall stops,

$$\begin{aligned}\langle\mathcal{N}_{\mathbf{k}}\rangle_\rho=\langle\mathcal{N}_{\mathbf{k}}\rangle_{T=0}&(1+2n_{\mathbf{k}})-2\sin^2(2.56\tau)\cosh^2(0.45\tau)\\&\times(n_{\mathbf{k}}-n_{\mathbf{j}})-\sin^2(2{,}56\tau)(n_{\mathbf{k}}+n_{\mathbf{j}})+n_{\mathbf{k}}, \quad (84)\end{aligned}$$

where $n_{\mathbf{k}}=1/(e^{\beta E_{\mathbf{k}}}-1)$ is the Bose mean occupation number for $t<0$. For the mode $\mathbf{j}$ we find

$$\begin{aligned}\langle\mathcal{N}_{\mathbf{j}}\rangle_\rho=\langle\mathcal{N}_{\mathbf{j}}\rangle_{T=0}&(1+2n_{\mathbf{j}})-2\sin^2(2.56\tau)\cosh^2(0.45\tau)\\&\times(n_{\mathbf{j}}-n_{\mathbf{k}})-\sin^2(2.56\tau)(n_{\mathbf{j}}+n_{\mathbf{k}})+n_{\mathbf{j}}. \quad (85)\end{aligned}$$

Again, we obtain the result that the effect of the temperature is to increase the number of photons in the pure vacuum case with thermal factors now depending on t. For $\tau_f\gg 1$ we have $\langle\mathcal{N}_{\mathbf{k}}\rangle_{T=0}\approx\langle\mathcal{N}_{\mathbf{j}}\rangle_{T=0}$ [see Eq. (56)]. Therefore the total number of photons created inside the cavity becomes

$$\langle\mathcal{N}_{\mathrm{total}}\rangle_\rho\cong(1+n_{\mathbf{k}}+n_{\mathbf{j}})\langle\mathcal{N}_{\mathrm{total}}\rangle_{T=0}. \quad (86)$$

For a cubic cavity of size $L=1$ cm at room temperature $T\approx 290$ K, we obtain the result that the total number in created photons is approximately 300 times that of pure vacuum ($T=0$).

VI. CONCLUSIONS

We have calculated the photon production inside a three-dimensional oscillating cavity, using MSA to deal with the resonant effects. We have taken into account that, even though the spectrum of the cavity is nonequidistant, the different modes may be coupled, and this coupling affects the rate of photon creation.

We have found resonant effects when the external frequency is equal to the sum of the frequencies of two unperturbed modes $\Omega=\omega_{\mathbf{s}}+\omega_{\mathbf{p}}$. When $\omega_{\mathbf{s}}\neq\omega_{\mathbf{p}}$, the number of photons in both modes grows exponentially. When $\Omega=2\omega_{\mathbf{k}}$, the usual "parametric resonance case," the number of photons in the mode $\mathbf{k}$ also grows exponentially, along with the number of photons of other modes coupled to $\mathbf{k}$. When the mode $\mathbf{k}$ is coupled to one mode, the rate of photon creation decreases by a factor of 2 with respect to the uncoupled case.

We have also analyzed slightly off-resonance situations. Using an extension of the MSA we showed that the number of photons in the relevant modes also increases exponentially if certain threshold conditions are satisfied. These conditions imply that the external frequency should be almost equal to the resonant frequency.

As an important example, we have described in detail the case of a cubic cavity. The fundamental mode (1,1,1) is coupled to the mode (5,1,1) when the external frequency is $\Omega=2\omega_{(1,1,1)}$. The number of photons created in both modes grows as $e^{0.9\tau}$. Neglecting the mode coupling, one would erroneously conclude that the number of photons in the mode

(1,1,1) grows as $e^{1.8\tau}$, and that there no exponential growth for the mode (5,1,1). The mode coupling in three-dimensional cavities has not been taken into account in the previous literature.

We have also computed the enhancement of the dynamical Casimir effect for an initial thermal state. The main result is contained in Eq. (82). Only when the Bogoliubov coefficients are approximately diagonal does one recovers the usual result, i.e., the number of photons created in a given mode at temperature T is the $T=0$ result times the thermal distribution factor.

For simplicity, we have considered a quantum scalar field. The generalization to the case of an electromagnetic field is not completely straightforward. We state here the main results; the details will be described in a future work. Assuming that the potential vector satisfies the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, the Maxwell equations read $\Box\mathbf{A}=0$. The boundary conditions are the usual (perfect conductors) on the static walls. On the moving mirror, these boundary conditions must be imposed in a Lorentz frame in which the mirror is instantaneously at rest (see [10]). This implies that, at $x=L_x(t)$, the potential vector must satisfy $A_y(t,L_x(t),y,z)=A_z(t,L_x(t),y,z)=0$.

As for the case of the scalar field, one can expand the potential vector in an instantaneous basis, and use MSA to deal with the secular terms. The resonant conditions coincide with Eqs. (26)–(28) in the present paper. The differential equations for the (now polarization-dependent) Bogoliubov coefficients are different for the transverse electric (TE) and transverse magnetic (TM) cases. However, it can be shown that, in the parametric resonant case, the rate of photon production in both polarizations TE and TM is the same and coincides with the rate for the scalar field computed in this paper.

Finally, we would like to comment briefly about the possibility of observing the dynamical Casimir effect experimentally. Considering cavities of dimensions of the order of 1 cm, the external frequency should be at least 1 GHz in order to have resonant photon creation. This is not trivial, the upper limit being around 100 MHz with the present techniques [21]. Another serious technical problem is that, as already mentioned, in order to have resonance the external frequency should be tuned with high accuracy to the resonant frequency. Although extremely difficult, an experimental verification of the photon production seems not completely unrealistic.

ACKNOWLEDGMENTS

We are grateful to Paulo A. Maia Neto for reading the manuscript. D.D. thanks Luis M. Bettencourt for discussions. The work of M.C. and F.D.M. was supported by Universidad de Buenos Aires, Conicet, and Agencia Nacional de Promoción Científica y Tecnológica, Argentina.

[1] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948); V. M. Mostepanenko and N. N. Turnov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997); M. Bordag, *The Casimir Effect 50 Years Later* (World Scientific, Singapore, 1999); P. Milonni, *The Quantum Vacuum* (Academic Press, San Diego, 1994).
[2] S.K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).
[3] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998).
[4] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **90**, 958 (1993); **90**, 2105 (1993); **90**, 4505 (1993); **90**, 7285 (1993).
[5] A. Lambrecht, M.T. Jaekel, and S. Reynaud, Phys. Rev. Lett. **77**, 615 (1996).
[6] J.Y. Ji, H.H. Jung, J.W. Park, and K.S. Soh, Phys. Rev. A **56**, 4440 (1997); V.V. Dodonov *et al.*, Phys. Lett. A **149**, 225 (1990).
[7] D.A.R. Dalvit and F.D. Mazzitelli, Phys. Rev. A **57**, 2113 (1998); **59**, 3049 (1999).
[8] C.K. Cole and W.C. Schieve, Phys. Rev. A **52**, 4405 (1995); V.V. Dodonov, A.B. Klimov, and D.E. Nikonov, J. Math. Phys. **34**, 2742 (1993).
[9] V.V. Dodonov and A.B. Klimov, Phys. Rev. A **53**, 2664 (1996).
[10] D.F. Mundarain and P.A. Maia, Phys. Rev. A **57**, 1379 (1998).
[11] J.-Y. Li *et al.*, J. Phys. A **31**, 457 (1998).
[12] G. Plunien, R. Schützhold, and G. Soff, Phys. Rev. Lett. **84**, 1882 (2000).
[13] A. Lambrecht, M.T. Jaekel, and S. Reynaud, Europhys. Lett. **43**, 147 (1998).
[14] C.K. Law, Phys. Rev. A **51**, 2537 (1995).
[15] C. M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[16] L.Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[17] V.V. Dodonov, Phys. Lett. A **213**, 219 (1996).
[18] V.V. Dodonov, Phys. Rev. A **58**, 4147 (1998); Phys. Lett. A **244**, 517 (1998).
[19] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon Press, Oxford, 1987).
[20] G. Plunien, B. Muller, and W. Greiner, Phys. Rep. **134**, 87 (1986).
[21] R. Onofrio (private communication).