Resummation of the divergent perturbation series for a hydrogen atom in an electric field

Ulrich D. Jentschura*
Laboratoire Kastler-Brossel, Unité de Recherche associée au CNRS No. C8552, Case 74, Université Pierre et Marie Curie, F-75252 Paris Cedex 05, France
National Institute of Standards and Technology, Mail Stop 8401, Gaithersburg, Maryland 20899-8401
Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany
(Received 7 August 2000; revised manuscript received 7 December 2000; published 31 May 2001)

We consider the resummation of the perturbation series describing the energy displacement of a hydrogenic bound state in an electric field (known as the Stark effect or the LoSurdo-Stark effect), which constitutes a divergent formal power series in the electric-field strength. The perturbation series exhibits a rich singularity structure in the Borel plane. Resummation methods are presented that appear to lead to consistent results even in problematic cases where isolated singularities or branch cuts are present on the positive and negative real axis in the Borel plane. Two resummation prescriptions are compared: (i) a variant of the Borel-Padé resummation method, with an additional improvement due to utilization of the leading renormalon poles, and (ii) a contour-improved combination of the Borel method with an analytic continuation by conformal mapping, and Padé approximations in the conformal variable. The singularity structure in the case of the LoSurdo-Stark effect in the complex Borel plane is shown to be similar to (divergent) perturbative expansions in quantum chromodynamics.

DOI: 10.1103/PhysRevA.64.013403 PACS number(s): 32.60.+i, 32.70.Jz, 11.15.Bt, 11.10.Jj

I. INTRODUCTION

Consider the energy shift of the ground state of atomic hydrogen in an electric field of field strength F that we assume to lie along the z axis: the energy displacement can be expressed in perturbation theory as a formal power series. The first nonvanishing perturbation [in atomic units, see also Eq. (29) below] is the second-order effect

$$F^2 \sum_{m \neq 1S} \frac{\langle \phi_{1S}|z|\phi_m\rangle\langle \phi_m|z|\phi_{1S}\rangle}{E_{1S}-E_m},$$

where the sum over m runs over the entire spectrum, including the continuum, but excluding the $1S$ ground state, and E_m is the nonrelativistic (Schrödinger) energy of the mth state. A well known, but perhaps surprising result says that the coefficients of the terms of order F^4, F^6, F^8, ... grow so rapidly that the series in F ultimately diverges, irrespective of how small the field strength. The convergence radius of the factorially divergent perturbation series is zero. The resummation of the divergent series is problematic in the considered case, because the Borel transform, from which the physically correct, finite result is obtained by evaluating the Laplace-Borel integral—see Eq. (11) in Sec. III below—exhibits a rich singularity structure in the complex plane.

The purpose of this paper is to present numerical evidence that divergent perturbation series whose Borel transform exhibits a rich singularity structure in the complex plane, can be resummed to the complete, physically relevant result. The resummation methods use as input data only a finite number of perturbative coefficients. Problematic singularities on the positive real axis in the Borel plane are treated by appropriate integration prescriptions. In particular, it is shown that the Borel transform of the divergent perturbation series for the LoSurdo-Stark effect involves two cuts in the Borel plane, generated essentially by the divergent alternating and nonalternating subcomponents of the perturbation series. This singularity structure is also expected of the (divergent) perturbation series in quantum field theory, notably quantum chromodynamics (in this case, the alternating and nonalternating factorially divergent components correspond in their mathematical structure to the ultraviolet and infrared renormalons[1]).

We present results that suggest that the integration contours and resummation techniques discussed here may be of relevance, at least in part, to theories with degenerate minima. As a byproduct of these investigations, numerical pseudoeigenvalues are obtained for the LoSurdo-Stark effect; selected field strengths and atomic states are considered.

The LoSurdo-Stark effect and its associated divergent perturbative expansion, including the the nonperturbative, nonanalytic imaginary contributions, have attracted considerable attention, both theoretically and experimentally [1–34]. Experiments have been performed in field strengths up to a couple of MV/cm [35–38]. One might be tempted to say that the autoionization decay width could be interpreted as a paradigmatic example for a nonperturbative effect that exhibits fundamental limitations to the validity of perturbation theory (unless the perturbative expansion is combined with

*Electronic address: ulj@nist.gov

[1]The term ''renormalon,'' as now commonly used in particle physics and large-order perturbation theory, stands for a factorially divergent subcomponent of a perturbation series. In quantum field theory, this divergent subcomponent is associated with a specific class of Feynman diagrams (for example, so-called ''bubble diagrams''). For a comprehensive discussion, see M. Beneke, Phys. Rep. **317**, 1 (1999) and references therein. ''Bubble diagrams'' are illustrated in Fig. 1 *ibid.*

resummation methods). We briefly summarize here: The Rayleigh-Schrödinger perturbation series for the LoSurdo-Stark effect [39,40] can be formulated to arbitrarily high order [1]. The perturbative coefficients grow factorially in absolute magnitude [2], and the radius of convergence of the perturbation series about the origin is zero. The perturbation series is a divergent, asymptotic expansion in the electric-field strength F, i.e., about zero electric field. This means that the perturbative terms at small coupling first decrease in absolute magnitude up to some minimal term. After passing through the minimal term, the perturbative terms increase again in magnitude, and the series ultimately diverges.

By the use of a *resummation* method, it is possible to assign a finite value to an otherwise divergent series, and various applications of resummation methods in mathematics and physics have been given, e.g., in Refs. [41–45]. When a divergent series is resummed, the superficial precision limit set by the minimal term can be overcome, and more accurate results can be obtained as compared to the optimal truncation of the perturbation series (see also the numerical results in the tables of Ref. [46]). The divergent perturbation series of the LoSurdo-Stark effect has both alternating and nonalternating components (as explained in Sec. II below). The resummation of nonalternating series or of a series that have a leading or subleading divergent nonalternating component, corresponds to a resummation ''on the cut'' in the complex plane [41,42].

Rather mathematically motivated investigations regarding the Borel summability of the divergent perturbation series for the LoSurdo-Stark effect were performed in Refs. [12,47], and it was established that the perturbation series of the LoSurdo-Stark effect is Borel summable in the distributional sense (for the definition of ''distributional Borel summability'' we refer to Ref. [48]). Here, to supplement the mathematically motivated investigations, we consider the calculation of transforms of the divergent series, which use as input data only a finite number of perturbative coefficients and exhibit apparent convergence to the complete, physically relevant result.

In the remarkable investigation [25], whose significance may not have been sufficiently noticed in the field of large-order perturbation theory, it was not only shown that it is possible to perform the required analytic continuation of the Borel transform beyond its radius of convergence by employing Padé approximants, but that it is also possible to reconstruct the full physical result, including the imaginary contribution that corresponds to the autoionization decay width, by integration in the complex plane.

In Sec. II, we discuss the singularity structure of the Borel transform in the complex plane. The structure of a doubly cut plane has been postulated for quantum chromodynamic perturbation series [49,50], and this structure has been exploited to devise resummation prescriptions based on conformal mappings [49–55]. Here, we present results that suggest that the convergence of the transforms obtained by conformal mapping can be improved if Padé approximants in the conformal variable are used (see also Ref. [55]). We also discuss improvements of the ''pure'' Borel-Padé method (these additional improvements take advantage of leading renormalon poles). Also, in comparison to the investigation [25], we use here a slightly modified, but equivalent integration contour for the evaluation of the generalized Borel integral (see Ref. [46] and Sec. III below). Our version of the integration contour exhibits the additional terms that have to be added to the otherwise recommended principal-value prescription [49,50,56].

As stressed above, it has been another main motivation for the current investigation to establish the singularity structure of the Borel transform in the complex plane, and to demonstrate the analogy of the singularity structure of the perturbation series for the LoSurdo-Stark effect to quantum chromodynamic perturbation series. We also consider a divergent perturbation series generated by a model problem for theories with degenerate minima. In the particular model case discussed here, a perturbation series with real coefficients is summed to a *real* result—in contrast to the LoSurdo-Stark effect, there is no imaginary part involved in this case. One of the three alternative integration contours introduced in Ref. [46] has to be employed.

This paper is organized as follows: In Sec. II, we give a brief outline of the perturbative expansion for the LoSurdo-Stark effect. In Secs. III and IV, we describe the resummation methods that are used to obtain the numerical results presented in Sec. V. In Sec. VI, we consider theories with degenerate minima. We conclude with a summary of the results in Sec. VII. Finally, the connection of the current paper to quantum-field-theoretic perturbation series and to double-well oscillators are discussed in the Appendixes A and B.

II. PERTURBATION SERIES FOR THE LoSURDO-STARK EFFECT

In the presence of an electric field, the SO(4) symmetry of the hydrogen atom is broken, and parabolic quantum numbers n_1, n_2, and m are used for the classification of the atomic states [57]. For the Stark effect, the perturbative expansion of the energy eigenvalue $E(n_1,n_2,m,F)$ reads [see Eq. (59) of Ref. [1]],

$$E(n_1,n_2,m,F) \sim \sum_{N=0}^{\infty} E^{(N)}_{n_1 n_2 m} F^N, \tag{1}$$

where F is the electric-field strength. For $N\to\infty$, the leading large-order factorial asymptotics of the perturbative coefficients have been derived in Ref. [16] as

$$E^{(N)}_{n_1 n_2 m} \sim A^{(N)}_{n_1 n_2 m} + (-1)^N A^{(N)}_{n_2 n_1 m}, \quad N\to\infty, \tag{2}$$

where $A^{(N)}_{n_i n_j m}$ is given as an asymptotic series,

$$A^{(N)}_{n_i n_j m} \sim K(n_i,n_j,m,N) \times \sum_{k=0}^{\infty} a_k^{n_i n_j m} (2\,n_j + m + N - k)!. \tag{3}$$

The quantities $a_k^{n_i n_j m}$ are constants. The K coefficients in Eq. (3) are given by

$$K(n_i,n_j,m,N) = -[2\pi n^3 n_j!(n_j+m)!]^{-1} \times \exp\{3(n_i-n_j)\}6^{2n_j+m+1}(3n^3/2)^N. \quad (4)$$

Here, the principal quantum number n as a function of the parabolic quantum numbers n_1, n_2, and m is given by [see Eq. (65) in Ref. [1]]

$$n = n_1 + n_2 + |m| + 1. \quad (5)$$

According to Eq. (2), the perturbative coefficients $E^{(N)}_{n_1 n_2 m}$, for large-order $N\to\infty$ of perturbation theory, can be written as a sum of a nonalternating factorially divergent series [first term in Eq. (2)] and of an alternating factorially divergent series [second term in Eq. (2)]. Because the $a_k^{n_i n_j m}$ in Eq. (3) are multiplied by the factorial $(2n_i+m+N-k)!$, we infer that for large perturbation-theory order N, the term related to the $a_0^{n_i n_j m}$ coefficient $(k=0)$ dominates. Terms with $k\geq 1$ are suppressed in relation to the leading term by a relative factor of $1/N^k$ according to the asymptotics

$$\frac{(2n_j+m+N-k)!}{(2n_j+m+N)!} \sim \frac{1}{N^k}\left[1+O\left(\frac{1}{N}\right)\right] \quad (6)$$

for $N\to\infty$. The leading $(k=0)$ coefficient has been evaluated in Ref. [2] as

$$a_0^{n_i n_j m} = 1. \quad (7)$$

According to Eqs. (2), (3), and (7), for states with $n_1<n_2$, the nonalternating component of the perturbation series dominates in large order of perturbation theory, whereas for states with $n_1>n_2$, the alternating component is dominant as $N\to\infty$. For states with $n_1=n_2$, the odd-N perturbative coefficients vanish [16], and the even-N coefficients necessarily have the same sign in large order [see Eq. (2)]. According to Eq. (2), there are subleading divergent nonalternating contributions for states with $n_1>n_2$, and there exist subleading divergent alternating contributions for states with $n_1<n_2$. This complicates the resummation of the perturbation series.

III. BOREL-PADÉ RESUMMATION

The resummation of the perturbation series (1) by a combination of the Borel transformation and Padé approximants proceeds as follows. First we define the parameter

$$\lambda = 2\max(n_1,n_2) + m + 1. \quad (8)$$

The large-order growth of the perturbative coefficients [see Eqs. (2) and (3)] suggests the definition of the (generalized) Borel transform [see Eq. (4) in Ref. [58]]

$$E_{\rm B}(z) \equiv E_{\rm B}(n_1,n_2,m,z) = \mathcal{B}^{(1,\lambda)}[E(n_1,n_2,m);z] = \sum_{N=0}^{\infty} \frac{E^{(N)}_{n_1 n_2 m}}{\Gamma(N+\lambda)} z^N, \quad (9)$$

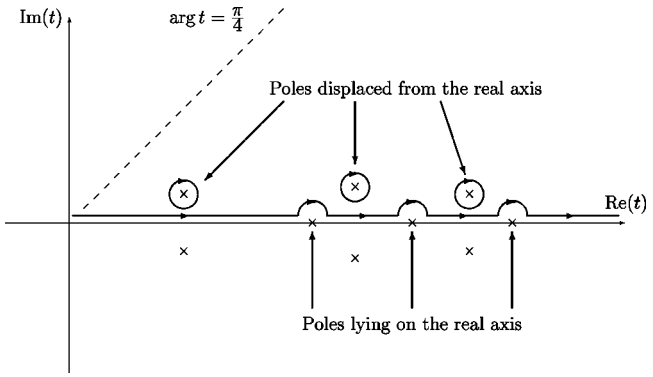

FIG. 1. Integration contour C_{+1} for the evaluation of the generalized Borel integral defined in Eq. (11). Poles displaced from the real axis are evaluated as full poles, whereas those poles that lie on the real axis are treated as half poles.

where we consider the argument z of $E_{\rm B}(z)$ as a complex variable and λ is defined in Eq. (8). The additive constant (in this case λ) in the argument of the Gamma function is chosen in accordance with the notion of an ''asymptotically improved'' resummation (see also Ref. [58]). It is observed that the additive constant λ can be shifted by a small integer without affecting the convergence of the Borel resummed series. Because the perturbative coefficients $E^{(N)}_{n_1 n_2 m}$ diverge factorially in absolute magnitude, the Borel transform $E_{\rm B}(z)$ has a finite radius of convergence about the origin. The evaluation of the (generalized) Laplace-Borel integral [see Eq. (11) below] therefore requires an analytic continuation of $E_{\rm B}(z)$ beyond the radius of convergence. The ''classical'' Borel integral is performed in the z range $z\in(0,\infty)$, i.e., along the positive real axis [see, e.g., Eqs. (8.2.3) and (8.2.4) of Ref. [41]]. It has been suggested in [56] that the analytic continuation of Eq. (9) into regions where F retains a nonvanishing, albeit infinitesimal, imaginary part can be achieved by evaluating Padé approximants. Using the first $M+1$ terms in the power expansion of the Borel transform $E_{\rm B}(z)$, we construct the Padé approximant (we follow the notation of Ref. [43])

$$\mathcal{P}_M(z) = [[\![M/2]\!]/[\![(M+1)/2]\!]]_{E_{\rm B}}(z), \quad (10)$$

where $[\![x]\!]$ denotes the largest positive integer smaller than x. We then evaluate the (modified) Borel integral along the integration contour C_{+1} shown in Fig. 1 in order to construct the transform $\mathcal{T}E_M(F)$ where

$$\mathcal{T}E_M(F) = \int_{C_{+1}} dt\, t^{\lambda-1} \exp(-t)\mathcal{P}_M(Ft). \quad (11)$$

The successive evaluation of transforms $\mathcal{T}E_M(F)$ in increasing transformation order M is performed, and the apparent convergence of the transforms is examined. This procedure is illustrated in Tables I and II of Ref. [46]. In the current evaluation, a slightly modified scheme is used for selecting the poles in the upper-right quadrant of the complex plane as compared to Ref. [46].

The contour C_{+1} is supposed to encircle all poles at $t=z_i$ in the upper-right quadrant of the complex plane with $\arg z_i < \pi/4$ in the mathematically negative sense. That is to say, the contribution of all poles z_i with $\mathrm{Re}\, z_i > 0$, $\mathrm{Im}\, z_i > 0$ and $\mathrm{Re}\, z_i > \mathrm{Im}\, z_i$,

$$-2\pi\,\mathrm{i}\sum_i \operatorname*{Res}_{t=z_i} t^{\lambda-1}\exp(-t)\mathcal{P}_M(Ft),$$

is added to the principal value (P.V.) of the integral (11) carried out in the range $t\in(0,\infty)$. Note the further restriction ($\mathrm{Im}\, z_i < \mathrm{Re}\, z_i$ or equivalently $\arg z_i < \pi/4$) with regard to the selection of poles in comparison to the previous investigation [46]. In practical calculations, this modification is observed not to affect the numerical values of the transforms $\mathcal{T}E_M(F)$ defined in Eq. (11) in higher transformation order $M \geqslant 10$ [i.e., for large M, see also Eq. (14) below], because the poles are observed to cluster near the real axis in higher transformation order, and so the contribution of poles with $\pi/4 < \arg z_i < \pi/2$ gradually vanishes. We have,

$$\mathcal{T}E_M(F) = \mathrm{P}\int_0^\infty \mathrm{d}t\, t^{\lambda-1}\exp(-t)\mathcal{P}_M(F\,t) - 2\pi\mathrm{i}\sum_i \operatorname*{Res}_{t=z_i} t^{\lambda-1}\exp(-t)\mathcal{P}_M(Ft). \quad (12)$$

The principal-value prescription [first term in Eq. (12)] for the evaluation of the Laplace-Borel integral has been recommended in Refs. [56,59]. This prescription leads to a real (rather than complex) result for the energy shift and cannot account for the width of the quasistationary state. The additional pole contributions [second term in Eq. (12)] are responsible for complex-valued (imaginary) corrections that lead, in particular, to the decay width.

By contour integration (Cauchy Theorem) and Jordan's Lemma, one can show that the result obtained along C_{+1} is equivalent to an integration along the straight line with $\arg z = \pi/4$,

$$\mathcal{T}E_M(F) = c^\lambda \int_0^\infty \mathrm{d}t\, t^{\lambda-1}\exp(-c\,t)\mathcal{P}_M(Fct), \quad (13)$$

where $c=\exp(i\pi/4)$. This contour has been used in Ref. [25] (see also p. 815 in Ref. [60]). The factor $\exp(-c\,t)$ and the asymptotic behavior of the Padé approximant $\mathcal{P}_M(F\,c\,t)$ as $t\to\infty$ together ensure that the integrand falls off sufficiently rapidly so that the Cauchy Theorem and Jordan's Lemma can be applied to show the equivalence of the representations (12) and (13).

The representation (13) illustrates the fact that the integration in the complex plane along C_{+1} analytically continues the resummed result in those cases where the evaluation of the standard Laplace-Borel integral is not feasible due to poles on the real axis. The representations (11) and (12) serve to clarify the role of the additional terms that have to be added to the result obtained by the principal-value prescription in order to obtain the full physical result, including the nonperturbative, nonanalytic contributions. Note that, as stressed in Ref. [46], the pole contributions in general do not only modify the imaginary, but also the real part of the resummed value for the perturbation series.

Formally, the limit of the sequence of the $\mathcal{T}E_M(F)$ as $M\to\infty$, provided it exists, yields the nonperturbative result inferred from the perturbative expansion (1),

$$\lim_{M\to\infty}\mathcal{T}E_M(F) = E(F) \equiv E(n_1,n_2,m,F). \quad (14)$$

Because the contour C_{+1} shown in Fig. 1 extends into the complex plane, the transforms $\mathcal{T}E_M(F)$ acquire an imaginary part even though the perturbative coefficients in Eq. (1) are real.

In the context of numerical analysis, the concept of incredulity [61] may be used for the analysis of the convergence of the transforms $\mathcal{T}E_M(F)$ of increasing order M. If a certain number of subsequent transforms exhibit apparent numerical convergence within a specified relative accuracy, then the calculation of transforms is stopped, and the result of the last calculated transformation is taken as the numerical limit of the series under investigation. It has been observed in Refs. [46,56] that for a number of physically relevant perturbation series, the apparent numerical convergence of the transforms (11), with increasing transformation order, leads to the physically correct results.

It is observed that the rate of convergence of the transforms (11) can be enhanced if instead of the unmodified Padé approximants (10) leading renormalon poles are explicitly used for the construction of modified approximants. For the ground state, this entails the following replacement in Eq. (11):

$$\mathcal{P}_M(z)\to\mathcal{P}'_M(z),$$

where

$$\mathcal{P}'_M(z) = \frac{1}{1-z^2}\left[\left[\frac{M+4}{2}\right]\Big/\left[\frac{M-3}{2}\right]\right]_{E'_{\mathrm{B}}(\zeta)}(z), \quad (15)$$

where $E'_{\mathrm{B}}(\zeta)=(1-\zeta^2)E_{\mathrm{B}}(\zeta)$. For the excited state with quantum numbers $n_1=3$, $n_2=0$, and $m=1$, we replace

$$\mathcal{P}_M(z)\to\mathcal{P}''_M(z),$$

where

$$\mathcal{P}''_M(z) = \frac{1}{1-z}\left[\left[\frac{M+2}{2}\right]\Big/\left[\frac{M-1}{2}\right]\right]_{E''_{\mathrm{B}}}(z), \quad (16)$$

where $E''_{\mathrm{B}}(\zeta)=(1-\zeta^2)E_{\mathrm{B}}(\zeta)$. The resummation method by a combination of Borel and Padé techniques—current section—will be referred to as ''method I'' throughout the current paper.

IV. DOUBLY-CUT BOREL PLANE AND RESUMMATION BY CONFORMAL MAPPING

According to Eqs. (2) and (3), the perturbative coefficient $E^{(N)}_{n_1n_2m}$, for large N, can be written as the sum of an alternating and of a nonalternating divergent series. In view of

Eqs. (4) and (7), we conclude that the series defined in Eq. (9),

$$E_{\rm B}(z)=\sum_{N=0}^{\infty}\frac{E^{(N)}_{n_1n_2m}}{\Gamma(N+\lambda)}z^N,$$

has a radius of convergence

$$s=\frac{2}{3n^3} \tag{17}$$

about the origin, where n is the principal quantum number [see Eq. (5)]. Therefore, the function

$$\mathcal{E}_{\rm B}(w)=\sum_{N=0}^{\infty}\frac{E^{(N)}_{n_1n_2m}s^N}{\Gamma(N+\lambda)}w^N, \tag{18}$$

has a unit radius of convergence about the origin. It is not *a priori* obvious if the points $w=-1$ and $w=+1$ represent isolated singularities or branch points. The asymptotic properties (2) and (3) together with Eq. (6) suggest that the points $w=-1$ and $w=+1$ do not constitute poles of finite order. We observe that the leading factorial growth of the perturbative coefficients in large perturbation order N is divided out in the Borel transform (18), which is a sum over N. The perturbative coefficient $E^{(N)}_{n_1n_2m}$ can be written as an asymptotic series over k [see Eq. (3)]. We interchange the order of the summations over N and k, we use Eq. (6) and take advantage of the identity

$$\sum_{N=0}^{\infty}\frac{w^N}{N^k}={\rm Li}_k(w). \tag{19}$$

The Borel transform $\mathcal{E}_{\rm B}(w)$ can then be written as a sum over terms of the form $T_k(w)$ where for $k\to\infty$,

$$T_k(w)\sim C(n_i,n_j,m)a_k^{n_in_jm}\ {\rm Li}_k(w). \tag{20}$$

The coefficient $C(n_i,n_j,m)$ is given by

$$C(n_i,n_j,m)=-[2\pi n^3 n_j!(n_j+m)!]^{-1}\times\exp\{3\,(n_i-n_j)\}6^{2\,n_j+m+1}. \tag{21}$$

These considerations suggest that the points $w=-1$ and $w=+1$ represent essential singularities (in this case, branch points) of the Borel transform $\mathcal{E}_{\rm B}(w)$ defined in Eq. (18). For the analytic continuation of $\mathcal{E}_{\rm B}(w)$ by conformal mapping, we write w as

$$w=\frac{2y}{1+y^2}, \tag{22}$$

(this conformal mapping preserves the origin of the complex plane). Here, we refer to w as the Borel variable, and we call y the conformal variable. We then express the Mth partial sum of Eq. (18) as

$$\mathcal{E}^M_{\rm B}(w)=\sum_{N=0}^{M}\frac{E^{(N)}_{n_1n_2m}s^N}{\Gamma(N+\lambda)}w^N=\sum_{N=0}^{M}C_Ny^N+\mathcal{O}(y^{M+1}), \tag{23}$$

where the coefficients C_N are uniquely determined [see, e.g., Eqs. (36) and (37) of Ref. [49]]. We define the partial sum of the Borel transform, re-expanded in terms of the conformal variable y, as

$$\mathcal{E}'^{M}_{\rm B}(y)=\sum_{N=0}^{M}C_Ny^N. \tag{24}$$

We then evaluate (lower-diagonal) Padé approximants to the function $\mathcal{E}'^{M}_{\rm B}(y)$,

$$\mathcal{E}''^{M}_{\rm B}(y)=[[M/2]/[[(M+1)/2]]_{\mathcal{E}'^{M}_{\rm B}}(y). \tag{25}$$

We define the following transforms,

$$\mathcal{T}''E_M(F)=s^\lambda\int_{C_{+1}}{\rm d}w\ w^{\lambda-1}\ \exp(-w)\mathcal{E}''^{M}_{\rm B}[y(w)]. \tag{26}$$

At increasing M, the limit as $M\to\infty$, provided it exists, is then again assumed to represent the complete, physically relevant solution,

$$E(F)=\lim_{M\to\infty}\mathcal{T}''E_M(F). \tag{27}$$

We do not consider the question of the existence of this limit here (for an outline of questions related to these issues we refer to Ref. [50]; potential problems at excessively strong coupling are discussed in Sec. II C of Ref. [62]).

Inverting Eq. (22) yields [see Eq. (26)]

$$y(w)=\frac{\sqrt{1+w}-\sqrt{1-w}}{\sqrt{1+w}+\sqrt{1-w}}. \tag{28}$$

The conformal mapping given by Eqs. (22) and (28) maps the doubly cut w plane with cuts running from $w=1$ to $w=\infty$ and $w=-1$ to $w=-\infty$ unto the unit circle in the complex y plane. The cuts themselves are mapped to the edge of the unit circle in the y plane.

In comparison to the investigations [49] and [50], we use here a different conformal mapping defined in Eqs. (22) and (28) which reflects the different singularity structure in the complex plane [cf. Eq. (27) in Ref. [49]]. We also mention the application of Padé approximants for the numerical improvement of the conformal mapping performed according to Eq. (25). In comparison to a recent investigation [55], where the additional Padé improvement in the conformal variable is also used, we perform here the analytic continuation by a mapping whose structure reflects the double cuts suggested by the asymptotic properties of the perturbative coefficients given in Eqs. (2), (3), and (6) [cf. Eq. (5) in Ref. [55]].

The method introduced in this section will be referred to as ''method II.'' It is one of the motivations for the current paper to contrast and compare the two methods I and II. A

comparison of different approaches to the resummation problem for the series with both alternating and nonalternating divergent components appears useful, in part because the conformal mapping (without further Padé improvement) has been recommended for the resummation of quantum chromodynamic perturbation series [49,50].

We do not consider order-dependent mappings here [51–54]. For an order-dependent mapping to be constructed, the conformal mapping in Eq. (22) has to be modified, and a free parameter ρ has to be introduced. The coefficients C_N in the accordingly modified Eq. (24) then become ρ dependent. The free parameter ρ is chosen so that the ρ-dependent coefficient $C_M(\rho)$ of order M vanishes. Consequently, the choice of ρ depends on the order M of perturbation theory, and in this way, the mapping becomes order dependent. Certain complications arise because ρ cannot be chosen arbitrarily, but has to be selected, roughly speaking, as the first zero of the ρ-dependent coefficient $C_M(\rho)$ for which the absolute magnitude of the derivative $C'_M(\rho)$ is small (this is explained in Ref. [60], p. 886). It is conceivable that with a judicious choice of ρ, further acceleration of the convergence can be achieved, especially when the order-dependent mapping is combined with a Padé approximation as it is done here in Eq. (25) with our order-*independent* mapping. In the current paper, we restrict the discussion to the conformal order-independent mapping (22) that is nevertheless optimal in the sense discussed in Refs. [49,50].

V. NUMERICAL CALCULATIONS

In this section, the numerical results based on the resummation methods introduced in Secs. III and IV are presented. Before we describe the calculation in detail, we should note that relativistic corrections to both the real and the imaginary part of the energy contribute at a relative order of $(Z\alpha)^2$ compared to the leading nonrelativistic effect that is treated in the current paper (and in the previous work on the subject, see, e.g., Refs. [16,25]). Therefore, the theoretical uncertainty due to relativistic effects can be estimated to be, at best, 1 part in 10^4 (for an outline of the relativistic and quantum electrodynamic corrections in hydrogen see Refs. [63–69]). Measurements in very high fields are difficult [35]. At the achievable field strengths to date (less than 0.001 a.u. or about 5 MV/cm), the accuracy of the theoretical prediction exceeds the experimental precision, and relativistic effects do not need to be taken into account.

The perturbative coefficients $E^{(N)}_{n_1 n_2 m}$ defined in Eq. (1) for the energy shift can be inferred, to arbitrarily high order, from the Eqs. (9), (13–15), (28–33), (59–67), and (73) in Ref. [1]. The atomic unit system is used in the sequel, as is customary for this type of calculation [1,6,9,11]. The unit of energy is $\alpha^2 m_e c^2 = 27.211$ eV where α is the fine-structure constant, and the unit of the electric field is the field strength felt by an electron at a distance of one Bohr radius $a_{\rm Bohr}$ to a nucleus of elementary charge, which is $1/(4\pi\epsilon_0)(e/a^2_{\rm Bohr}) = 5.142\times 10^3$ MV/cm (here, ϵ_0 is the permittivity of the vacuum).

We consider the resummation of the divergent perturbative expansion (1) for two states of atomic hydrogen. These are the ground state ($n_1=n_2=m=0$) and an excited state with parabolic quantum numbers $n_1=3$, $n_2=0$, $m=1$. We list here the first few perturbative coefficients for the states under investigation. For the ground state, we have (in atomic units),

$$E_{000}(F) = -\frac{1}{2} - \frac{9}{4}F^2 - \frac{3\,555}{64}F^4 - \frac{2\,512\,779}{512} \times F^6 - \frac{13\,012\,777\,803}{16\,384}F^8 + \cdots . \quad (29)$$

The perturbation series for the state $n_1=3$, $n_2=0$, $m=1$ is alternating, but has a subleading nonalternating component [see Eq. (2)]. The first perturbative terms read

$$E_{301}(F) = -\frac{1}{50} + \frac{45}{2}F - \frac{31\,875}{2}F^2 + \frac{54\,140\,625}{4}F^3 - \frac{715\,751\,953\,125}{16}F^4 + \cdots . \quad (30)$$

Note that for $F=0$, the unperturbed nonrelativistic energy is recovered, which is $-1/(2n^2)$ in atomic units. In contrast to the real perturbative coefficients, the energy pseudoeigenvalue (resonance) $E(n_1,n_2,m,F)$ has a real and an imaginary component,

$$E(n_1,n_2,m,F) = \mathrm{Re}\, E_{n_1 n_2 m}(F) - \frac{i}{2}\Gamma_{n_1 n_2 m}(F), \quad (31)$$

where $\Gamma_{n_1 n_2 m}(F)$ is the autoionization width.

Using a computer algebra system [70,71], the first 50 nonvanishing perturbative coefficients are evaluated for the ground state, and for the state with parabolic quantum numbers $n_1=3$, $n_2=0$, $m=1$, we evaluate the first 70 nonvanishing perturbative coefficients. The apparent convergence of the transforms defined in Eqs. (11) and (26) in higher order is examined. In the case of the Borel-Padé transforms defined in Eq. (11), use is made of the replacements in Eqs. (15) and (16) [''leading poles are being put in by hand'']. This procedure leads to the numerical results listed in Tables I and II. The numerical error of our results is estimated on the basis of the highest and lowest value of the four highest-order transforms.

An important result of the comparison of the methods introduced in Secs. III and IV is the following: Both methods appear to accomplish a resummation of the perturbation series to the physically correct result. Method I (Borel+Padé with leading renormalon poles, see Sec. III) and method II (Borel+Padé-improved conformal mapping, see Sec. IV) appear to lead to results of comparable accuracy.

To date, a rigorous theory of the performance of the resummation methods for divergent series of the type discussed in this paper (with alternating and nonalternating components) does not exist. The *logarithmic* singularities introduced by the branch points of higher-order polylogarithms [see the index k in Eq. (19)] are difficult to approximate with

TABLE I. Real and imaginary part of the energy pseudoeigenvalue $E_{000}(F)$ for the ground state of atomic hydrogen (parabolic quantum numbers $n_1=0$, $n_2=0$, $m=0$).

F(a.u.)	Re $E_{000}(F)$	$\Gamma_{000}(F)$
0.04	$-0.503\,771\,591\,013\,654\,2(5)$	$3.892\,699\,990(1)\times10^{-6}$
0.06	$-0.509\,203\,451\,088(2)$	$5.150\,775\,0(5)\times10^{-4}$
0.08	$-0.517\,560\,50(5)$	$4.539\,63(5)\times10^{-3}$
0.10	$-0.527\,419\,3(5)$	$1.453\,8(5)\times10^{-2}$
0.12	$-0.537\,334(5)$	$2.992\,7(5)\times10^{-2}$
0.16	$-0.555\,24(5)$	$7.131(5)\times10^{-2}$
0.20	$-0.570\,3(5)$	$1.212(5)\times10^{-1}$
0.24	$-0.582\,6(1)$	$1.767(5)\times10^{-1}$
0.28	$-0.591\,7(5)$	$2.32(3)\times10^{-1}$
0.32	$-0.600(5)$	$2.92(3)\times10^{-1}$
0.36	$-0.604(5)$	$3.46(3)\times10^{-1}$
0.40	$-0.608(5)$	$4.00(5)\times10^{-1}$

the rational functions employed in the construction of Padé approximants. A solution to the problem of approximating the logarithmic singularities, based on the finite number of perturbative coefficients, would probably lead to further optimizimation of the rate of convergence of the transformed series. Within the current scheme of evaluation, the problematic logarithmic singularities may be responsible, at least in part, for certain numerical instabilities at higher transformation order, e.g., in the result for $\mathcal{T}'E_{70}(F=2.1393\times10^{-4})$ in Eq. (32) below.

For the atomic state with quantum numbers $n_1=3$, $n_2=0$, and $m=1$, the evaluation of the transforms $\mathcal{T}E_M(F)$ defined in Eq. (11) (method I) and of the transforms $\mathcal{T}'E_M(F)$ defined in Eq. (26) (method II) in transformation order $M=67,68,69,70$ for a field strength of $F=2.1393\times10^{-4}$. Method I leads to the following results,

$$\mathcal{T}E_{67}(F=2.1393\times10^{-4})=-0.015\,860\,468\,199\,2 - \mathrm{i}\,0.529\,048\times10^{-6},$$

$$\mathcal{T}E_{68}(F=2.1393\times10^{-4})=-0.015\,860\,468\,200\,9 - \mathrm{i}\,0.529\,047\times10^{-6},$$

$$\mathcal{T}E_{69}(F=2.1393\times10^{-4})=-0.015\,860\,468\,198\,9 - \mathrm{i}\,0.529\,048\times10^{-6},$$

and

$$\mathcal{T}E_{70}(F=2.1393\times10^{-4})=-0.015\,860\,468\,194\,5 - \mathrm{i}\,0.529\,015\times10^{-6}. \quad (32)$$

Method II yields the following data,

$$\mathcal{T}'E_{67}(F=2.1393\times10^{-4})=-0.015\,860\,468\,200\,4 - \mathrm{i}\,0.529\,047\times10^{-6},$$

$$\mathcal{T}'E_{68}(F=2.1393\times10^{-4})=-0.015\,860\,468\,200\,3 - \mathrm{i}\,0.529\,047\times10^{-6},$$

$$\mathcal{T}'E_{69}(F=2.1393\times10^{-4})=-0.015\,860\,468\,200\,4 - \mathrm{i}\,0.529\,047\times10^{-6},$$

and

TABLE II. Real part and imaginary part of the energy pseudoeigenvalue $E_{301}(F)$ for the excited state with parabolic quantum numbers $n_1=3$, $n_2=0$, $m=1$. The field strength F is given in atomic units. The data are compared to Ref. [9]. Discrepancies are observed at large electric-field strength.

	Real part of the resonance Re $E_{301}(F)$		Autoionization decay width $\Gamma_{301}(F)$	
F (a.u.)	Ref. [9]	Our results	Ref. [9]	Our results
1.5560×10^{-4}	$-0.016\,855\,237\,2$	$-0.016\,855\,237\,140\,761\,7(5)$	0.42×10^{-9}	$0.421\,683(5)\times10^{-9}$
1.9448×10^{-4}	$-0.016\,179\,388\,5$	$-0.016\,179\,388\,257\,0(5)$	$0.143\,8\times10^{-6}$	$0.143\,773(5)\times10^{-6}$
2.1393×10^{-4}	$-0.015\,860\,468$	$-0.015\,860\,468\,20(1)$	$0.105\,7\times10^{-5}$	$0.105\,09(5)\times10^{-5}$
2.5282×10^{-4}	$-0.015\,269\,204$	$-0.015\,269\,293(1)$	$0.175\,60\times10^{-4}$	$0.176\,39(5)\times10^{-4}$
2.9172×10^{-4}	$-0.014\,740\,243$	$-0.014\,742\,60(3)$	$0.976\,51\times10^{-4}$	$0.999\,96(9)\times10^{-4}$
3.3061×10^{-4}	$-0.014\,242\,49$	$-0.014\,260\,2(3)$	$0.278\,53\times10^{-3}$	$0.295\,4(2)\times10^{-3}$

$$\mathcal{T}'E_{70}(F=2.1393\times10^{-4}) = -0.015\,860\,468\,203\,3 - \mathrm{i}\,0.529\,046\times10^{-6}. \tag{33}$$

Numerical results obtained by resummation are presented in Tables I and II for a variety of field strengths and for the two atomic states under investigation here. Results are compared to the numerical calculation [9]. which yields very accurate data for all experimentally accessible electric-field strengths to date. In addition, it should be noticed that the inaccuracies at excessively large field of Ref. [9] have been pointed out by the same authors in Ref. [27]. However, not all atomic states considered in Ref. [9] were treated in the later investigation [27]. Our data for the ground state indicated in Table I are consistent with the numerical results obtained in Ref. [27]. However, it should be noted that the later work [27] leaves out the excited state with quantum numbers $n_1=3$, $n_2=0$, and $m=1$ for which results are given here in Table II. To the best of our knowledge, the numerical discrepancy with Ref. [9] for the excited state with quantum numbers $n_1=3$, $n_2=0$, and $m=1$ has not been recorded in the literature. We do not claim here that it would have been impossible to discern this discrepancy with the other methods that have been devised for the theoretical LoSurdo-Stark problem. Notably, it appears likely that the approach from Ref. [27] or the method presented in Ref. [19] could easily be generalized to the particular excited state considered here, and that such a generalization would lead to very accurate results. We merely include Table II here in order to illustrate the utility of the rather unconventional resummation method for the regime of large coupling, where even rather sophisticated numerical methods, which avoid the intricacies of a small-field perturbative expansion, have been shown to be problematic [9,27]. We confirm that the numerical data given in Ref. [9] are accurate up to a field strength of $F\approx0.1$ for the ground state and up to $F\approx3\times10^{-4}$ for the excited ($n=5$) state with $n_1=3$, $n_2=0$, and $m=1$.

VI. MODEL EXAMPLE FOR DEGENERATE MINIMA

We consider the generating functional in a zero-dimensional theory (in this case, the usual path integral reduces to an ordinary integral). First, we briefly consider the Φ^4 theory in zero dimensions [see Eqs. (9–177) ff. in Ref. [72]]; the generating functional reads

$$Z(\Phi)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\Phi^2-g\Phi^4\right]. \tag{34}$$

The strictly alternating divergent asymptotic expansion in powers of g for $g\to0$ reads,

$$Z(\Phi)\sim\sum_{N=0}^{\infty}\frac{4^N\Gamma(2N+1/2)}{\sqrt{\pi}\Gamma(N+1)}(-g)^N. \tag{35}$$

On using the known asymptotics valid for $N\to\infty$, which in this case yield the ''large-order'' asymptotics of the perturbative coefficients,

$$\frac{\Gamma(2N+1/2)}{\Gamma(N+1)}\sim\frac{4^N}{\sqrt{2\pi}}\Gamma(N)\left[1+\mathcal{O}\left(\frac{1}{N}\right)\right], \tag{36}$$

it is easy to explicitly establish the factorial divergence of the series (see also p. 888 of Ref. [60]). The generating functional in zero dimensions has been proposed as a paradigmatic example for the divergence of perturbation theory in higher order. It can be resummed easily to the nonperturbative result; in particular, it is manifestly Borel summable, and no singularities are present on the positive real axis.

Complications are introduced by degenerate minima. As a second example, we consider the modified generating functional (compare with Eq. (2.6) on p. 15 of Ref. [73] and with Eq. (40.1) on p. 854 of Ref. [60]):

$$\begin{aligned}Z'(\Phi)&=\int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\Phi^2(1-\sqrt{g}\Phi)^2\right]\\&=\int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\Phi^2+\sqrt{g}\Phi^3-\frac{1}{2}g\Phi^4\right].\end{aligned} \tag{37}$$

The expansion of the exponential in powers of the coupling g leads to a divergent asymptotic series,

$$\begin{aligned}Z'(\Phi)&=\sum_{N=0}^{\infty}\frac{1}{N!}\int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}\mathrm{e}^{-1/2\Phi^2}\left(\sqrt{g}\Phi^3-\frac{1}{2}g\Phi^4\right)^N\\&=\sum_{N=0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}\mathrm{e}^{-1/2\Phi^2}\sum_{j=0}^{N}\frac{(-1)^j}{\Gamma(2N-j+1)}\\&\quad\times\binom{2N-j}{j}(\sqrt{g}\phi^3)^{2(N-j)}\left(\frac{g\phi^4}{2}\right)^j\\&=\sum_{N=0}^{\infty}2\sqrt{\pi}\,\frac{(-1)^N C_{2N}^{N+1/2}(1)}{\Gamma(N-1/2)}g^N\\&=\sum_{N=0}^{\infty}\frac{8^N\Gamma(2N+1/2)}{\sqrt{\pi}\Gamma(N+1)}g^N,\end{aligned} \tag{38}$$

where $C_M^N(x)$ denotes a Gegenbauer (ultraspherical) polynomial. Note that terms of half-integer power of g entail an odd power of the field and vanish after integration. The first few terms of the asymptotic expansion read,

$$\begin{aligned}Z'(\Phi)=1&+6g+210g^2+13\,860\,g^3+1\,351\,350g^4\\&+174\,594\,420g^5+28\,109\,701\,620g^6\\&+5\,421\,156\,741000g^7+1\,218\,404\,977\,539\,750g^8\\&+O(g^9).\end{aligned} \tag{39}$$

For the perturbative coefficients

$$C_N=\frac{8^N\Gamma(2N+1/2)}{\sqrt{\pi}\Gamma(N+1)}, \tag{40}$$

we establish the following asymptotics,

$$C_N \sim \frac{1}{\pi\sqrt{2}} N^{-1} 32^N \Gamma(N+1). \tag{41}$$

Due to the nonalternating character of the expansion (38), it is not Borel summable in the ordinary sense. Rather, it is Borel summable in the distributional sense [48,47]. Here, we present numerical evidence supporting the summability of the divergent expansion (39) based on a finite number of perturbative coefficients. The final integration is carried out along the contour C_0 introduced in Ref. [46] [see also Eq. (44) below]. The same contour has also been used for the resummation of divergent perturbation series describing renormalization group (anomalous dimension) γ functions [55]. As explained in Ref. [46], the integration along C_0, which is based on the mean value of the results obtained above and below the real axis, leads to a *real* final result if all perturbative coefficients are real.

In particular, the resummation of the divergent expansion (39) is accomplished as follows. We first define the Borel transform of the generating functional by [see Eq. (4) in Ref. [58] and the discussion after Eq. (9)]

$$Z'_{\mathrm{B}}(z) \equiv \mathcal{B}^{(1,1)}[Z';z] = \sum_{N=0}^{\infty} \frac{C_N}{\Gamma(N+1)} z^N. \tag{42}$$

Padé approximants to this Borel transform are evaluated,

$$\mathcal{P}'_M(z) = [[\![M/2]\!]/[\![(M+1)/2]\!]]_{Z'_{\mathrm{B}}}(z), \tag{43}$$

where $[\![x]\!]$ denotes the largest positive integer smaller than x. We then evaluate the (modified) Borel integral along the integration contour C_0 introduced in Ref. [46]; specifically, we define the transform $\mathcal{T}Z_M(g)$

$$\mathcal{T}Z_M(g) = \int_{C_0} \mathrm{d}t \, \exp(-t) \mathcal{P}'_M(g\,t). \tag{44}$$

In this case, poles above and below the real axis must be considered, and the final result involves no imaginary part. The particular case of $g=0.01$ is considered. Values for the partial sums of the perturbation series (39) and the transforms defined in Eq. (44) are shown in Table III. The transforms exhibit apparent convergence to six decimal places in 20th order, whereas the partial sums of the perturbation series diverge. Between the second and fourth terms of the perturbation series (the fourth term constitutes the minimal term), the partial sums provide approximations to the exact result. It might seem surprising that the minimal term in the perturbative expansion is reached already in fourth order, although the coupling assumes the small value $g=0.01$. This behavior immediately follows from the large geometric factor in Eq. (41) which leads to a ''resultative coupling strength parameter'' of $g_{\mathrm{res}}=0.32$. ''Nonperturbative effects'' of the order of $\exp(-1/g_{\mathrm{res}})$ provide a fundamental limit to the accuracy obtainable by optimal truncation of the perturbation series; this is consistent with the numerical data in Table III.

TABLE III. Resummation of the asymptotic series for the generating functional of a zero-dimensional theory with degenerate minima given in Eqs. (38) and (39). We have $g=0.01$. Results in the third column are obtained by the method indicated in Eq. (44) along the integration contour C_0 (see Ref. [46]). The partial sums in the second column are obtained from the asymptotic series (38).

M	Partial sum	$\mathcal{T}Z_M(g=0.01)$
2	1.081 000	1.102 326
3	1.094 860	1.096 141
4	1.108 373	1.089 875
5	1.125 832	1.090 695
6	1.153 942	1.092 000
7	1.208 154	1.091 596
8	1.329 994	1.091 389
9	1.642 718	1.091 553
10	2.545 239	1.091 545
11	5.438 230	1.091 503
12	1.5×10^{1}	1.091 525
13	5.5×10^{1}	1.091 527
14	2.2×10^{2}	1.091 519
15	9.5×10^{2}	1.091 523
16	4.5×10^{3}	1.091 523
17	2.2×10^{4}	1.091 521
18	1.2×10^{5}	1.091 522
19	6.9×10^{5}	1.091 522
20	4.1×10^{6}	1.091 522
exact	1.091 522	1.091 522

We have also investigated the resummation of the divergent series (39) via a combination of a conformal mapping and Padé aproximants in the conformal variable. The situation is analogous to the LoSurdo-Stark effect: Results are consistent with those presented in Table III obtained by the ''pure'' Borel-Padé and in this case slightly more accurate. The radius of convergence of the Borel transform $Z'_{\mathrm{B}}(z)$ defined in Eq. (42) is $s=1/32$ [cf. Eq. (17) for the LoSurdo-Stark effect], and the appropriate conformal mapping in this case reads

$$w = \frac{4y}{(1+y)^2} \tag{45}$$

[cf. Eq. (22)]. The inverse reads

$$y(w) = \frac{1-\sqrt{1-w}}{1+\sqrt{1-w}}$$

[cf. Eq. (28)]. The conformal mapping (45) maps the complex w plane with a cut along $(1,\infty)$ unto the unit circle in the complex y plane. While the zero-dimensional model example given in Eq. (37) does not exhibit all problematic features of degenerate anharmonic double-well oscillators, certain analogies can be established; these comprise in particular the need to evaluate the mean value of Borel transforms above and below the real axis (see Appendix B).

VII. CONCLUSION

We discuss the resummation of the divergent perturbation series of the LoSurdo-Stark effect, and of a divergent model series describing a zero-dimensional theory with degenerate minima, using two methods. Method I, which uses a variant of the contour-improved Borel-Padé technique introduced in Ref. [25], is described in Sec. III. The integration contour is modified so that the additional terms that have to be added to the principal value of the Laplace-Borel integral are clearly identified [see also the discussion in Ref. [46] and Eq. (12)]. Use is made of the leading infrared renormalon pole. Method II, which comprises an analytic continuation by conformal mapping with additional improvement by Padé approximants in the conformal variable [see Eq. (25)], is discussed in Sec. IV. This method is a variant of the method introduced in Refs. [49,50] which has been shown to accelerate convergence of perturbative quantum chromodynamics (by optimal conformal mapping of the Borel plane). We find that *both* methods accomplish a resummation of the divergent perturbation series (1) for the LoSurdo-Stark effect, and the decay width of the quasistationary states is obtained (see Sec. V. Numerical results are given in Tables I and II). A main result of the current paper is the demonstration of the analogous mathematical structure (doubly cut Borel plane) of the perturbative expansion for the LoSurdo-Stark effect and perturbative expansions in quantum chromodynamics (renormalon theory [74]). The series investigated here exhibit a nontrivial singularity structure in the Borel plane. In particular, we encounter poles and branch cuts on the positive real axis.

In quantum electrodynamics, we encounter nonperturbative effects in the electron-positron pair-production amplitude in a background electric field [72,75–77]. The vacuum-to-vacuum amplitude acquires an imaginary part, whose magnitude is related to the production rate per space-time volume of fermion-antifermion pairs. This nonperturbative, imaginary contribution can be inferred from the perturbative expansion of the effective action by contour-improved resummation (see Ref. [46] and the discussion in Appendix A). Nonperturbative effects typically involve a nonanalytic factor of $\exp(-1/g)$ where g is an appropriate coupling parameter for the physical system under investigation (in the case of the LoSurdo-Stark effect, the coupling parameter is the electric-field strength F). The existence of nonperturbative contributions is intimately linked with the failure of the Carleman criterion for a particular perturbation series (see for example Ref. [78], Theorems XII.17 and XII.18 and the definition on p. 43 in Ref. [79], p. 410 in Ref. [41], or the elucidating discussion in Ref. [80]). The Carleman criterion determines, roughly speaking, if nonanalytic contributions exist for a given effect that is described by a specified perturbation series.

The current paper illustrates the utility of resummation methods in those cases where perturbation theory breaks down at large coupling. As explained in Sec. V, even in situations where the perturbation series diverges strongly, it can still be used to obtain meaningful physical results if it is combined with a suitable resummation method. In a relatively weak field, it is possible to obtain more accurate numerical results by resummation than by optimal truncation of the perturbation series (see also Ref. [46]). In a strong field, it is possible to obtain physically correct results by resummation even though the perturbation series diverges strongly (see the discussion in Sec. V and the data in Tables I, II, and III). By resummation, the perturbation series, which is inherently a weak-coupling expansion, can be given a physical interpretation even in situations where the coupling is large. Returning to the analogy to quantum field theory, one might be tempted to suggest that physically complete results are obtained after regularization, renormalization, *and* resummation.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor G. Soff and Professor J. Zinn-Justin for many insightful discussions, and G. Soff for continuous encouragement. Helpful discussions with Professor P. J. Mohr, E. Caliceti, E. J. Weniger, J. Sims, I. Nándori, and S. Roether are also gratefully acknowledged. The author would like to thank the Deutscher Akademischer Austauschdienst (DAAD) for organizational support, and he would like to thank the Laboratoire Kastler-Brossel, where major parts of this work were performed, and the National Institute for Standards and Technology for their kind hospitality.

APPENDIX A: DIVERGENT PERTURBATION SERIES IN QUANTUM FIELD THEORY

We briefly indicate aspects of certain divergent perturbation series in quantum field theory, in particular, the quantum electrodynamic (QED) effective action and the associated pair-production amplitude for electron-positron pairs. We use natural units in which the reduced Planck's constant, the permittivity of the vacuum, and the speed of light (in field-free vacuum) assume the value of unity ($\hbar=\epsilon_0=c=1$). The one-loop QED effective action for an arbitrary electric and magnetic field per space-time volume reads (this result can be found, e.g., in Eq. (4-123) of Ref. [72], upon inclusion of an additional counterterm; see also Refs. [76,81])

$$S=\lim_{\epsilon,\eta\to 0^+}-\frac{1}{8\pi^2}\int_0^{\mathrm{i}\infty+\eta}\frac{\mathrm{d}s}{s^3}\mathrm{e}^{-(m_\mathrm{e}^2-\mathrm{i}\epsilon)s}\times\left[(es)^2ab\coth(eas)\cot(ebs)-\frac{1}{3}(es)^2(a^2-b^2)-1\right]. \tag{A1}$$

The specification of the infinitesimal quantity η is necessary, strictly speaking, in order to avoid the singularities of the coth function along the imaginary axis. By a and b we denote the *secular invariants*,

$$a=\sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}^2},$$

$$b=\sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}^2},$$

$$\mathcal{F}=\frac{1}{4}F^{\mu\nu}F_{\mu\nu}=\frac{1}{2}(\boldsymbol{B}^2-\boldsymbol{E}^2),$$

$$\mathcal{G}=\frac{1}{4}F^{\mu\nu}(*F)_{\mu\nu}=-\boldsymbol{E}\cdot\boldsymbol{B}.$$

If the relativistic invariant $\mathcal{G}$ is positive, then it is possible to transform to a Lorentz frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are *antiparallel*. In the case $\mathcal{G}<0$, it is possible to choose a Lorentz frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are *parallel*. Irrespective of the sign of $\mathcal{G}$, we have in the specified frame

$$a=|\boldsymbol{B}| \quad \text{and} \quad b=|\boldsymbol{E}|.$$

Then, in the special Lorentz frame, the effective action reads [we keep all infinitesimal contributions]

$$S=\lim_{\epsilon,\eta\to 0^+}-\frac{1}{8\pi^2}\int_0^{\mathrm{i}\infty+\eta}\frac{\mathrm{d}s}{s^3}\mathrm{e}^{-(m_\mathrm{e}^2-\mathrm{i}\epsilon)s}\times\Bigg[(es)^2|\boldsymbol{B}||\boldsymbol{E}|\coth(e|\boldsymbol{B}|s)\cot(e|\boldsymbol{E}|s)-\frac{1}{3}(es)^2(\boldsymbol{B}^2-\boldsymbol{E}^2)-1\Bigg]. \tag{A2}$$

The particular cases of a pure magnetic and a pure electric field follow from the above integral representation by considering appropriate limits ($|\boldsymbol{E}|\to 0$ and $|\boldsymbol{B}|\to 0$, respectively). These particular cases are of some interest because they can be used as model series for divergent alternating and nonalternating asymptotic perturbation series [44–46,58]. In the case of a pure magnetic field $B=|\boldsymbol{B}|$, the result reads (see, e.g., Eq. (5) in Ref. [45])

$$S_B=-\frac{e^2B^2}{8\pi^2}\int_0^\infty\frac{\mathrm{d}s}{s^2}\left\{\coth s-\frac{1}{s}-\frac{s}{3}\right\}\exp\left(-\frac{m_\mathrm{e}^2}{eB}s\right), \tag{A3}$$

where m_e is the mass of the electron. This result can be expressed as a divergent asymptotic perturbation series in the coupling parameter $g_B=e^2B^2/m_\mathrm{e}^4$. For the pure electric field, the result reads

$$S_E=\frac{e^2E^2}{8\pi^2}\int_0^\infty\frac{\mathrm{d}s}{s^2}\left\{\coth s-\frac{1}{s}-\frac{s}{3}\right\}\exp\left[-\mathrm{i}\left(\frac{m_\mathrm{e}^2}{eE}-\mathrm{i}\epsilon\right)s\right]$$
$$=-\frac{e^2E^2}{8\pi^2}\int_{0+\mathrm{i}\epsilon}^{\infty+\mathrm{i}\epsilon}\frac{\mathrm{d}s}{s^2}\left\{\cot s-\frac{1}{s}+\frac{s}{3}\right\}\exp\left[-\frac{m_\mathrm{e}^2}{eE}s\right] \tag{A4}$$

[Eq. (7) of Ref. [46] and the expression before Eq. (10) of Ref. [45] contain typographical errors]. We take the opportunity to supplement the proportionality factor for the expression in Eq. (7) of Ref. [46] to yield the effective action per space-time volume element; it reads $e^2E^2/(8\pi^2)$. As evident from the Eq. (A4), the integration of the Borel-Padé transform for the electric-field case should be carried out along the contour C_{+1} shown here in Fig. 1. When this contour is used, then a sign change results for the imaginary contributions in Table I of Ref. [46] (the sign change of the imaginary part, according to the choice of the integration contour, has been discussed at length in Ref. [46]). The magnitude of the imaginary part yields the pair-production amplitude. The contour C_{+1} is used in the current investigation (and in the context of the related brief discussion in Ref. [46]) for the calculation of nonperturbative imaginary effects, i.e., the autoionization decay width of atomic states (LoSurdo-Stark effect).

The divergent asymptotic perturbation series for the cases of the magnetic and electric field, generated by the expansion of the results in Eqs. (A3) and (A4), can be found in Eqs. (6) and (7) of Refs. [45] (B field, alternating series, coupling parameter $g_B=e^2B^2/m_\mathrm{e}^4$) and in Eqs. (8) and (9) of Ref. [46] (E field, nonalternating series, coupling parameter $g_E=e^2E^2/m_\mathrm{e}^4$). The singularity structure of the Borel transform of the series for the magnetic-field case has been determined in Ref. [58] as a sequence of singularities corresponding to alternating, factorially divergent components (these correspond in their mathematical structure to the so-called ultraviolet renormalons in quantum chromodynamics). The perturbation series for the LoSurdo-Stark effect contains both nonalternating and alternating components so that its resummation represents a comparatively more interesting task. The same applies to the more complex perturbation series calculated in Ref. [82] for the renormalization group γ function, whose resummation—at strong coupling—has been discussed in Refs. [55,82] (in this case, there is no imaginary part involved). We are not aware of any *a priori* reasoning to determine the absence or presence of imaginary contributions in a particular physical problem (see also the discussion in Refs. [83]).

APPENDIX B: BOREL SUMMABILITY IN PROBLEMATIC CASES

Consider the one-dimensional double-well Hamiltonian

$$H(g)=p^2+x^2(1-gx)^2. \tag{B1}$$

For $g=0$, the Hamiltonian describes harmonic oscillations. For $g>0$, we have degenerate minima of the potential at $x=0$ and at $x=1/g$, and to each eigenvalue of the unperturbed harmonic oscillator, we have to associate two eigenvalues belonging to opposite-parity wave functions with respect to $x=1/(2g)$. The difference of the two eigenvalues is nonanalytic in g. Two different approaches have been developed to circumvent this problem and to allow for a treatment based on the resummation of perturbation series.

The first approach [60] is based essentially on generalized Bohr-Sommerfeld quantization formulas and leads to an expansion of the ground-state energy eigenvalue in terms of

$$E(g)=\sum_{n=0}^\infty E_l^{(0)}g^l+\sum_{n=1}^\infty\frac{1}{\sqrt{\pi g}}\,\mathrm{e}^{-1/6g}\times\sum_{k=0}^{n-1}[\ln(-2/g)]^k\sum_{l=0}^\infty E_{kl}^{(n)}g^l, \tag{B2}$$

where the upper index of the E coefficients labels the multi-instanton contributions (the zero-instanton contribution corresponds to the ''usual'' perturbative expansion). The odd-instanton contributions lead to a separation of the ground state and the first excited state that have opposite parity but the same naive perturbation series $\Sigma_{n=0}^{\infty} E_l^{(0)} g^l$. The series $\Sigma_{n=0}^{\infty} E_l^{(0)} g^l$ is nonalternating and therefore not Borel summable; however, possible imaginary contributions must be suppressed for physical reasons because the ground-state energy is real. The suppression can be enforced explicitly by defining the sum as the arithmetic mean of the values obtained above and below the real axis, or it can be motivated by the following observation [60]: We define the sum of $\Sigma_{n=0}^{\infty} E_l^{(0)} g^l$ for negative g and carry out an analytic continuation to positive g; this leads to an imaginary part, which in this case, cancels with the imaginary part generated by the $\ln(-2/g)$ coming from the two-instanton contribution ($n=2$, $k=1$). Note that for the model example discussed in Sec. VI, only one alternative is feasible—the explicit cancellation—because no additional terms are present that could lead to cancellations.

The second approach [84,85] involves contour integrations and makes use of projection operators in order to ''select'' states of specified parity (this approach has been shown to be applicable as well to the problematic Herbst-Simon hamiltonian [86] that involves a vanishing perturbation series). Specifically, we can write the perturbed energy eigenvalue as

$$E(g) = \frac{\mathrm{Re}\, F_1(g,g)}{\mathrm{Re}\, F_0(g,g)}, \tag{B3}$$

where

$$F_j(g,\gamma) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \mathrm{d}z\, z^j \langle \psi^{\pm}(g)| \frac{1}{H(\gamma)-z} |\psi^{\pm}(g)\rangle, \tag{B4}$$

and where $\psi^{\pm}(g)$ are *test functions* with a definite parity with respect to $1/(2g)$. The closed contour Γ has radius unity; it is chosen to encircle one and only one shifted resonance of the perturbed oscillator, while the test functions select the state with the desired parity. Specifically, we have $\psi^{\pm}(g) = P^{\pm}(g)\psi$ where ψ is the eigenvector of the unperturbed hamiltonian, and the projection operators are

$$[P^{\pm}(g)\psi](x) = \frac{1}{2}[\psi(x) \pm \psi(1/g - x)]. \tag{B5}$$

The functions $F_j(g,\gamma)$ may be expressed as asymptotic series,

$$F_j(g,\gamma) = \sum_{N=0}^{M} [a_{j,N} + \mathrm{i}\, b_{j,N}(g)]\, \gamma^N + \mathcal{O}(\gamma^{M+1}). \tag{B6}$$

The authors of Refs. [84,85] define $\Phi_j^{\mathrm{R}}(g,\gamma)$ to be the Borel sum of $\Sigma_{N=0}^{\infty} a_{j,N}\gamma^N$ for g, $|\gamma|$, and $\arg\gamma$ small and positive, and they establish a corresponding relation for $\Phi_j^{\mathrm{I}}(g,\gamma)$ and $\Sigma_{N=0}^{\infty} b_{j,N}\gamma^N$. According to (unnumbered) equations on p. 626 of Ref. [84], the final result in this case is given in terms of the mean values—each obtained above and below the real axis—of the two Borel sums Φ_j^{R} and Φ_j^{I},

$$\mathrm{Re}\, F_j(g,\gamma) = \frac{1}{2}[\Phi_j^{\mathrm{R}}(g,\gamma) + \overline{\Phi_j^{\mathrm{R}}(g,\overline{\gamma})}] + \frac{\mathrm{i}}{2}[\Phi_j^{\mathrm{I}}(g,\gamma) - \overline{\Phi_j^{\mathrm{I}}(g,\overline{\gamma})}], \tag{B7}$$

where $\overline{z}$ denotes the complex conjugate of z. The value $\mathrm{Re}\, F_j(g,g)$ is then determined by (unique) analytic continuation $\gamma \to g$ from $\mathrm{Re}\, F_j(g,\gamma)$. In our simplified model example, we have $b_{j,N}(g) = 0$ (all perturbative coefficients are real). The need to evaluate the arithmetic mean of Borel sums above and below the real axis appears to arise naturally in the context of double-well problems.

[1] H. Silverstone, Phys. Rev. A **18**, 1853 (1978).
[2] L. Benassi, V. Grecchi, E. Harrell, and B. Simon, Phys. Rev. Lett. **42**, 704 (1979).
[3] H.S. Silverstone and P.M. Koch, J. Phys. B **12**, L537 (1979).
[4] J.R. Oppenheimer, Phys. Rev. **31**, 66 (1928).
[5] M.H. Alexander, Phys. Rev. **178**, 34 (1969).
[6] M. Hehenberger, H.V. McIntosh, and E. Brändas, Phys. Rev. A **10**, 1494 (1974).
[7] N.A. Gushina and V.K. Nikulin, Chem. Phys. **10**, 23 (1975).
[8] P. Froelich and E. Brändas, Phys. Rev. A **12**, 1 (1975).
[9] R.J. Damburg and V.V. Kolosov, J. Phys. B **9**, 3149 (1976).
[10] T. Yamabe, A. Tachibana, and H.J. Silverstone, Phys. Rev. A **16**, 877 (1977).
[11] R.J. Damburg and V.V. Kolosov, J. Phys. B **11**, 1921 (1978).
[12] S. Graffi and V. Grecchi, Commun. Math. Phys. **62**, 83 (1978).
[13] I.W. Herbst and B. Simon, Phys. Rev. Lett. **41**, 67 (1978).
[14] L. Benassi, V. Grecchi, E. Harrell, and B. Simon, Phys. Rev. Lett. **42**, 1430 (1979).
[15] R.J. Damburg and V.V. Kolosov, J. Phys. B **12**, 2637 (1979).
[16] H. Silverstone, B.G. Adams, J. Čížek, and P. Otto, Phys. Rev. Lett. **43**, 1498 (1979).
[17] J.E. Avron, B.G. Adams, J. Čížek, M. Clay, M.L. Glasser, P. Otto, J. Paldus, and E. Vrscay, Phys. Rev. Lett. **43**, 691 (1979).
[18] B.G. Adams, J.E. Avron, J. Čížek, P. Otto, J. Paldus, R.K. Moats, and H.J. Silverstone, Phys. Rev. A **21**, 1914 (1980).
[19] L. Benassi and V. Grecchi, J. Phys. B **13**, 911 (1980).
[20] E. Luc-Koenig and A. Bachelier, J. Phys. B **13**, 1743 (1980).
[21] E. Luc-Koenig and A. Bachelier, J. Phys. B **13**, 1769 (1980).
[22] H.J. Silverstone, E. Harrell, and C. Grot, Phys. Rev. A **24**, 1925 (1981).
[23] A. Maquet, S.I. Chu, and W.P. Reinhardt, Phys. Rev. A **27**, 2946 (1983).
[24] D. Farrelly and W.P. Reinhardt, J. Phys. B **16**, 2103 (1983).
[25] V. Franceschini, V. Grecchi, and H.J. Silverstone, Phys. Rev. A **32**, 1338 (1985).
[26] W.L. Glab, K. Ng, D. Yao, and M.H. Nayfeh, Phys. Rev. A **31**, 3677 (1985).

[27] V.V. Kolosov, J. Phys. B **20**, 2359 (1987).
[28] V.V. Kolosov, J. Phys. B **22**, 833 (1989).
[29] G. Alvarez and H.J. Silverstone, Phys. Rev. Lett. **63**, 1364 (1989).
[30] G. Alvarez, R.J. Damburg, and H.J. Silverstone, Phys. Rev. A **44**, 3060 (1991).
[31] F.M. Fernández, J. Phys. A **25**, 495 (1992).
[32] C.A. Nicolaides and S.I. Themelis, Phys. Rev. A **45**, 349 (1992).
[33] G. Alvarez and H.J. Silverstone, Phys. Rev. A **50**, 4679 (1994).
[34] J. Zamastil, J. Čižek, and L. Skála, Phys. Rev. Lett. **84**, 5683 (2000).
[35] P.M. Koch, Phys. Rev. Lett. **41**, 99 (1978).
[36] T. Bergeman *et al.*, Phys. Rev. Lett. **53**, 775 (1984).
[37] W.L. Glab and M.N. Nayfeh, Phys. Rev. A **31**, 530 (1985).
[38] H. Rottke and K.H. Welge, Phys. Rev. A **33**, 301 (1986).
[39] A. LoSurdo, Atti R. Accad. Naz. Lincei, Mem. Cl. Sci. Fis., Mat. Nat. **22**, 664 (1913).
[40] J. Stark, Berl. Ber. **1913**, 932 (1913).
[41] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[42] E.J. Weniger, Comput. Phys. Rep. **10**, 189 (1989).
[43] G.A. Baker and P. Graves-Morris, *Padé Approximants*, 2nd ed. (Cambridge University Press, Cambridge, 1996).
[44] G.V. Dunne and T.M. Hall, Phys. Rev. D **60**, 065002 (1999).
[45] U.D. Jentschura, J. Becher, E.J. Weniger, and G. Soff, Phys. Rev. Lett. **85**, 2446 (2000).
[46] U.D. Jentschura, Phys. Rev. D **62**, 076001 (2000).
[47] E. Caliceti, V. Grecchi, and M. Maioli, Commun. Math. Phys. **157**, 347 (1993).
[48] E. Caliceti, V. Grecchi, and M. Maioli, Commun. Math. Phys. **104**, 163 (1986).
[49] I. Caprini and J. Fischer, Phys. Rev. D **60**, 054014 (1999).
[50] I. Caprini and J. Fischer, Phys. Rev. D **62**, 054007 (2000).
[51] R. Seznec and J. Zinn-Justin, J. Math. Phys. **20**, 1398 (1979).
[52] J.C. LeGuillou and J. Zinn-Justin, Ann. Phys. (N.Y.) **147**, 57 (1983).
[53] R. Guida, K. Konishi, and H. Suzuki, Ann. Phys. (N.Y.) **241**, 152 (1995).
[54] R. Guida, K. Konishi, and H. Suzuki, Ann. Phys. (N.Y.) **249**, 109 (1996).
[55] U.D. Jentschura and G. Soff, e-print hep-ph/0006089; J. Phys. A **34**, 1451 (2001).
[56] M. Pindor, e-print hep-th/9903151; Report No. IFT 9809002, Inst. of Theor. Phys. of Warsaw Univ. 1998 (unpublished).
[57] L.D. Landau and E.M. Lifshitz, *Quantenmechanik (Band III der Lehrbuchreihe über Theoretische Physik von L.D. Landau und E.M. Lifshitz)* (Akademie-Verlag, Berlin, 1979).
[58] U.D. Jentschura, E.J. Weniger, and G. Soff, J. Phys. G **26**, 1545 (2000).
[59] P.A. Rączka, Phys. Rev. D **43**, R9 (1991).
[60] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Clarendon Press, Oxford, 1996).
[61] A. van Wijngaarden, BIT Numer. Math. **6**, 66 (1966).
[62] J.C. LeGuillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).
[63] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, *Quantenelektrodynamik (Band IVb der Lehrbuchreihe über Theoretische Physik von L.D. Landau und E.M. Lifshitz)*, 1st ed. (Akademie-Verlag, Berlin, 1973).
[64] W.R. Johnson and G. Soff, At. Data Nucl. Data Tables **33**, 405 (1985).
[65] P.J. Mohr, in *Atomic, Molecular, and Optical Physics Handbook*, edited by G.W.F. Drake (AIP, Woodbury, NY, 1996), pp. 341–351.
[66] U.D. Jentschura and K. Pachucki, Phys. Rev. A **54**, 1853 (1996).
[67] U.D. Jentschura, G. Soff, and P.J. Mohr, Phys. Rev. A **56**, 1739 (1997).
[68] P.J. Mohr, G. Plunien, and G. Soff, Phys. Rep. **293**, 227 (1998).
[69] M.I. Eides, H. Grotch, and V.A. Shelyuto, e-print hep-ph/0002158; Phys. Rep. **342**, 63 (2001).
[70] S. Wolfram, *Mathematica-A System for Doing Mathematics by Computer* (Addison-Wesley, Reading, MA, 1988).
[71] Certain commercial equipment, instruments, or materials are identified in this paper to foster understanding. Such identification does not imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the materials or equipment identified are necessarily the best available for the purpose.
[72] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, NY, 1980).
[73] J. C. LeGuillou and J. Zinn-Justin, *Large-Order Behaviour of Perturbation Theory* (North-Holland, Amsterdam, 1990).
[74] M. Beneke, Phys. Rep. **317**, 1 (1999).
[75] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[76] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[77] W. Greiner and J. Reinhardt, *Quantum Electrodynamics* (Springer, New York, 1992).
[78] S. Graffi, V. Grecchi, and B. Simon, Phys. Lett. B **32**, 631 (1970).
[79] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic Press, New York, 1978).
[80] J. Fischer, Int. J. Mod. Phys. A **12**, 3625 (1997).
[81] W. Dittrich and H. Gies, *Probing the Quantum Vacuum*, Vol. 166 of Tracts in Physics (Springer, Berlin, 2000).
[82] D. Broadhurst and D. Kreimer, Phys. Lett. B **475**, 63 (2000).
[83] J. Fischer, Acta Phys. Pol. B **27**, 2549 (1997).
[84] E. Caliceti, V. Grecchi, and M. Maioli, Commun. Math. Phys. **113**, 625 (1988).
[85] E. Caliceti, V. Grecchi, and M. Maioli, Commun. Math. Phys. **176**, 1 (1996).
[86] I.W. Herbst and B. Simon, Phys. Lett. B **78**, 304 (1978).