Heisenberg-type structures of one-dimensional quantum Hamiltonians

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We construct a Heisenberg-like algebra for the one-dimensional infinite square-well potential in quantum mechanics. The ladder operators are realized in terms of physical operators of the system as in the harmonic-oscillator algebra. These physical operators are obtained with the help of variables used in a recently developed noncommutative differential calculus. This ''square-well algebra'' is an example of an algebra in a large class of generalized Heisenberg algebras recently constructed. This class of algebras also contains q oscillators as a particular case. We also discuss the physical content of this large class of algebras.

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The one-dimensional quantum harmonic oscillator is a special system in physics for several well-known reasons. The algebra related to it, the Heisenberg algebra, is a reference tool in the second quantization approach, and its structure, having as generators the Hamiltonian and the ladder operators, is used in several areas of physics.

In recent years, there hase been a intense activity on deformed algebras [1]. A deformed algebra is a nontrivial generalization of a well-known algebra through the introduction of one or more complex parameters, such that, in a certain limit of the parameters the well-known algebra is recovered. There have been several attempts to generalize Heisenberg algebra, and a particular deformation of Heisenberg algebra, known as q oscillators [2], has attracted considerable attention [3–5]. Nevertheless, in all generalizations of Heisenberg algebra, a clear comprehension of the physical problem under consideration is always lacking.

Recently, a generalization of the Heisenberg algebra was constructed depending on a general functional of one generator of the algebra, $f(J_0)$ [6,7], the characteristic function of the algebra. For linear f it was shown that the algebra corresponds to q oscillators, the Heisenberg algebra being obtained in the limit when the deformation parameter $q\rightarrow 1$. The representations of the algebra, when f is any analytical function, was shown to be obtained through the study of the stability of the fixed points of f and of their composed functions, exhibiting an unsuspected link between algebraic and dynamical system formalisms.

We show here that this generalization of the Heisenberg algebra together with a noncommutative differential calculus, developed to be used in space-time discrete networks [8–10], are appropriate to describe algebraic aspects of a simple quantum-mechanical system: the one-dimensional infinite square-well potential. The generators of the Heisenberg-type algebra that describes the one-dimensional square-well potential are the Hamiltonian and the ladder operators. The ladder operators are realized in terms of physical operators of the system in a similar way to what happens in the harmonic oscillator.

As will be clear in what follows, the approach we have developed here exhibits the physical content of the class of generalized Heisenberg algebras constructed in [7], i.e., this class of algebras describes the Heisenberg-type algebras of a class of one-dimensional quantum systems having energy eigenvalues (ϵ_n) written as $\epsilon_{n+1}=f(\epsilon_n)$, where $f(x)$ is a different function for each physical system. This function $f(x)$ is exactly the characteristic function of the algebra.

In the final remarks we will show that the introduction of a Heisenberg-type algebra having as generators the Hamiltonian of a one-dimensional quantum system and ladder operators, realized in terms of physical operators of the physical system, opens the possibility of constructing alternative formalisms of second quantization with possible applications ranging from condensed matter to quantum field theories. We sketch there the construction of a nonstandard nonrelativistic free quantum field theory based on the square-well potential.

The generalization of the Heisenberg algebra recently developed in [6,7] can be described by the generators J_0, $J_\pm$, satisfying the relations

$$J_0 J_+ = J_+ f(J_0), \tag{1}$$

$$J_- J_0 = f(J_0) J_-, \tag{2}$$

$$[J_+, J_-] = J_0 - f(J_0), \tag{3}$$

where, by hypothesis $J_- = J_+^\dagger$, $J_0^\dagger = J_0$, and $f(J_0)$ is a general analytic function of J_0 that we call the characteristic function of the algebra. It can easily be shown that the Jacobi identity of this algebra is trivially satisfied $\forall$ f analytic function. The above algebraic relations are constructed in order that, in the representation theory, the nth eigenvalue of the operator J_0 is given by the nth iteration, through the function f, of an initial value α_0. The operator

$$C = J_+ J_- - J_0 = J_- J_+ - f(J_0), \tag{4}$$

is a Casimir operator of the algebra. The representation theory of the algebra can be analyzed assuming that we have an irreducible representation of the algebra given by Eqs. (1)–(3). Consider the state $|0\rangle$ with the lowest eigenvalue of the Hermitian operator J_0,

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$$J_0|0\rangle=\alpha_0|0\rangle. \tag{5}$$

For each value of α_0 and the parameters of the algebra we have a different vacuum that for simplicity will be denoted by $|0\rangle$. As $|0\rangle$ is the vacuum, we require

$$J_-|0\rangle=0. \tag{6}$$

As consequence of the algebraic relations [(1)–(3), (5), (6)] we obtain for a general functional f

$$J_0|m\rangle=f^m(\alpha_0)|m\rangle, \quad m=0,1,2,\ldots, \tag{7}$$

$$J_+|m\rangle=N_m|m+1\rangle, \tag{8}$$

$$J_-|m\rangle=N_{m-1}|m-1\rangle, \tag{9}$$

where $N_m^2=f^{m+1}(\alpha_0)-\alpha_0$ and we have used $f^0(\alpha_0)=\alpha_0$. Note that $f^m(\alpha_0)$ denotes the mth iterate of f,

$$\alpha_m\equiv f^m(\alpha_0)=f(\alpha_{m-1}). \tag{10}$$

Equations (7)–(9) define the general conditions for an n-dimensional representation of the algebra. In order to solve it, i.e., to construct the conditions under which we have finite- and infinite-dimensional representations, we have to specify the functional $f(J_0)$. The Heisenberg algebra is the simplest particular case of algebra [(1)-(3)] and we can see that if we choose $f(J_0)=J_0+1$ the algebra given by Eqs. (1)–(3) becomes the Heisenberg algebra. In [7] we used linear and quadratic functionals, leading to multiparametric deformations of the Heisenberg algebra. Also, we showed in [7] that it is the iteration aspect of the algebra that allows us to find their representations through the analysis of the stability of the fixed points of the function f and their composed functions [6,7].

Here, in this paper, we shall use the inverse approach utilized in [6,7], where it was studied the representation theory for general functions f. Now, we consider a simple one-dimensional quantum system with a known spectrum and obtain the characteristic function $f(x)$ of the associated Heisenberg-type algebra for this physical problem. Moreover, we also realize the generators of the algebra in terms of the physical operators of the system. To implement this program we shall need the formalism of the noncommutative differential calculus mainly studied by Dimakis *et al.* [8–10].

In [8] a formalism was developed for a one-dimensional spacial lattice with finite spacing, i.e., a discrete space. We shall summarize here an analogous formalism for a momentum space instead of the position space. The reason is that in many physical problems the momentum space is already discretized. In the one-dimensional infinite square-well potential, for example, which will be analyzed below, the allowed values for the (adimensional) momenta are only the positive integers, as it is well known. Thus, the noncommutative differential calculus approach seems to be appropriate to be used in the momentum space. The formulas used here are analogous to the formulas used in [8], and the reader should see this paper for a more detailed exposition and explanation of the noncommutative calculus (remembering again that their formulas were derived for a discrete position space). Therefore, let us consider a one-dimensional lattice in a momentum space where the momenta are allowed only to take discrete values, say p_0, p_0+a, p_0+2a, p_0+3a, etc., with $a>0$. The noncommutative differential calculus is based on the expression

$$[p,dp]=dp\,a, \tag{11}$$

implying that

$$f(p)dg(p)=dg(p)f(p+a) \tag{12}$$

for all functions f and g. Let us introduce partial derivatives by

$$d\,f(p)=dp(\partial_p f)(p)=(\bar{\partial}_p f)(p)dp, \tag{13}$$

where the left and right discrete derivatives are given by

$$(\partial_p f)(p)=\frac{1}{a}[f(p+a)-f(p)], \tag{14}$$

$$(\bar{\partial}_p f)(p)=\frac{1}{a}[f(p)-f(p-a)], \tag{15}$$

and satisfy

$$(\bar{\partial}_p f)(p)=(\partial_p f)(p-a). \tag{16}$$

The Leibniz rule for the right discrete derivative can be written as

$$(\partial_p fg)(p)=(\partial_p f)(p)g(p)+f(p+a)(\partial_p g)(p), \tag{17}$$

with a similar formula for the left derivative [8].

Let us now introduce the momentum shift operators

$$A=1+a\,\partial_p, \tag{18}$$

$$\bar{A}=1-a\,\bar{\partial}_p, \tag{19}$$

which increases (decreases) the momentum value by a

$$(Af)(p)=f(p+a), \tag{20}$$

$$(\bar{A}f)(p)=f(p-a), \tag{21}$$

and satisfies

$$A\,\bar{A}=\bar{A}A=1, \tag{22}$$

where 1 means the identity on the algebra of functions of p. Let us now introduce the momentum operator [8]

$$(Pf)(p)=p\,f(p) \tag{23}$$

$(P^\dagger=P)$, which returns the value of the variable of the function f. Clearly,

$$AP=(P+a)A, \tag{24}$$

$$\overline{A}P=(P-a)\overline{A}. \tag{25}$$

Integrals can also be defined in this formalism but it is rather a technical point and the interested reader can find in [8] a detailed explanation on the subject. Here we will only use the definition of a definite integral of a function f from p_d to p_u (p_u being equal to p_d+Ma, where M is a positive integer) as

$$\int_{p_d}^{p_u} dp\, f(p)=a\sum_{k=0}^{M} f(p_d+k\,a). \tag{26}$$

Using Eq. (26), an inner product of two (complex) functions f and g can be defined as

$$\langle f,g\rangle=\int_{p_d}^{p_u} dp\, f(p)^* g(p), \tag{27}$$

where the asterisk indicates the complex conjugation of the function f. Clearly, the norm $\langle f,f\rangle\geqslant 0$ is zero only when f is identically null. The set of equivalent classes of normalizable functions f ($\langle f,f\rangle$ is finite) is a Hilbert space and it can be shown that the operators A and $\overline{A}$ are well defined in this space [8]. We have

$$\langle f,Ag\rangle=\langle \overline{A}f,g\rangle, \tag{28}$$

where

$$\overline{A}=A^\dagger, \tag{29}$$

$A^\dagger$ being the adjoint operator of A. Equations (22) and (29) show that A is a unitary operator. It is also possible to define a position operator X given as $X=(\partial_p+\overline{\partial_p})/2i$ [8]. With this very short adapted review of the noncommutative differential calculus we can go further and, together with the generalization of the Heisenberg algebra, analyze the physical example of the quantum-mechanical infinite one-dimensional square-well potential.

Thus, let us assume a one-dimensional system with zero potential between zero and L and infinite elsewhere. As it is well known, the spectrum of the Hamiltonian ($H=cP^2$, $c=1/2m$, $\hbar=1$) with the above boundary conditions is proportional to n^2, where $n=1,2,3,\ldots$. The momentum is quantized and proportional to n. Therefore, we can see the momentum space as a one-dimensional periodic lattice with constant spacing $a=\pi/L$, clearly a candidate to apply the noncommutative differential calculus reviewed before. We then take the momentum operator in the Hamiltonian $H=cP^2$, with the above boundary conditions, as defined in Eq. (23).

The Hamiltonian's eigenvalue associated with the $(n+1)$th level is proportional to $(n+1)^2$ and we can write

$$e_{n+1}=b(n+1)^2=(\sqrt{e_n}+\sqrt{b})^2, \tag{30}$$

where e_n is the eigenvalue of the Hamiltonian associated with the nth level and $b=\pi^2/2mL^2$. As J_0 is related to the Hamiltonian [6,7] and their eigenvalues satisfy the iterations given by a function f in Eqs. (1)–(3), we see that if we choose this function as

$$f(x)=(\sqrt{x}+\sqrt{b})^2, \tag{31}$$

the J_0 in Eqs. (1)–(3) has eigenvalues equal to the energy eigenvalues of the square-well potential. Equations (1)–(3) can then be rewritten for this case as

$$[J_0,J_+]=2\sqrt{b}\,J_+\sqrt{J_0}+b\,J_+, \tag{32}$$

$$[J_0,J_-]=-2\sqrt{b}\sqrt{J_0}J_- -b\,J_-, \tag{33}$$

$$[J_+,J_-]=-2\sqrt{b}\sqrt{J_0}-b. \tag{34}$$

The square root of the generator J_0 is well defined since this is a Hermitian and positive definite operator.

We then have an algebra [Eqs. (32)–(34)] where, by construction, the eigenvalues of J_0, e_n, are the energy eigenvalues of the quantum-mechanical one-dimensional infinite square-well potential and $J_\pm$ act as ladder operators. In order to have a complete description similar to the case of the one-dimensional harmonic oscillator, we must realize the operators $J_{(\pm,0)}$ in terms of physical operators. We propose for this problem the following realization:

$$J_+=(cP^2-b)^{1/2}\overline{A}, \tag{35}$$

$$J_-=A(cP^2-b)^{1/2}, \tag{36}$$

$$J_0=c\,P^2. \tag{37}$$

Clearly, J_0 is the Hamiltonian and can be written, analogously to the harmonic-oscillator case, as an ordered product of ladder operators

$$J_+J_-=J_0-b, \tag{38}$$

as according to Eq. (22), $A\overline{A}=1$. Comparing Eqs. (4) and (38) we see that b is the Casimir of the representation for the square-well potential. Using Eqs. (24) and (25) it is straightforward to check that these operators indeed satisfy the commutation relations given by Eqs. (32)–(34). We stress that the operators P and X are the momentum and position operators in the momentum space for the one-dimensional infinite square-well potential. Moreover, as will be seen below it is possible to write the operators P and X in terms of the ladder operators $J_\pm$ and the operator J_0.

Fock space representations of the algebra generated by J_0 and $J_\pm$, Eqs. (32)–(34), are obtained considering eigenstates of J_0, with fixed values of the momentum. Let us call $|n\rangle$ the eigenstate of J_0 whose momentum is associated with the quantum number n, $n=1,2,3,\ldots$. The eigenvalue α_n that appears in Eqs. (5)–(10) can be set as $\alpha_n=b(n+1)^2$ and

Eqs. (7)–(9) can be rewritten, after a trivial rename of the states $|n\rangle$, such that the lowest-energy state corresponds to $|1\rangle$, as

$$J_0|n\rangle = b\, n^2|n\rangle, \quad n=1,2,\ldots, \tag{39}$$

$$J_+|n\rangle = \sqrt{b(n+1)^2-b}\,|n+1\rangle, \tag{40}$$

$$J_-|n\rangle = \sqrt{bn^2-b}\,|n-1\rangle, \tag{41}$$

$$P|n\rangle = a\, n|n\rangle, \;\; \overline{A}|n\rangle = |n+1\rangle, \tag{42}$$

where $N_n^2 = b(n+1)^2 - b$. Note that, $J_-|1\rangle = 0$ as it happens in the standard notation of the square-well potential since the lowest-energy state is represented by the state $|1\rangle$.

Hence, we see that an algebraic formalism similar to the harmonic-oscillator algebra was constructed for another physical problem: the one-dimensional infinite square-well potential in quantum mechanics. The main point here is that the Hamiltonian itself is one of the generators of the algebra, together with the ladder operators. In other physical realizations of the ladder operators [12], the Hamiltonian is not, in general, one of the generators of the algebra.

Generally speaking, suppose we have an arbitrary one-dimensional quantum system such that two successive energy eigenvalues ϵ_n can be related as

$$\epsilon_{n+1} = f(\epsilon_n), \tag{43}$$

where $f(x)$ is a different function for each physical system. If we assume that the generator J_0 of the class of Heisenberg-type algebras in Eqs. (1)–(3) is the Hamiltonian operator of this one-dimensional quantum system, Eq. (10) tells us that the algebra in Eqs. (1)–(3) with f appearing in Eq. (43) describes the algebraic structure of this quantum system. Moreover, from Eqs. (8) and (9) we see that J_+ and J_- are the ladder operators of this quantum system. In summary, the Heisenberg-type algebras [7] given in Eqs. (1)–(3) describe the algebraic structure of one-dimensional quantum systems having successive eigenvalues related by Eq. (43) where the characteristic function of the algebra is the function $f(x)$ appearing in Eq. (43).

Once the Heisenberg-type structure of a one-dimensional quantum system is understood, the next step, as was performed here for the square-well potential, is to realize the ladder operators of the algebra in terms of the physical operators of the system, such that the algebra is still satisfied and is the product J_+J_- proportional to the Hamiltonian of the one-dimensional quantum system under consideration. This program could supply an alternative approach to quantum field theory as indicated in what follows.

Using the momentum operator P defined on a lattice, Eq. (23), and the associated lattice derivatives we can define two type coordinate operators as

$$X = \frac{1}{2i}(\overline{\partial_p} + \partial_p), \tag{44}$$

$$Q = \overline{\partial_p} - \partial_p, \tag{45}$$

where ∂_p and $\overline{\partial_p}$ are the left and right discrete derivatives defined in Eqs. (14) and (15). Of course, in the continuous limit $(a\to 0)$ the operator Q is identically null since ∂_p and $\overline{\partial_p}$ represent, in this limit, the same derivative. It can be checked that the operators P, X, and Q generate an algebra on the momentum lattice that reduces to the standard Heisenberg algebra when $a\to 0$. With the help of Eqs. (18) and (19) and (35) and (36) we can rewrite X and Q in terms of the ladder operators of the suqare-well algebra as

$$X = \frac{i}{2a}(S^{-1}A^\dagger - AS^{-1}), \tag{46}$$

$$Q = \frac{1}{a}(-2 + S^{-1}A^\dagger + A\,S^{-1}), \tag{47}$$

where $S = (cP^2 - b)^{1/2}$. Using an independent copy of the operators Q and X for each point of a three-dimensional lattice we can define two fields and two momentum fields that can be used to construct a free quantum field theory Hamiltonian. This Hamiltonian can be written as

$$H = \sum_{\vec{k}} J_+(\vec{k})J_-(\vec{k}) = \sum_{\vec{k}} S_{\vec{k}}^2 = \sum_{\vec{k}} [cP_{\vec{k}}^2 - b(\vec{k})], \tag{48}$$

where $P_{\vec{k}}$, for each $\vec{k}$, is the momentum operator for a particle with mass m in a square-well potential. This is a nonrelativistic free quantum field theory and the details of construction of a relativistic free quantum field theory can be found in [11].

The eigenvectors of H form a complete set and span the Hilbert space of this system. The eigenvectors are

$$|1\rangle,\;\; J_+(\vec{k})|1\rangle,\;\; J_+(\vec{k})J_+(\vec{k}')|1\rangle \quad \text{for} \quad \vec{k}\neq\vec{k}',\;\; [J_+(\vec{k})]^2|1\rangle,\ldots. \tag{49}$$

This Hilbert space has a different interpretation with respect to the standard spin-0 quantum field theory based on the harmonic oscillator. While in the standard quantum field theory the creation operator creates one particle of mass m each time it is applied to the vacuum, in this nonrelativistic quantum field theory one reads from Eq. (48) that the creation operator in this case creates excited states of a particle in a box. This could provide an alternative quantum field theory phenomenological approach to hadronic interactions.

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