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Finite-temperature hydrodynamic modes of trapped quantum gases

András Csordás
Research Group for Statistical Physics of the Hungarian Academy of Sciences, Pázmány Péter Sétány 1/A, 1117 Budapest, Hungary

Robert Graham
Fachbereich Physik, Universität-GH, 45117 Essen, Germany

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The hydrodynamic equations of an ideal fluid formed by a dilute quantum gas in a parabolic trapping potential are studied analytically and numerically. Due to the appearance of internal modes in the fluid stratified by the trapping potential, the spectrum of low-lying modes is found to be dense in the high-temperature limit, with an infinitely degenerate set of zero-frequency modes. The spectrum for Bose fluids and Fermi fluids is obtained and discussed.

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I. INTRODUCTION

The successful trapping of dilute Bose and Fermi gases in magnetic traps, and their subsequent cooling to temperatures below quantum degeneracy [1–3], has made the study of their hydrodynamics a subject of high current interest. The basic hydrodynamic equations for a fluid formed by such gases in local thermodynamic equilibrium are well known. In the limit where the fluid can be considered ideal, they are the continuity equation for the mass-density ρ and velocity field $\vec{u}$,

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\rho\vec{u}=0, \tag{1.1}$$

the Euler equation for the velocity field,

$$\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}=-\frac{1}{\rho}\vec{\nabla}P+\vec{f}(\vec{x}), \tag{1.2}$$

with the external force per unit mass $\vec{f}=-(1/m)\vec{\nabla}V(\vec{x})$ and the pressure P related to the internal energy density ε by $P=\frac{2}{3}\varepsilon$. This satisfies

$$\frac{\partial P}{\partial t}+\vec{u}\cdot\vec{\nabla}P=-\frac{5}{3}[\vec{\nabla}(\vec{u}P)-\rho\vec{u}\cdot\vec{f}]. \tag{1.3}$$

The thermodynamic equilibrium distributions of the density $\rho_0(\vec{x})$ and pressure $P_0(\vec{x})$ with $\vec{\nabla}P_0=\rho_0\vec{f}$ are given by the ideal quantum-gas expressions at constant temperature $\beta=1/k_BT$ and chemical potential μ,

$$\rho_0(\vec{x})=m\int\frac{d^3k}{(2\pi)^3}f_{\mp}(\vec{k},\vec{x}),$$

$$P_0(\vec{x})=\frac{2}{3}\int\frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m}f_{\mp}(\vec{k},\vec{x}), \tag{1.4}$$

with the single-particle distribution

$$f_{\mp}(\vec{k},\vec{x})=\frac{1}{e^{\beta[(\hbar^2k^2/2m)+V(\vec{x})-\mu]}\mp 1}. \tag{1.5}$$

of the Bose-Einstein (upper sign) or Fermi-Dirac (lower sign) form.

The derivation of these equations from the Boltzmann equation is well known, see Ref. [4]. In recent years a number of papers have been devoted to the study of solutions of these equations linearized around the equilibrium state in parabolic traps. Griffin, Wu and Stringari [5] derived a closed equation for the velocity fluctuations, and gave explicit solutions for surface waves of a Bose gas in an isotropic trap and those of a classical gas, also in the axially symmetric anisotropic trap. In the latter case they also gave solutions for modes corresponding to irrotational flow. A further study of the hydrodynamic regime of a trapped Bose gases was presented in Ref. [6]. Fermi gases were considered by Bruun and Clark [7]. Besides considering the low-temperature limit for a degenerate Fermi gas, these authors gave an analytical solution for the mode spectrum in an isotropic trap in the high-temperature limit, and identified one branch of the dispersion relation as ‘‘internal waves’’ driven by the inhomogeneous trap potential. This is a point which we intend to examine further in the present paper. Amoruso *et al.* [8] also derived special solutions to the linearized hydrodynamic equations for the low-temperature limit of the degenerate Fermi gas. In a recent paper [9], we also studied this low-temperature regime for Fermi gases, and gave solutions for the completely anisotropic parabolic trap. The present paper will therefore concentrate on temperatures of the order of the degeneracy temperature or above it. In a number of papers effects beyond the scope of Eqs. (1.1)–(1.5) were also considered. Vichi and Stringari [10] considered the effects of mean field due to interactions on the collective oscillations of Fermi gases in a trap, while Pethick and co-workers [11–13], Vichi [14], and Guéry-Odelin *et al.* [15] discussed the collisional damping of collective modes in Bose gases and Fermi gases respectively. In Refs. [11,12] a simple interpolation formula was proposed between the mode frequencies ω_c in the collisionless regime and the hydrodynamic regime, ω_h, of the form

$$\omega^2=\omega_c^2+\frac{\omega_h^2-\omega_c^2}{1-i\omega\tau}, \tag{1.6}$$

where $\tau^{-1}=(8\pi a^2/m)\langle\rho_0 v\rangle$ is the mean collision rate. This description was further examined in Ref. [15]. Equation (1.6) is based on general considerations of nonequilibrium thermodynamics [16]. Damping of the hydrodynamics in Bose gases was also studied by Griffin and co-workers in a series of papers; see Ref. [17], where further references can be found.

In the present paper we will not be concerned with damping effects. Instead our goal is to study further the collision-dominated dissipationless hydrodynamic regime in harmonic traps with arbitrary anisotropy. We do this by giving a systematic treatment of the linearized hydrodynamic equations based on Eqs. (1.1)–(1.5), applicable (within our basic assumptions) in the whole temperature range from the high temperature domain, where the Boltzmann limit

$$f_{\mp} \simeq \exp\left[-\beta\left(\frac{\hbar^2 k^2}{2m} + V(\vec{x}) - \mu\right)\right]$$

applies, to the regime close to the degeneracy temperature for bosons and down to nearly vanishing temperature for fermions. We shall discuss a class of exact solutions of the dissipationless equations applicable to the whole temperature domain covered by the theory, generalizing results previously obtained for traps with axial symmetry. It can be shown that in the high-temperature limit of a classical Boltzmann gas the linearized hydrodynamic equations in a completely anisotropic trap are integrable and separable in elliptic coordinates, just like their low-temperature counterparts [18,9]. However, at lower temperatures where effects of quantum statistics become important, the integrability and separability are lost, which manifests itself, e.g., in effects of avoided level crossings.

Of special interest in the present paper, besides the common sound modes, will be the phenomenon of internal waves, which are characteristic of fluids whose equilibrium state is stratified by an external potential. Internal waves in trapped Fermi gases were mentioned in Ref. [7], but have not yet been investigated in detail for trapped quantum gases. For the discussion of internal waves in classical contexts like waves in the atmosphere see Ref. [19].

II. LINEARIZED HYDRODYNAMIC EQUATIONS AND HILBERT SPACE OF THEIR SOLUTIONS

In the present section we give the five linearized hydrodynamic equations whose solution is the central theme of this paper. In previous work on these equations they were reduced to a set of three wave equations for the velocity field, which we shall also give for completeness, and some special solutions of this latter set of equations were given. However, appropriate boundary conditions are hard to formulate for the velocity field, and therefore it is not clear, so far, what function space is spanned by the solutions, and whether a scalar product can be placed on this function space, and if so which it is. This question is of particular practical and theoretical relevance for the present problem, because in general the solutions have to be constructed numerically by converting the differential operators to matrices using a basis and the scalar product in the solution space. It is important to choose the correct scalar product because, as we shall see, the problem possesses a dense-lying discrete spectrum of low-lying states; it is *a priori* far from clear whether all these states are needed to span the complete space of solutions, and if not, how the correct states are to be distinguished. It is our aim here to devote particular attention to this open problem, and to present an answer. The way to achieve this will be to deviate from the previous line of approach by deriving, instead of three coupled wave equations for the components of the velocity field, two coupled wave equations for the pressure and the density. For these we shall construct a scalar product in which the wave operator is self-adjoint, so that its eigenfunctions form a complete set in a well-defined Hilbert space.

A. Linearized hydrodynamic equations

Let us introduce small deviations $\delta\rho$ and δP of density and pressure from equilibrium,

$$\rho = \rho_0(\vec{x}) + \delta\rho(\vec{x},t), P = P_0(\vec{x}) + \delta P(\vec{x},t), \tag{2.1}$$

where $\rho_0(\vec{x})$ and $P_0(\vec{x})$ solve the time-independent hydrodynamic equations with vanishing velocity field $\vec{u}_0 = 0$, namely, $\vec{\nabla} P_0(\vec{x}) = \rho_0(\vec{x})\vec{f}(\vec{x})$, which defines the mechanical equilibrium condition. In principle there are many equilibrium profiles $\rho_0(\vec{x}), P_0(\vec{x})$ satisfying this requirement. In our present context the physically relevant one is the thermodynamic equilibrium of maximum local entropy. The entropy maximum is achieved by the special profiles $\rho_0(\vec{x})$ and $P_0(\vec{x})$ corresponding to a state with a uniform temperature T and a chemical potential μ. Using Eqs. (1.4) $\rho_0(\vec{x})$ and $P_0(\vec{x})$ can be written as

$$\rho_0(\vec{x}) = A_{\mp} m \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} F_{\mp}\left(\frac{3}{2}, \frac{V(\vec{x}) - \mu}{k_B T}\right),$$

$$P_0(\vec{x}) = A_{\mp} k_B T \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} F_{\mp}\left(\frac{5}{2}, \frac{V(\vec{x}) - \mu}{k_B T}\right), \tag{2.2}$$

where the upper (lower) sign refers to bosons (fermions) and $A_- = 1, A_+ = 2$. Equations (2.2), but with space-dependent $T(\vec{x})$ and $\mu(\vec{x})$, also apply to states of local thermodynamic equilibrium, in which the system is always in the hydrodynamic limit. They can then be taken to define two of the four fields $P(\vec{x})$, $\rho(\vec{x})$, $T(\vec{x})$, and $\mu(\vec{x})$ in terms of the other two. In the fermionic case we need to assume the presence of two equally populated hyperfine sub-states, the collisions between which can then ensure the local thermodynamic equilibrium. The Bose-Einstein integrals $F_-(s,\zeta)$ and Fermi-Dirac integrals $F_+(s,\zeta)$ are defined by

$$F_{\mp}(s,\zeta) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^{x+\zeta} \mp 1} dx, \tag{2.3}$$

satisfying the familiar recursion relation $\partial F_{\mp}(s,\zeta)/\partial\zeta = -F_{\mp}(s-1,\zeta)$. In the present case ζ, and therefore also $F_{\mp}(s,\zeta)$, is space dependent via $\zeta=\zeta(\vec{x})=[V(\vec{x})-\mu]/k_BT$. However, we shall usually suppress the ζ and $\vec{x}$ dependences in our notation for simplicity, and just write $F_{\mp}(s)$.

Let us see under what conditions this thermodynamic equilibrium state is stable against mechanical perturbations. Displacing a volume element of fluid mechanically in the direction of increasing pressure, i.e., in the direction of $\vec{f}$, its volume is compressed adiabatically, so that its density is increased, per unit displacement, by $(\partial\rho_0/\partial P_0)_S\vec{\nabla}P_0$, whereas the density in the ambient equilibrium-gas changes by $\vec{\nabla}\rho_0$ in the same displacement. A restoring force per unit volume in the direction opposite to the displacement,

$$\vec{f}\cdot[(\partial\rho_0/\partial P_0)_S\vec{\nabla}P_0-\vec{\nabla}\rho_0]<0, \tag{2.4}$$

must result for a mechanically stable state.[1] Using the relation $(\partial\rho_0/\partial P_0)_S=3\rho_0/5P_0$, valid for ideal quantum gases, and Eqs. (2.2) and (2.3) we may rewrite Eq. (2.4) as

$$\vec{f}^2\frac{m^2}{(k_BT)^2}\left[\frac{3}{5}\frac{F_{\mp}^2\left(\frac{3}{2}\right)}{F_{\mp}\left(\frac{5}{2}\right)}-F_{\mp}\left(\frac{1}{2}\right)\right]<0. \tag{2.5}$$

This condition is satisfied for bosons for $\zeta=(V(\vec{x})-\mu)/k_BT>0$, and for fermions for all positive and negative values of ζ.

The hydrodynamic equations linearized in $\vec{u}$, $\delta\rho$, and δP are then given by

$$\rho_0(\vec{x})\partial_t\vec{u}=-\vec{\nabla}\delta P+\delta\rho\vec{f}(\vec{x}), \tag{2.6}$$

$$\partial_t\delta P=-\frac{5}{3}\vec{\nabla}\cdot(P_0(\vec{x})\vec{u})+\frac{2}{3}\rho_0(\vec{x})\vec{f}(\vec{x})\cdot\vec{u}, \tag{2.7}$$

$$\partial_t\delta\rho=-\vec{\nabla}\cdot(\rho_0(\vec{x})\vec{u}). \tag{2.8}$$

Eliminating δP from Eqs. (2.6) and (2.7), then using the continuity equation (2.8) to eliminate $\delta\rho$, and finally using the fact that $\vec{\nabla}\rho_0$ is parallel to $\vec{f}$, one immediately obtains the closed wave equation for the velocity field,

$$\partial_t^2\vec{u}=\frac{5}{3}\frac{P_0(\vec{x})}{\rho_0(\vec{x})}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})+\vec{\nabla}(\vec{u}\cdot\vec{f})+\frac{2}{3}\vec{f}(\vec{\nabla}\cdot\vec{u}), \tag{2.9}$$

which has been the starting point of previous works [5,7,11,12]. As already mentioned, we do not find this to be the most convenient starting point for our present study, as the boundary conditions on the velocity field at infinity, the nature of the function space formed by the solutions, the hermiticity or self-adjointness of the wave operator, and hence the nature of the spectrum of eigenvalues all remain unclear, even if some particular solutions, e.g., those satisfying $\vec{\nabla}\times\vec{u}=0$, can be constructed. To circumvent this obstacle, we therefore proceed by eliminating the velocity field from Eqs. (2.6)–(2.8) by taking the time derivative of Eqs. (2.8) and (2.7) and inserting Eq. (2.6).

It is convenient to define the function

$$F_{\mp}(\zeta)=\frac{F_{\mp}\left(\frac{5}{2},\zeta\right)}{F_{\mp}\left(\frac{3}{2},\zeta\right)}. \tag{2.10}$$

The bosonic function $F_-(\zeta)$ is only defined for $\zeta>0$. It increases monotonically from 0.513... at $\zeta=0$, where it has an infinite slope, to 1 for $\zeta\to\infty$. The fermionic function $F_+(\zeta)$ is defined for all real ζ. It is monotonically decreasing toward 1, first with a slope of roughly -0.4 for large negative ζ then exponentially for positive ζ.

In terms of $F_{\mp}(\zeta)$ and its derivative $F'_{\mp}(\zeta)$, with respect to its argument ζ, we can write

$$\frac{P_0(\vec{x})}{\rho_0(\vec{x})}=\frac{k_BT}{m}F_{\mp}((V(\vec{x})-\mu)/k_BT). \tag{2.11}$$

In the following we suppress the subscript $\mp$ and also the argument $(V(\vec{x})-\mu)/k_BT$ of F and F' for notational simplicity; after some calculation we obtain

$$\partial_t^2\delta\rho=\nabla^2\delta P-\vec{f}\cdot\vec{\nabla}\delta\rho-(\vec{\nabla}\cdot\vec{f})\delta\rho, \tag{2.12}$$

$$\begin{aligned}\partial_t^2\delta P={}&\frac{5}{3}\frac{k_BT}{m}F\nabla^2\delta P-\left(\frac{5}{3}F'+\frac{2}{3}\right)\vec{f}\cdot\vec{\nabla}\delta P\\&-\frac{5}{3}\frac{k_BT}{m}F\vec{f}\cdot\vec{\nabla}\delta\rho+\left[\left(\frac{5}{3}F'+\frac{2}{3}\right)\vec{f}^2\right.\\&\left.-\frac{5}{3}\frac{k_BT}{m}F(\vec{\nabla}\cdot\vec{f})\right]\delta\rho.\end{aligned} \tag{2.13}$$

So far we have gained in simplicity compared to Eq. (2.9), because we have now only two coupled wave equations instead of three. More important, however, is the fact that it is physically clear that the density and pressure perturbations must go to zero in the limit of large distances from the center of a confining trap. It should be noted that the same cannot be said for the velocity field. Indeed, it is clear from Eq. (2.6) that for $|\vec{x}|\to\infty$ where $\rho_0(\vec{x})\to 0$ the velocity field $\vec{u}$ is not necessarily bounded by the hydrodynamic equations. However, in spite of the improvement of the formulation of the linearized hydrodynamics we have achieved so far, the self-adjointness of the wave operator $\underline{\mathbf{H}}$ defined by writing Eqs. (2.12) and (2.13) in the form

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[1] A restoring force does not result if the whole fluid-layer on an equipotential surface is displaced in the same way orthogonal to the equipotential surface; because then no ambient fluid remains, which could give rise to the buoyancy force [Eq. (2.4)]. Instead a new mechanical equilibrium is reached. This mechanism gives rise to the zero-frequency modes discussed later.

$$\partial_t^2\begin{pmatrix}\delta P\\ \delta\rho\end{pmatrix}=-\underline{\mathbf{H}}\cdot\begin{pmatrix}\delta P\\ \delta\rho\end{pmatrix}\tag{2.14}$$

remains to be clarified. Can a scalar product be found in which the operator $\underline{\mathbf{H}}$ is Hermitian? This is the question to which we turn next.

B. Scalar product and hermiticity of the wave operator

In order to find a useful scalar product on the space of solutions of Eqs. (2.12) and (2.13) we proceed as follows. First we find a Lagrangian for Eqs.(2.12) and (2.13), which must be a quadratic functional of $\delta\rho(\vec{x},t)$ and $\delta P(\vec{x},t)$. It can be found by making a general ansatz and comparing the coefficient-functions of the resulting Euler-Lagrange equations with those in Eqs. (2.12) and (2.13). From the Lagrangian density we can pass to the associated ‘‘energy density’’

$$\mathcal{H}=(\partial_t\delta\rho)\frac{\partial\mathcal{L}}{\partial(\partial_t\delta\rho)}+(\partial_t\delta P)\frac{\partial\mathcal{L}}{\partial(\partial_t\delta P)}-\mathcal{L},\tag{2.15}$$

whose space integral

$$E=\int d^3x\mathcal{H}\tag{2.16}$$

must be conserved by time-translation invariance. We shall then define the scalar product $\langle\mathbf{P}_1|\mathbf{P}_2\rangle$ in such a way that the conserved energy takes the form

$$E=\langle\mathbf{P}|\underline{\mathbf{H}}\mathbf{P}\rangle\tag{2.17}$$

for vectors $\mathbf{P}$ satisfying the time-independent wave-equation $\underline{\mathbf{H}}\mathbf{P}=\omega^2\mathbf{P}$. For the Lagrangian density, after some calculation we find

$$\mathcal{L}=\frac{\beta}{2}(\partial_t\delta P)^2+\alpha(\partial_t\delta P)(\partial_t\delta\rho)-\frac{5}{3}\frac{k_BT}{m}\frac{F\alpha}{2}(\partial_t\delta\rho)^2+(\text{terms without time derivatives}),\tag{2.18}$$

with the coefficients α,β

$$\alpha=-\frac{K}{(2+5F')F_{\mp}\left(\frac{3}{2},\frac{V-\mu}{k_BT}\right)},\qquad\beta=-\frac{m}{k_BT}\frac{1+F'}{F}\alpha.\tag{2.19}$$

The coefficient K in the relation for α is arbitrary, and can be used for normalization. The kinetic energy contained in $\mathcal{H}$ is the same as in $\mathcal{L}$. It is now useful to employ the vector notation already defined in Eq. (2.14) by defining

$$\mathbf{P}(\vec{x},t)=\begin{pmatrix}\delta P(\vec{x},t)\\ \delta\rho(\vec{x},t)\end{pmatrix}.\tag{2.20}$$

With the harmonic time dependence

$$\mathbf{P}(\vec{x},t)=\mathbf{P}(\vec{x})\cos(\omega t+\varphi),\tag{2.21}$$

the conserved quantity E can be written as

$$E=\omega^2\sin^2(\omega t+\varphi)\int d^3x\left[\frac{\beta}{2}(\delta P)^2+\alpha\,\delta P\,\delta\rho-\frac{5}{3}\frac{k_BT}{m}\frac{F\alpha}{2}(\delta\rho)^2\right]+[\text{terms proportional to }\cos^2(\omega t+\varphi)].\tag{2.22}$$

In order to meet our goal [Eq. (2.17)], we should define the scalar product $\langle\mathbf{P}_1|\mathbf{P}_2\rangle$ in such a way that the coefficients of $\sin^2$ and $\cos^2$ in Eq. (2.22) both become equal to $\omega^2\langle\mathbf{P}|\mathbf{P}\rangle$. We therefore need only the coefficient of the $\sin^2$ term to conclude that the norm becomes

$$\langle\mathbf{P}|\mathbf{P}\rangle=\int d^3x\left[\frac{\beta}{2}\delta P^2+\alpha\,\delta P\,\delta\rho-\frac{5}{6}\frac{k_BT}{m}F\alpha(\delta\rho)^2\right].\tag{2.23}$$

(That the coefficient of the $\cos^2$ term is the same follows without further calculation from the fact that in a harmonic system the average values of the kinetic and potential energies are always the same.) Using relations (2.19) for α and β, it can be checked that the norm is positive, as required, if $\alpha<0$, (which can always be achieved by the choice of the constant K), and $\beta>0$, which requires the inequality $1+F'>0$, and $-\frac{5}{3}(k_BT/m)F\alpha\beta-\alpha^2>0$, which in turn requires the stronger inequality

$$5F'((V(\vec{x})-\mu)/k_BT)+2>0.\tag{2.24}$$

Using definition (2.10) of $F=F_{\mp}$, it is easy to check that Eq. (2.24) is equivalent to the stability condition [Eq. (2.4)] of the thermodynamic equilibrium state.

It is useful for numerical purposes to transform the scalar product [Eq. (2.23)] to the more symmetrical form

$$\langle\mathbf{P}_1|\mathbf{P}_2\rangle=\int d^3x(u_1^*u_2+v_1^*v_2)\tag{2.25}$$

by the linear transformation

$$\mathbf{P}=\begin{pmatrix}\delta P\\ \delta\rho\end{pmatrix}=\begin{pmatrix}\sqrt{\dfrac{-5k_BTF}{\alpha m(5F'+2)}} & 0\\ \sqrt{\dfrac{-9m}{\alpha 5k_BTF(5F'+2)}} & \sqrt{\dfrac{-3m}{\alpha 5k_BTF}}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix},\tag{2.26}$$

diagonalizing and normalizing the kinetic term in the Lagrangian $\mathcal{L}$ (and hence also in the energy density $\mathcal{H}$). In the new variables u and v, the wave equation now reads

$$-\frac{\partial^2}{\partial t^2}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix} -\frac{5}{3}\frac{k_BT\vec{\nabla}\cdot F\vec{\nabla}}{m}+\frac{(\vec{\nabla}\cdot\vec{f})}{6}+\frac{m}{60k_BT}\frac{\vec{f}^2}{F} & \frac{\vec{\nabla}\cdot\sqrt{5F'+2}\vec{f}}{\sqrt{3}}+\frac{\sqrt{3}m}{30k_BT}\frac{\sqrt{5F'+2}\vec{f}^2}{F}\\ -\frac{\sqrt{5F'+2}\vec{f}\cdot\vec{\nabla}}{\sqrt{3}}+\frac{\sqrt{3}m}{30k_BT}\frac{\sqrt{5F'+2}\vec{f}^2}{F} & \frac{m}{5k_BT}\frac{(5F'+2)\vec{f}^2}{F}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}, \tag{2.27}$$

with a matrix wave operator which is manifestly Hermitian in the scalar product [Eq. (2.25)], provided the surface integral

$$\int d\vec{S}\cdot\left[-\frac{5}{3}\frac{k_BT}{m}(u_1^*F\vec{\nabla}u_2-u_2F\vec{\nabla}u_2^*)+\sqrt{\frac{5F'+2}{3}}\vec{f}(u_1^*v_2-u_2v_1^*)\right] \tag{2.28}$$

vanishes at $|\vec{x}|\to\infty$. Since the coefficient functions α and β in Eq. (2.23) grow for $|\vec{x}|\to\infty$ like $\exp(V(\vec{x})-\mu/k_BT)$, the fluctuations $\delta P(\vec{x})$ and $\delta\rho(\vec{x})$ for all solutions must vanish sufficiently rapidly for $|\vec{x}|\to\infty$.

C. Zero-frequency modes and isothermal modes

For arbitrary temperature T and trap potential $V(\vec{x})$, Eqs. (2.12) and (2.13) possess a class of exact time-independent solutions, which depend on an arbitrary function $G(V(\vec{x}))$, and its derivative $G'=dG/dV$, namely

$$\delta\rho(\vec{x})=-\varepsilon G'(V(\vec{x})),\qquad \delta P(\vec{x})=\frac{\varepsilon}{m}G(V(\vec{x})). \tag{2.29}$$

ε is a parameter which is sufficiently small to make the linearized theory consistent. The norm [Eq. (2.23)] of these solutions $|\mathbf{P}_0\rangle$ is

$$\langle\mathbf{P}_0|\mathbf{P}_0\rangle=\varepsilon^2\int d^3x\frac{|\alpha|}{2m}\left[\frac{1+F'}{k_BTF}G^2+2GG'+\frac{5}{3}k_BTFG'^2\right]. \tag{2.30}$$

Since all functions under the integral depend on $\vec{x}$ only via $V(\vec{x})$, the integration $\int d^3x\ldots$ can be replaced by $\mathrm{const}\int dV\sqrt{V}\ldots$, if V is a homogeneous function of second order of $\vec{x}$, e.g., a parabolic potential. The scalar product exists and is positive under condition (2.24), if $G(V)$ vanishes sufficiently rapidly, e.g., like $\exp(-V/k_BT)$, for $V\to\infty$. Then these solutions belong to the Hilbert space and have to be considered. Physically, they appear because of the coexistence of a continuum of mechanical equilibrium states and the thermodynamic equilibrium. As we shall see in Sec. III, these states are not isolated from all the other states but occur, for any local wave number, as the end point of a spectral branch of states if the local wave number is turned in the direction of $\vec{f}=-(1/m)\vec{\nabla}V$.

For harmonic potentials there are some further exact solutions of wave equations (2.12) and (2.13) which hold for all temperatures in the fermionic case and the temperature $T>T_c$ in the bosonic case. They are obtained by extending ansatz (2.29) for the zero-frequency modes according to

$$\begin{pmatrix}\delta P(\vec{x},t)\\ \delta\rho(\vec{x},t)\end{pmatrix}=\epsilon x^\alpha y^\beta z^\gamma\begin{pmatrix}\frac{1}{m}G(V(\vec{x}))\\ -G'(V(\vec{x}))\end{pmatrix}e^{-i\omega t}, \tag{2.31}$$

with α, β, $\gamma=0$ or 1. Inserting this ansatz into Eq. (2.12), and using the property that for $\alpha=\beta=\gamma=0$ Eq. (2.31) is a zero-frequency mode, after a simple calculation we find

$$(\omega^2-\alpha\omega_1^2-\beta\omega_2^2-\gamma\omega_3^2)G'(V)=0. \tag{2.32}$$

Next we also insert the ansatz into Eq. (2.13) and by a similar calculation obtain

$$[\omega^2+(\alpha\omega_1^2+\beta\omega_2^2+\gamma\omega_3^2)(\tfrac{5}{3}F'+\tfrac{2}{3})]G(V)+\tfrac{5}{3}k_BT(\alpha\omega_1^2+\beta\omega_2^2+\gamma\omega_3^2)FG'(V)=0. \tag{2.33}$$

Equation (2.32) determines the mode frequencies as

$$\omega_{\alpha\beta\gamma}=\sqrt{\alpha\omega_1^2+\beta\omega_2^2+\gamma\omega_3^2}, \tag{2.34}$$

while Eq. (2.33), for α, β, and γ not all equal to zero, fixes the yet undetermined function $G(V)$ in ansatz (2.31) as

$$G(V)=\mathrm{const}\,F\left(\frac{3}{2},\frac{V(\vec{x})-\mu}{k_BT}\right). \tag{2.35}$$

It follows from Eq. (2.31) that $\delta P(\vec{x},t)$ and $\delta\rho(\vec{x},t)$ for these modes are related by

$$\delta P(\vec{x},t)=\frac{k_BT}{m}\frac{F\left(\frac{3}{2},\frac{V(\vec{x})-\mu}{k_BT}\right)}{F\left(\frac{1}{2},\frac{V(\vec{x})-\mu}{k_BT}\right)}\delta\rho(\vec{x},t), \tag{2.36}$$

which is the relation between changes of pressure and density implied by the local thermodynamic equilibrium [Eq. (2.2)] if the temperature is kept constant. These isothermal modes were already found in Ref. [5] for the special case of isotropic and axially symmetric parabolic trap potentials.

Mode (2.34) contains as special cases, the three Kohn modes $\omega_{100}=\omega_1$, $\omega_{010}=\omega_2$, and $\omega_{001}=\omega_3$, corresponding to oscillations of the center of mass of the trapped gas. It is interesting to note that collisionless Kohn modes of the form $\delta\rho(\vec{x},t)=\epsilon(\partial\rho_0(\vec{x})/\partial x_i)e^{-i\omega t}$, with the same frequencies ω_i, also exist. It therefore follows from the phenomenological formula [Eq. (1.6)] that these modes are not damped by the relaxation mechanisms present in the system, in agreement with the general statement made by the Kohn theorem.

For fermions, results (2.34), (2.35), and (2.31) for the frequencies and mode functions apply to all temperatures, and can therefore also be extrapolated to $T\to 0$. Indeed, for $T\to 0$, modes with the frequency (2.34) were found in Ref. [9]. In order also to compare the mode functions, we use the Bethe-Sommerfeld expansion to evaluate the Fermi-integrals asymptotically, $F_+(s,z)\sim(-z)^s/s!$ for $z\to-\infty$, and for $T\to 0$ find that

$$\delta P=\frac{2}{3}\,\frac{\mu-V(\vec{x})}{m}\,\delta\rho\sim x^{\alpha}y^{\beta}z^{\gamma}(\mu-V(\vec{x}))^{3/2}, \qquad (2.37)$$

in agreement with Ref. [9].

Finally, another set of solutions for arbitrary temperature is found by generalizing an ansatz of Ref. [5] for a trap without axial symmetry by putting $u_i(\vec{x},t)=A_i x_i e^{-i\omega t}$ with three constants A_i. Inserting directly into Eqs. (2.7) and (2.8), we obtain the density and pressure modes:

$$\delta\rho(\vec{x},t)=-\frac{im}{\omega}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\left[F\left(\frac{3}{2}\right)\sum_{i=1}^{3}A_i - F\left(\frac{1}{2}\right)\frac{m}{k_BT}\sum_{i=1}^{3}A_i\omega_i^2x_i^2\right]e^{-i\omega t} \qquad (2.38)$$

$$\delta P(\vec{x},t)=\frac{ik_BT}{\omega}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\left[-\frac{5}{3}F\left(\frac{5}{2}\right)\sum_{i=1}^{3}A_i - F\left(\frac{3}{2}\right)\frac{m}{k_BT}\sum_{i=1}^{3}A_i\omega_i^2x_i^2\right]e^{-i\omega t}. \qquad (2.39)$$

A comparison of Eqs. (2.38) and (2.39) with the local equilibrium relations Eqs. (1.4)] reveals that the temperature oscillates in these modes with a spatially constant amplitude proportional to $\Sigma_{i=1}^{3}A_i$. Finally, using the results (2.38) and (2.39) in Eq. (2.6) for momentum conservation, we arrive at the eigenvalue problem

$$\omega^2A_i=2A_i\omega_i^2+\frac{2}{3}\omega_i^2\sum_{j=1}^{3}A_j, \quad i=1,2,3. \qquad (2.40)$$

The eigenvector $\vec{A}$ and the eigenvalue ω^2 are clearly temperature independent, and follow from the cubic secular equation

$$(\omega^2)^3-\frac{8}{3}(\omega_x^2+\omega_y^2+\omega_z^2)(\omega^2)^2+\frac{20}{3}(\omega_x^2\omega_y^2+\omega_x^2\omega_z^2+\omega_y^2\omega_z^2)\omega^2-16\omega_x^2\omega_y^2\omega_z^2=0. \qquad (2.41)$$

In the special case of an axially symmetric trap, the cubic equation can be reduced to a quadratic one, and a result first obtained in Ref. [5] is recovered.

III. SHORT WAVELENGTH SOLUTIONS

The two coupled wave equations derived in Sec. II in various forms are difficult to solve for arbitrary temperature in a system which is made spatially inhomogeneous by an external potential $V(\vec{x})\neq 0$. An exception, however, are waves of wavelengths, which are short on the spatial scale on which $V(\vec{x})$, and hence also $P_0(\vec{x}),\rho_0(\vec{x})$, vary. Such waves, in the representation with $u(\vec{x},t),v(\vec{x},t)$, can be written as

$$\begin{pmatrix}u(\vec{x},t)\\ v(\vec{x},t)\end{pmatrix}=e^{-i\omega t}\begin{pmatrix}a_0(\vec{x})\\ b_0(\vec{x})\end{pmatrix}e^{iS(\vec{x},t)}. \qquad (3.1)$$

The eikonal $S(\vec{x},t)$ defines the local wave vector by the relation $\vec{k}(\vec{x},t)=\vec{\nabla}S(\vec{x},t)$. The amplitudes a_0,b_0, and $\vec{k}$ vary slowly in space, on the same scale as $V(\vec{x})$, $P_0(\vec{x})$, and $\rho_0(\vec{x})$. The frequency ω is independent of $\vec{x}$ and t. Inserting the ansatz into Eq. (2.27), and neglecting derivatives of a_0,b_0, and $\vec{k}$, and also assuming that $\omega^2\gg|\vec{\nabla}\cdot\vec{f}|$, we obtain the secular equation as the vanishing of the determinant:

$$\begin{vmatrix}\frac{5}{3}\frac{k_BT}{m}F\vec{k}^2+\frac{1}{60}\frac{m}{k_BT}\frac{\vec{f}^2}{F}-\omega^2 & \frac{i}{\sqrt{3}}\sqrt{5F'+2}(\vec{k}\cdot\vec{f})+\frac{\sqrt{3}}{30}\frac{\sqrt{5F'+2}}{F}\frac{m}{k_BT}\vec{f}^2\\ -\frac{i}{\sqrt{3}}\sqrt{5F'+2}(\vec{k}\cdot\vec{f})+\frac{\sqrt{3}}{30}\frac{\sqrt{5F'+2}}{F}\frac{m}{k_BT}\vec{f}^2 & \frac{5F'+2}{5F}\frac{m}{k_BT}\vec{f}^2-\omega^2\end{vmatrix}=0. \qquad (3.2)$$

We note that terms with $\vec{f}^2$ and $\vec{k}\cdot\vec{f}$ are essential to retain in this approximation, together with the k^2 terms, because $|\vec{f}|$ grows at large distances from the trap center, at least for parabolic traps, and provides the physically crucial confining mechanism. On the other hand, $a_0(\vec{x}),b_0(\vec{x})$, and $\vec{\nabla}\cdot\vec{f}$ do not grow in a similar way, and are therefore consistently negligible.

From Eq. (3.2), we deduce the local dispersion law for waves of short wavelength $2\pi/k$:

$$\omega^2=\omega_\pm^2(\vec{k},\vec{x})$$

$$=\frac{1}{2}\left[\frac{5}{3}\frac{k_BT}{m}Fk^2+\frac{m}{k_BT}\left(\frac{1}{60F}+\frac{5F'+2}{5}\right)\vec{f}^2\right]\pm\sqrt{\frac{1}{4}\left[\frac{5}{3}\frac{k_BT}{m}Fk^2+\frac{m}{k_BT}\left(\frac{1}{60F}+\frac{5F'+2}{5}\right)\vec{f}^2\right]^2-\frac{1}{3}(5F'+2)(\vec{f}\times\vec{k})^2}. \tag{3.3}$$

On the same level of approximation the pressure and density oscillations are related by

$$\delta P(\vec{k},x)=\frac{1}{k^2}[\omega^2-i\vec{f}\cdot\vec{k}]\delta\rho(\vec{k},x), \tag{3.4}$$

as follows from Eq. (2.12).

The dispersion law [Eq. (3.3)] contains a great deal of physics, and will now be discussed. First we note that for $\vec{f}\neq 0$ there are two branches of the dispersion law, one of high frequency and another one of lower frequency, which are both physical. Thus there are two different types of waves in these systems. Both branches correspond to frequencies $\omega^2\geqslant 0$ for all $\vec{k}$, i.e. to stable oscillation waves. Another simple observation is that the local dispersion relation is anisotropic, and depends on the angle between $\vec{f}$ and $\vec{k}$. The physical nature of the two branches is most easily seen by assuming that the angle between $\vec{f}$ and $\vec{k}$ is sufficiently small to permit the expansion of the square-root in Eq. (3.3) in the second term of its radicand. We then obtain to lowest nonvanishing order,

$$\omega_+^2=\frac{5}{3}\frac{k_BT}{m}Fk^2+\frac{m}{k_BT}\left(\frac{1}{60F}+\frac{5F'+2}{5}\right)\vec{f}^2, \tag{3.5}$$

$$\omega_-^2=\frac{5F'+2}{3}\,\frac{(\vec{f}\times\vec{k})^2}{\frac{5}{3}\frac{k_BT}{m}Fk^2+\frac{m}{k_BT}\left(\frac{1}{60F}+\frac{5F'+2}{5}\right)\vec{f}^2}. \tag{3.6}$$

The high-frequency branch is easily recognized in this limit as an adiabatic sound mode, in particular if the identity for the adiabatic sound velocity,

$$c_s^2=\left.\frac{\partial P_0}{\partial\rho_0}\right|_s=\frac{5}{3}\frac{P_0}{\rho_0}=\frac{5}{3}\frac{k_BT}{m}F, \tag{3.7}$$

is used, which is valid for the ideal quantum gases, with the definition of $F=F_\mp$ by Eqs. (2.10) and (2.11). The low-frequency branch reaches its lowest frequency $\omega_-=0$, in the present approximation, for all waves traveling locally in the direction parallel to the force of the trap $\vec{f}$, so that $\vec{f}\times\vec{k}=0$. The existence of such zero-frequency modes was already discussed in Sec. II. Looking at exact solutions at high temperature in Sec. II, we shall see them appear again.

The low-frequency branch for given $|\vec{k}|$ achieves its highest frequency if the wave propagates locally in directions orthogonal to $\vec{f}$. The maximum frequency for modes orthogonal to $\vec{f}$ is then reached for short wavelengths,

$$c_s^2k^2\gg\frac{m}{k_BT}\left(\frac{1}{60F}+F'+\frac{2}{5}\right)\vec{f}^2, \tag{3.8}$$

and given by

$$\omega_-^{\max}=\sqrt{\frac{5F'+2}{3}}\frac{|\vec{f}|}{c_s}. \tag{3.9}$$

Waves with the properties of the low-frequency branch solutions found here are typical of media which are stratified by an external force, and are called ‘‘internal waves.’ One of their surprising and counterintuitive properties is that in regions where Eq. (3.8) applies the group velocity $\vec{\nabla}_k\omega_-(\vec{k})$ is orthogonal to the wave vector $\vec{k}$. For a textbook discussion of such waves, see Ref. [19]. Indeed, the dispersion relations (3.5) and (3.6) can, e.g., be directly compared with Eqs. (53) and (54) given there.

IV. SOLUTION IN THE HIGH-TEMPERATURE REGION

The high-temperature regime is defined by $T\gg T_{deg}$, where T_{deg} is the degeneracy temperature at which the de Broglie wavelength becomes of the order of the mean particle distance. In this regime we can approximate

$$f_\mp(\vec{p},\vec{x})=e^{-[(p^2/2m)+V(\vec{x})-\mu]/k_BT} \tag{4.1}$$

and $\rho_0(\vec{x})=\rho_0(0)e^{-V(\vec{x})/k_BT}$, $P_0(\vec{x})=(k_BT/m)\rho_0(\vec{x})$. It is useful to introduce new dimensionless variables $P_1(\vec{x})$ and $\rho_1(\vec{x})$ via the definitions

$$\delta P(\vec{x},t)=\frac{k_BT}{m}\rho_0(\vec{x})P_1(\vec{x},t),$$

$$\delta\rho(\vec{x},t)=\rho_0(\vec{x})\rho_1(\vec{x},t), \tag{4.2}$$

and to rewrite the coupled wave equations for δP and $\delta\rho$ as coupled equations for P_1 and

$$P_1-\rho_1=T_1=\frac{\delta T}{T}, \tag{4.3}$$

where $\delta T(\vec{x},t)$ is the deviation of the temperature from equilibrium. Separating the time dependence $e^{-i\omega t}$, we arrive at

$$\begin{pmatrix} -\frac{k_BT}{m}\frac{5}{3}\nabla^2-\vec{f}\cdot\vec{\nabla}-\omega^2 & -\frac{5}{3}[(\vec{\nabla}\cdot\vec{f})+\vec{f}\cdot\vec{\nabla}]-\frac{m}{k_BT}\vec{f}^2 \\ -\frac{k_BT}{m}\frac{2}{3}\nabla^2 & -\frac{2}{3}[(\vec{\nabla}\cdot\vec{f}\,)+\vec{f}\cdot\vec{\nabla}]-\omega^2 \end{pmatrix}\begin{pmatrix} P_1 \\ P_1-\rho_1 \end{pmatrix}=0. \tag{4.4}$$

We specialize these equations to a harmonic potential:

$$V(\vec{x})=\frac{m}{2}(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2). \tag{4.5}$$

It is then clear, via term by term inspection, that there are polynomial solutions of Eq. (4.4), in which P_1 is polynomial in the Cartesian components of $\vec{x}$ of total order n and $P_1-\rho_1$ is a corresponding polynomial of total order $n-2$. This is so because then each term on the right-hand side of Eq. (4.4) either decreases the total order of the polynomial on the left-hand side by 2 via the operation of ∇^2, or keeps the same order of the polynomial. It follows that we can pick freely as the highest total power an integer n for P_1, and determine the eigenvalues ω_n^2 by comparing the coefficients of all terms with this highest total power, imposing the condition for nontrivial solvability. It is clear that we obtain polynomials of arbitrary total order in this way. Moreover these solutions lie in the Hilbert space because, via Eq. (4.2), δP and $\delta\rho$ fall off sufficiently rapidly for $|\vec{x}|\rightarrow\infty$, if P_1 and ρ_1 are polynomials in the Cartesian components of $\vec{x}$.

To see this explicitly, we specialize the scalar product [Eq. (2.23)] for the present high-temperature case by replacing $F[3/2,(V(\vec{x})-\mu)/k_BT]=\exp[-\mathbf{V}(\vec{x})-\mu)/k_BT]$, $F_{\mp}=1$, $F'_{\mp}=0$ which gives $\alpha=-(K/2)\exp[(V(\vec{x})-\mu)/k_BT]$, $\beta=-(m/k_BT)\alpha$, and the scalar product

$$\langle\mathbf{P}|\tilde{\mathbf{P}}\rangle=\frac{K}{4}\frac{k_BT}{m}(\rho_0(0))^2e^{-\mu/k_BT}\int d^3xe^{-V(\vec{x})/k_BT}\times\left[P_1\tilde{P}_1-P_1\tilde{\rho}_1-\rho_1\tilde{P}_1+\frac{5}{3}\rho_1\tilde{\rho}_1\right]. \tag{4.6}$$

It is also now clear that, for harmonic potentials $V(\vec{x})$, the polynomial solutions for P_1 and ρ_1 of all orders are complete in a space with this scalar product, a fact which is very familiar from the quantum mechanics of the harmonic oscillator.

To be specific let us consider a polynomial of x, y, and z for P_1 of total order n. It then has terms of highest total order of the form

$$P_1=\sum_{n_1,n_2,n_3}A_{n_1n_2n_3}x^{n_1}y^{n_2}z^{n_3}+(\text{lower order}), \tag{4.7}$$

with $n_1+n_2+n_3=n$, and $\binom{n+2}{2}=(n+2)(n+1)/2$ different coefficients$A_{n_1n_2n_3}$. For the same mode $P_1-\rho_1$ must be a polynomial of total order $n-2$, with $\binom{n}{2}=n(n-1)/2$ terms $x^{n_1}y^{n_2}z^{n_3}$ of highest total order $n_1+n_2+n_3=n-2$ with coefficients $B_{n_1n_2n_3}$.

The solvability condition for the linear homogeneous equations connecting all these coefficients then gives a secular equation for the eigenvalues ω_n^2 of order n^2+n+1 with just as many solutions. $(n+1)(n+2)/2$ of these modes can be considered as modes of sound waves, modified by the external potential, while the remaining $n(n-1)/2$ can be considered as modes of internal waves, modified by their coupling to sound waves. The second kind of mode therefore exists only for $n\geqslant 2$.

How many linearly independent zero-frequency modes appear at a given total order n? If n is odd, the polynomials for P_1 and ρ_1 have odd parity and cannot describe a zero-frequency mode, which must have even parity by Eq. (2.29). If n is even, on the other hand, there is precisely one linearly independent zero-frequency mode associated with that integer, which may be written in the form

$$\begin{pmatrix} \delta P(\vec{x}) \\ \delta\rho(\vec{x}) \end{pmatrix}=\text{const}\frac{\rho_0(\vec{x})}{m}\left[\frac{V(\vec{x})}{k_BT}\right]^{n/2-1}\begin{pmatrix} V(\vec{x}) \\ n/2 \end{pmatrix}. \tag{4.8}$$

The solution for the isotropic case $\omega_x=\omega_y=\omega_z=\omega_0$ in the high-temperature regime was already given by Bruun and Clark [7] based on the velocity equation (2.9). It is not clear, however, how the Hilbert space is defined in terms of the velocity variables. Indeed, the velocity-field turns out to be polynomial like ρ_1 and P_1, and is therefore not a square-integrable field. It is therefore worthwhile to check this case again, using our present description. In addition we also wish to find the mode functions for this case explicitly. Introducing a dimensionless frequency and space coordinate via $\Omega^2=\omega^2/\omega_0^2,\vec{r}=(m\omega_0^2/k_BT)^{1/2}\vec{x}$, and imposing a polynomial ansatz

$$P_1(\vec{r})=r^lY_{lm}(\Theta,\varphi)Q_{n_r}(r^2),$$

$$P_1(\vec{r})-\rho_1(\vec{r})=r^lY_{lm}(\Theta,\varphi)T_{n_r-1}(r^2), \tag{4.9}$$

with $n=2n_r+l$, and with Q_{n_r} and T_{n_r} polynomials of order n_r and $Y_{lm}(\Theta,\varphi)$ the spherical harmonics, we obtain the following.

(i) For $n_r=0$,

$$Q_0(r^2)=A_0^{(0)},\quad T_{-1}(r^2)=0, \tag{4.10}$$

$$\Omega^2=l. \tag{4.11}$$

These isothermal modes were first found by Griffin, Wu, and Stringari [5]

(ii) For $n_r \geqslant 1$,

$$Q_{n_r}(r^2)=\sum_{i=0}^{n_r} A_i^{(n_r)} r^{2i}, \quad T_{n_r-1}(r^2)=\sum_{i=0}^{n_r\text{-}1} B_i^{(n_r)} r^{2i}, \tag{4.12}$$

$$\Omega_{1,2}^2=\frac{\omega_{1,2}^2}{\omega_0^2} =\frac{1}{2}\left[\frac{5}{3}\left(l+2n_r+\frac{2}{5}\right) \pm\sqrt{\frac{25}{9}\left(l+2n_r+\frac{2}{5}\right)^2-\frac{8}{3}l(l+1)}\right]. \tag{4.13}$$

This result was first obtained by Bruun and Clark [7]. For the case $n_r=0$ we find that only one branch of solution (4.13) exists, namely the branch which yields $\Omega^2=l$, as seen from Eqs. (4.10) and (4.11). This fact also follows from a proper solution of Eq. (13) of Ref. [7] for $n_r=0$. In the case $n_r \geqslant 1, l=0$ one of the solutions [Eq. (4.13)] vanishes. This gives the zero-frequency solutions discussed in previous sections. The other solution at $\Omega^2=\omega_1^2/\omega_0^2=\frac{5}{3}(2n+\frac{2}{5})$ behaves normally, and describes a sound mode, as can, e.g., be seen from the factor 5/3 which is characteristic of the inverse of the adiabatic compressibility in ideal gases.

The coefficients of the polynomials defining the mode functions are obtained recursively, beginning with the term of the highest power. We obtain the confluent hypergeometric functions

$$T_{n_r-1}(r^2)=B_0^{(n_r)}\,{}_1F_1\left(1-n_r,\frac{2l+3}{2};\frac{r^2}{2}\right), \tag{4.14}$$

which is proportional to the generalized Laguerre polynomial $L_{n_r-1}^{[l+(1/2)]}(r^2/2)$, and

$$Q_{n_r}(r^2)=B_0^{(n_r)}\left\{\frac{3}{4n_r}\left[\Omega_{1,2}^2+\frac{2l}{3}\right]{}_1F_1\left(-n_r,\frac{2l+3}{2};\frac{r^2}{2}\right) -\frac{2l+1}{2n_r}\,{}_1F_1\left(-n_r,\frac{2l+1}{2};\frac{r^2}{2}\right)\right\}, \tag{4.15}$$

which is proportional to $(3\Omega_{1,2}^2+2l)L_{n_r}^{[l+(1/2)]}(r^2/2)-2(1+2l+2n_r)L_{n_r}^{[l-(1/2)]}(r^2/2)$. The physical modes are then

$$\delta P(\vec{x},t)=\frac{k_BT}{m}\rho_0(\vec{x})r^lY_{lm}(\Theta,\varphi)Q_{n_r}(r^2),$$

$$\frac{\delta T(\vec{x},t)}{T}=r^lY_{lm}(\Theta,\varphi)T_{n_r-1}(r^2). \tag{4.16}$$

It is quite remarkable that the spatial perturbation of the temperature in the two physically very different branches of the spectrum, the sound modes and the internal modes, is exactly the same, because $T_{n-1}(r^2)$ is independent of $\omega_\pm^2$ and therefore the same for both branches. (We note, however, that this is strictly true only in the Boltzmann limit.) The spatial distribution of the pressure and the density, on the other hand, is very different for both kinds of modes as, one would expect.

For $l=0$ and arbitrary n_r, we obtain the mode functions of the zero-frequency modes. They form the bottom of a ladder of rotational modes for each value of n. The mode functions are given by combinations of Hermite polynomials of the radial variable $(m\omega_0^2\vec{x}^2/k_BT)^{1/2}$ and form a complete set in the Hilbert space of radial functions defined by the scalar product [Eq. (4.6)]. This simply means that an *arbitrary* radial, i.e., an angle independent, mode function within our Hilbert space, is a zero-frequency mode.

Let us now turn to the case of an anisotropic harmonic trapping potential. Then the generator of rotations $\underline{\mathbf{1}}\vec{L}=-i\underline{\mathbf{1}}(\vec{x}\times\vec{\nabla})$ no longer commutes with the wave operator $\underline{\mathbf{H}}$ defined by the matrix-differential operator on the right-hand side of Eq. (4.4). On the other hand, as already discussed, for harmonic trapping potentials polynomial solutions continue to exist. Therefore, one must strongly suspect that a complete set of operators exists commuting with $\underline{\mathbf{H}}$ also exists in the fully anisotropic case. For Bose-Einstein condensed bosons and for fermions at temperature $T=0$, a similar situation prevails (but not at temperatures between the high- and low-temperature limits; cf. the discussion below), and two operators commuting with the corresponding wave-operator were constructed in our previous papers [9,18]. In the present high-temperature limit, corresponding conserved operators commuting with $\underline{\mathbf{H}}$ can again be found. In the case of axially symmetric traps with $\omega_x=\omega_y=\omega_\perp$, one of the conserved operators, namely, $\underline{\mathbf{1}}L_z$, still follows from symmetry, but one can check by direct calculation that a second conserved operator $\underline{\mathbf{B}}$ commuting with $\underline{\mathbf{H}}$ exists in the explicit forms $(\hat{R}+6/5)(P_1-\rho_1)=B(P_1-\rho_1)$ and $(\hat{R}-4/5)\rho_1=B\rho_1$, with

$$\hat{R}=\vec{x}\cdot\vec{\nabla}-\frac{k_BT}{m}\left(\frac{1}{\omega_x^2}\frac{\partial^2}{\partial x^2}+\frac{1}{\omega_y^2}\frac{\partial^2}{\partial y^2}+\frac{1}{\omega_z^2}\frac{\partial^2}{\partial z^2}\right), \tag{4.17}$$

where $\omega_x^2=\omega_y^2=\omega_\perp^2$. This is sufficient for the system to be integrable. The eigenvalue B is easily determined as $B=n-4/5$, where n is the polynomial order of the eigenfunction. The action of this additional conserved operator on ρ_1 can be used to show that all hydrodynamic modes satisfying the eigenvalue equations except the mode with $n=0$ are particle number conserving, and therefore belong to the physical spectrum. In particular, this also includes all the zero-frequency modes.

Let us also briefly discuss the solutions for axially symmetric traps, in order to make contact with results in the literature. It is useful to introduce scaled cylinder coordinates r, z, and φ via $\rho^2=(x_1^2+x_2^2)M\omega_\perp^2/k_BT$ and $z^2=x_3^2M\lambda\omega_\perp^2/k_BT$ $(\lambda=\omega_z^2/\omega_\perp^2)$. An appropriate ansatz solving the eigenvalue equation now is

$$P_1=\rho^{|m|}e^{im\varphi}z^{\alpha}Q(\rho^2,z^2),\quad P_1-\rho_1=\rho^{|m|}e^{im\varphi}z^{\alpha}T(\rho^2,z^2), \tag{4.18}$$

where Q and T are polynomials in ρ^2 and z^2 of order n' and $n'-1$, respectively, $\alpha=0,1$ determines the parity under inversion of the z axis, and m is the quantum number of angular momentum around the z axis. Let us mention some simple special cases.

For $n'=0$ we have $Q=A_0, T=0$, and obtain $\omega^2=(|m|+\lambda\alpha)\omega_\perp^2$, $(|m|=0,1,\ldots;\alpha=0,1)$. For $n'=1$ we obtain a cubic equation for ω^2. Specializing further to $m=0$ and the parity class $\alpha=0$, we find the explicit solutions $\omega_1^2=0$ with $A_0=\lambda A_1=(\lambda/2)B_0$, and $\omega_{2,3}^2=(\omega_\perp^2/3)[4\lambda+5\pm\sqrt{16(\lambda-1)^2+9}]$, first obtained in Ref. [5].

V. NUMERICAL DETERMINATION OF THE TEMPERATURE-DEPENDENT MODE-SPECTRUM

At intermediate temperatures the coupled wave equations (2.12) and (2.13) are generally not separable, and the spectrum can only be found numerically. Best suited for numerical work are the wave equations in the manifestly Hermitian form [Eq. (2.27)] with the scalar product in the simple form [Eq. (2.25)].

A numerical analysis will be performed for the axially symmetric case, choosing $\lambda=\omega_z^2/\omega_\perp^2=8$, for the anisotropy parameter, partially for historical reasons, as this was the geometry of the first TOP trap at JILA [20]. The number of particles is chosen as $N=10^6$. First the chemical potential is determined for the given particle number N as a function of temperature. This is done in the standard way, by integrating the first of Eqs. (1.4) to obtain $N(\mu,T)$, and solving for $\mu(N,T)$. We may remark here that it follows from the form of the potential that μ is a scaling function S of $N,\bar{\omega}=(\omega_x\omega_y\omega_z)^{1/3}$, and T of the form $\mu/\hbar\bar{\omega}=S(k_BT/\hbar\bar{\omega}N^{1/3})$.

In order to determine the spectrum, the wave operator in Eq. (2.27) is represented in the basis of the harmonic oscillator eigenfunctions with widths $\sqrt{k_BT/m\omega_z^2}$ and $\sqrt{k_BT/m\omega_\perp^2}$ in axial and radial directions, respectively. The basis is cut off at a finite size of order 100 both for u and v, and the resulting finite-dimensional Hermitian matrix is diagonalized. The size of the finite basis is varied in order that the eigenvalues obtained can be converged numerically. The truncation of the basis to a finite size introduces some spurious eigenmodes and eigenfrequencies, which can be distinguished and subsequently eliminated by the fact that they do not converge, but disappear and reappear somewhere else as the size of the basis is varied.

Some of the results are displayed in Figs. 1–5, which we now discuss. Figure 1 gives an overview, in the domain $94<k_BT/\hbar\bar{\omega}<195$, of the spectrum of eigenvalues ω^2, for a gas of bosons (but very similar results, not shown here, are also found for fermions). For clarity only modes with an azimuthal quantum number $m=0$ and even parity are shown. The basis used consists of oscillator eigenfunctions of order $2n_z$ in the z direction and order $2n_\rho$ in the radial direction, with an integer $n_\rho+n_z\leqslant 10$. All frequencies obtained by the diagonalization of the matrix of $\underline{\mathbf{H}}$ in this basis (except for the spurious ones introduced by the truncation of the basis) are shown in the figure. The chosen temperature domain extends from the BEC temperature at the chosen particle number to the high-temperature region. In fact, we have checked that our numerical code applied to the case of an isotropic trap gives a similar result, which for $k_BT/\hbar\bar{\omega}=195$ coincides with the analytically known eigenvalues [Eqs. (4.13)] in the

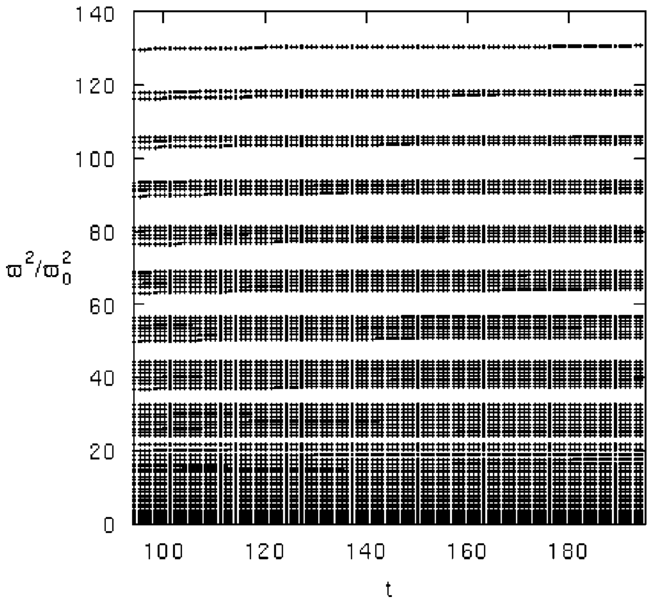

FIG. 1. Dimensionless squared hydrodynamic mode frequencies $(\omega/\bar{\omega})^2$ of a bosonic gas above the BEC transition as a function of the dimensionless temperature t, for $m=0$ modes of even parity with up to ten nodal surfaces; the anisotropy parameter $\lambda=(\omega_z/\omega_\perp)^2=8$; the number of atoms $N=10^6$; the total size of the basis is 132.

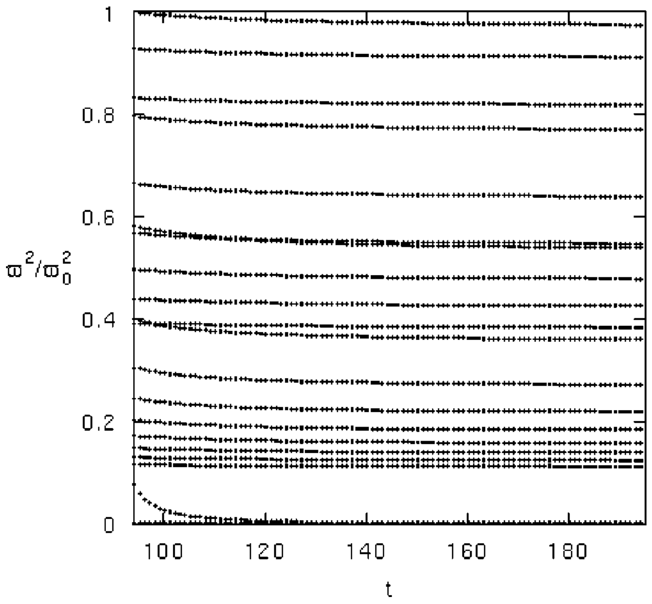

FIG. 2. Squared internal mode frequencies of Fig. 1 below the geometric mean trap frequency as functions of temperature, for the same parameters as in Fig. 1.

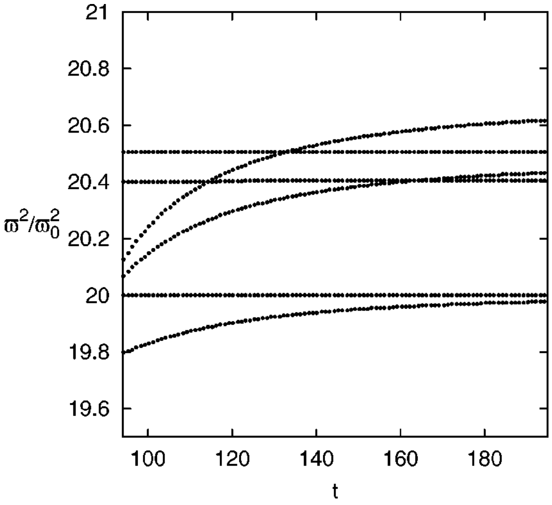

FIG. 3. Squared mode frequencies increasing with temperature and level crossings for a Bose gas with the same parameters as in Fig. 1, for an isotropic trap.

high-temperature limit with high precision. This fact offers us the possibility to assign the quantum numbers of the high-temperature domain to the whole corresponding temperature-dependent branch of frequencies.

An obvious feature of Fig. 1 is the bandlike structure of the spectrum, which is caused by the anisotropy of the trap: Because of the higher stiffness of the trap in the axial direction for the assumed value of $\lambda = 8$, nodes of the mode function in the axial direction are more costly in energy $\hbar\omega$, for soundlike modes, than nodes in the radial direction. Therefore, to the ''bands'' we can assign the quantum number n_z of nodes of sound-like modes in the axial direction with $n_z = 10$ in the highest band (which can consist of a single mode only in the subspace we consider in Fig. 1) and $n_z = 0$ in the lowest band. It should be noted that the two lowest bands are not split, and form a single broad band. Within a given band soundlike modes differ only by the radial quantum number n_ρ, counting the number of nodes in the radial direction. By our choice of a finite-dimensional basis, we are restricted to modes with quantum numbers $n_\rho + n_z \leqslant 10$. The presence of internal waves complicates the assignment of quantum numbers, because each pair of quantum numbers n_z and n_ρ appears twice, once at higher frequency for a soundlike mode and once at lower frequency for an internal wave mode. Thus internal modes with high values of n_z also give frequencies in the low-lying bands of Fig. 1. The number of eigenfrequencies in the bands depends, of course, on the size of the basis.

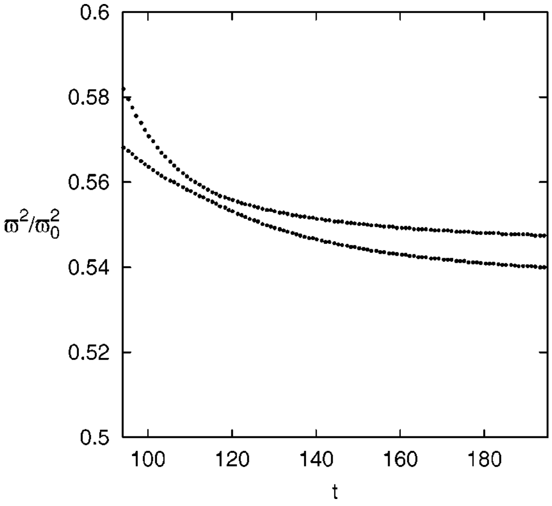

FIG. 4. Squared mode frequencies of internal waves decreasing with temperature, and an avoided level crossing for a Bose gas in an axially symmetric trap with the same parameters as in Fig. 1.

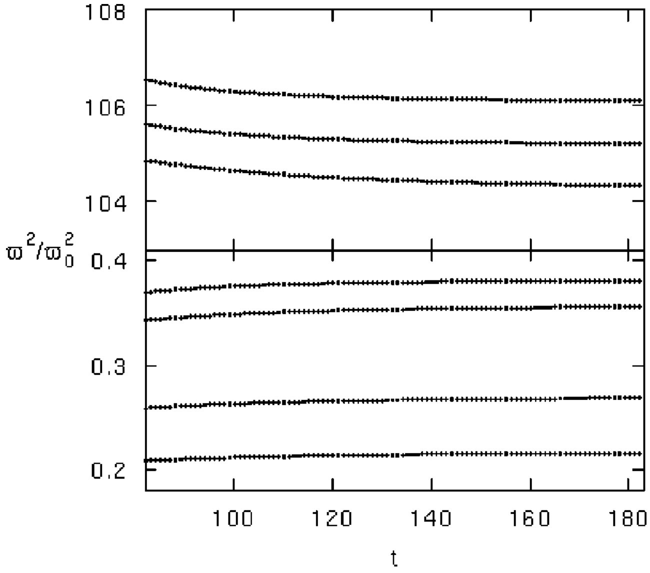

FIG. 5. Squared mode frequencies for a fermionic gas above the Fermi temperature as function of $t = k_B T/\hbar\overline{\omega}$ for sound modes (upper part) and for internal wave modes (lower part).

By these considerations we arrive at the following assignment of quantum numbers to the frequencies shown in Fig. 1. The mode with the largest frequency forms the band on the top, and must be a sound mode with $n_z = 10, n_\rho = 0$. The next lower band must contain two soundlike modes with $n_z = 9, n_\rho = 1$ and $n_z = 9, n_\rho = 0$. We see from the figure that this exhausts the number of modes (2) in the second band from the top, which can therefore not contain an internal wave mode. Similarly, in the third, fourth, fifth, etc. bands from the top, there are 3,4,5, etc. soundlike modes with $n_z = 8,7,6$, etc. and 3,4,5, etc. different values of n_ρ, respectively. In the figure we can follow this counting of different frequencies down to the third band from the bottom, thereby accounting for all eigenvalues in these bands. It follows that none of these bands contains any internal wave modes.

On the other hand, the lowest two bands, in particular the lowest one, contain many more different frequencies than the ten and 11 different sound-frequencies with $n_z = 1$ and 0, respectively. These must therefore be considered internal waves. Among the internal waves are also zero-frequency modes. For each even value of $n = n_z + n_\rho \geqslant 1$ in our subspace there is precisely one zero-frequency mode, so the eigenvalue $\omega^2 = 0$ is fivefold degenerate in our subspace. By their special nature the zero-frequency modes have quantum numbers $n_z = n_\rho$. Internal modes of high $n = 2(n_z + n_\rho)$, for which n_ρ differs only slightly from n_z, have frequencies close to zero

The discussion of the spectrum we have given depends on our arbitrary restriction of the number of nodal surfaces to a finite and not very large number. If modes with an arbitrary number of nodal surfaces are permitted, then the mode spectrum becomes dense, due to the existence of small internal wave frequencies from modes with arbitrarily high quantum numbers. This can already be seen from the analytically determined spectrum [Eq. (4.13)] for the isotropic case.

Within the subspace of Hilbert space we consider here, there are many internal mode frequencies below the geometric mean trap frequency, as can be seen with more clarity in Fig. 2. Only a small number of sound modes can occur in this regime. The mode frequencies displayed in Figs. 1 and 2 have a surprisingly weak temperature dependence throughout the range considered. For large T this can also be seen from the analytical results. For the sound modes the velocity of sound increases with temperature by roughly $\sim\sqrt{k_BT}$, but the wavelength of any given mode also increases with temperature by $\sim\sqrt{k_BT}$ due to the expansion of the size of the thermal cloud, so that both temperature dependences effectively cancel. However, for internal modes the compensation between the speed of sound and the wavelength does not work in the same way, as can, e.g., be seen from Eq. (3.6). In Fig. 3 we consider a magnification of a part of the frequency spectrum (in this instance for the isotropic case), where one can see an increase with temperature of some frequencies which freely cross other levels (which must have therefore different quantum numbers n,l), which are nearly temperature independent, and which belong to soundlike modes. However, in the axially symmetric case avoided crossings can also be seen, as shown in Fig. 4 for two internal wave modes at small frequency below the geometric mean trap frequency. This indicates that the conserved operator $\mathbf{B}$ of Sec. IV ceases to exist at intermediate-temperatures. It would be futile, therefore, to look for analytical solutions of the spectrum in the intermediate-temperature range, as the system appears to be nonintegrable.

Generally, the differences between the results for fermions and bosons in the region above the degeneracy temperature are qualitatively not very large. One difference due to quantum statistics can be seen in Fig. 5, where the temperature dependence of internal wave modes and sound modes is displayed for a Fermi gas. The frequencies of the internal wave modes curve downwards, and those of the sound modes curve upwards. In the Bose gas case the opposite tendency is found, as shown in Figs. 2 and 3.

VI. CONCLUSIONS

In the present paper we have made a systematic analysis of the hydrodynamic modes of quantum gases in a harmonic trap with general anisotropy in the collision-dominated nondissipative limit. Provided the hydrodynamic limit is applicable, the analysis applies for Fermi gases at all temperatures and for Bose gases at temperatures above Bose-Einstein condensation. Our results extend previous works by allowing for traps with arbitrary anisotropy, and by treating bosons and fermions side by side on an equal footing. In addition to analytical solutions in certain special cases and limits, we also present numerical solutions in the whole temperature domain covered by the theory. Our analysis is based on the reduction of the five conservation laws for the densities of mass, momentum, and energy to two coupled wave equations for the mass-density and the pressure. We have constructed a scalar product with a positive L_2 norm on the space of solutions, in which the two-component wave operator is Hermitian and, because of the stability of the hydrodynamic modes which we demonstrate, non-negative. However, a class of solutions with vanishing frequencies was found which is a consequence of the existence of mechanical equilibrium states in addition to the unique thermodynamic equilibrium which maximizes the entropy. A further class of exact solutions consisting of isothermal modes was identified among which are the center-of-mass modes required by the Kohn theorem. These results generalize earlier results by Griffin *et al.* [5] and Bruun and Clark [7] to traps without axial symmetry.

We studied the two coupled wave equations for pressure and density in the short-wavelength limit. Two different branches of solutions could be identified in this way: the high-frequency branch being associated with pressure-driven sound waves, the low-frequency branch with potential-driven internal waves. The explicit dispersion relation of the lower branch found in the short-wavelength limit, and its characteristic properties—like the existence of a maximal frequency, the anisotropy of the dispersion relation, and the orthogonality of the local group velocity and the local wave vector—makes the identification with internal waves manifest and unambiguous. We also examined the high-temperature limit of the two coupled wave equations, and demonstrated the existence of polynomial solutions by exhibiting a conserved operator whose eigenvalues fix the respective polynomial order, or, equivalently, the number of nodal surfaces of the solutions. We constructed the solutions for density and pressure also explicitly.

The wave equations were finally also solved numerically. Surprisingly, the mode spectrum found turned out to be quasicontinuous, which can be understood by the overlap of the spectrum of low-frequency internal waves of short wavelengths with the spectrum of sound waves with large wavelengths. Within a finite-dimensional subspace of the Hilbert space, defined by restricting from above the number of nodal surfaces, we found a bandlike structure of the eigenfrequencies in a strongly anisotropic trap, where the bands are labeled by the number of nodal surfaces orthogonal to the more strongly confined direction, while the modes within a given band differ by the number of nodes in the other directions. The number of internal wave modes in such a finite-dimensional subspace is also limited, and these modes are then found primarily in the lowest-lying band. We note that the restriction to a finite subspace of the Hilbert space is also physically motivated, since typical excitation mechanisms, such as the modulation of the trapping potential, will also only excite modes in a certain subspace with appreciable amplitude.

The analysis we presented is subject to some obvious limitations which we now discuss briefly. It is clear that only systems in the collision-dominated hydrodynamic limit have been considered here. For bosons a necessary requirement is therefore a large scattering length and a sufficiently large number density to ensure a large cross section for elastic collisions. For fermions this requires, besides a large number density, the simultaneous trapping of several hyperfine states, in order to allow for the interaction of different fermionic species by elastic collisions, which would be forbidden for a single species by the Pauli principle. Throughout we have neglected spin-wave excitations, which can also occur in the latter systems; this is permissible, because they decouple from the density waves by symmetry as long as the external potential is the same for all components. The collision rates in degenerate Fermi gases are suppressed by a Fermi-blocking factor compared to the classical collision rates [21], and scale in proportion to $(T/T_F)^2$. Therefore, at least in the low-temperature domain, it is necessary to use atomic species with particularly large positive or negative *s*-wave scattering lengths. Another limitation of our analysis is the neglect of mean-field effects of the interaction in comparison to the pressure term. This seems to be a rather good approximation for the experimentally realized trapped quantum gases, which behave like ideal quantum gases to a good approximation. The most severe limitation of our calculations is certainly the neglect of dissipation. The reason for this restriction (which was discussed further in Ref. [5] for bosons, and in Refs. [7,9] for fermions) lies not so much in the negligibility of dissipative effects for the physically excited modes, but in the particular purpose we set out to achieve in this paper, namely, to give an account of the mode spectrum in the whole temperature domain covered by the theory. This goal cannot be achieved so far with the inclusion of damping effects, but remains an interesting aim for future work. It seems clear that, with the inclusion of damping, the zero-frequency modes we have found will turn into purely overdamped modes. However, as exemplified for the Kohn modes, the phenomenological theory behind Eq. (1.6) also permits one to obtain a result for the damping of some modes (with the result that it is vanishing for the Kohn modes) whose hydrodynamic frequencies are only known in the absence of dissipation [11,12]. The requirement is that mode frequencies without damping are also known in the collisionless limit. Then an estimate of the collision time can be used in Eq. (1.6) to obtain an interpolation between the collision-dominated regime and the collisionless regime, including the damping due to the finite value of the collision time [11,12]. In the past this estimate proved to be quite useful in a comparison of the experimental results for bosons [22], and it may be hoped that further results along such lines, not only for bosons but also for rapidly developing experiments on trapped Fermi gases, may be obtained in the near future.

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