Comprehensive theory of the relative phase in atom-field interactions

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We explore the role played by the quantum relative phase in a well-known model of atom-field interaction, namely, the Dicke model. We introduce an appropriate polar decomposition of the atom-field relative amplitudes that leads to a truly Hermitian relative-phase operator, whose eigenstates correctly describe the phase properties, as we demonstrate by studying the positive operator-valued measure derived from it. We find the probability distribution for this relative phase and, by resorting to a numerical procedure, we study its time evolution.

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I. INTRODUCTION

The problem of the interaction of an atomic system with a radiation field is a keystone of quantum optics. Needless to say, it is impossible to obtain exact solutions to this problem and some approximations must be used; the most common being that the radiation field is quasimonochromatic and its frequency coincides almost exactly with one of the transition frequencies of the atoms (supposedly identical and with no direct interaction between them).

The two-level atom is the natural consequence of this hypothesis [1]. Such an object is an important tool because it allows us to describe the matter-field interaction in a very simple way, and the results constitute a first step to dealing with more realistic situations that could include losses, broadening of the atomic lines, etc. To put it bluntly, we can replace the whole atomic system by an effective two-level system that accounts for all the relevant details of the interaction. The Dicke model [2,3], describing the interaction of A identical two-level atoms with a single-mode field in a perfect cavity, is perhaps the archetype of this situation.

In the semiclassical version of this Dicke model, correlations are safely ignored and the field is interpreted to be a purely classical electric field [4–6]. Such an approximation has proven to be very successful and has the virtue of reducing the problem to the exclusive knowledge of the atomic dynamics, which is studied in terms of the inversion and the components of the atomic dipole in phase and in quadrature with the field (i.e., the Bloch vector). These quadratures are the dispersive and absorptive components of the dipole moment effective in coupling to the field.

For some phenomena, such as spontaneous emission by a fully excited atomic system, the quantization of the field is required. Then, one must take care also of the evolution of the field amplitudes, but the atomic dynamics is still explained in terms of inversion and dipole quadratures.

Classically, the interaction of matter with light is usually treated within the framework of the Lorentz model [7], which assumes that each electron-ion pair behaves as a single harmonic oscillator that couples to the field through its electric dipole moment. In spite of its simplicity, it is extraordinarily helpful in developing an intuitive feeling for the physical mechanisms involved. Although the dynamics of this model is sometimes expressed in quadrature components [1], the natural way of understanding its resonant behavior is in terms of the relative phase between the field and the atomic dipole [8].

While the quantum quadratures are well known, as are the associated eigenstates, the operator for this relative phase has resisted a quantum description. In this respect, we think that, in spite of its maturity and success, the Dicke model is apparently incomplete since it lacks a satisfactory description in terms of this relative phase, indispensable to compare with the classical world. The main goal of this paper is precisely the general description of that variable.

When focusing on the relative phase between two subsystems, we think the best way to proceed, much in the spirit of our previous work in the subject [9], is to try a polar decomposition of the quantum amplitudes, which parallels, as much as possible, the corresponding classical factorization. For the relative phase between two harmonic oscillators, this is quite straightforward procedure and leads to a unitary solution [10].

For the Dicke model, this polar decomposition seems to be more involved, mainly because, unlike for the case of two harmonic oscillators, the Hamiltonian cannot be cast in terms of su(2) operators, but rather in terms of some polynomial deformation of su(2). These nonlinear algebras have been examined very recently in quite a different physical contexts [11], and, by exploiting these results, it is possible to perform such a decomposition in an elegant way, obtaining a bona fide Hermitian operator representing the relative phase we wish to examine.

In this paper we use this operator to introduce the associated probability distribution and, then, the most relevant dynamical features of the Dicke model can be easily explained using this relevant variable.

II. QUANTUM DYNAMICS OF THE DICKE MODEL

The Dicke model describes the interaction of a collection of A identical two-level atoms with a quantum single-mode

field in a lossless cavity. The spatial dimensions of the atomic system are smaller than the wavelength of the field, so all the atoms feel the same field. However, the model neglects the dipole-dipole interaction between atoms (i.e., their wave functions do not overlap in the evolution).

The Hamiltonian for this model in the electric dipole and rotating-wave approximations reads as (in units $\hbar=1$)

$$H=H_0+H_{\rm int}, \tag{2.1}$$

with

$$H_0=\omega_{\rm f}N,$$

$$H_{\rm int}=\Delta S_3+g(a^\dagger S_-+aS_+). \tag{2.2}$$

Here,

$$N=a^\dagger a+S_3, \tag{2.3}$$

is the excitation number operator, g is the coupling constant (which, in this approximation, is the same for all the atoms and can be chosen as real), and $\Delta=\omega_{\rm f}-\omega_{\rm a}$ is the detuning between the atomic and field frequencies. The field mode is described by the annihilation and creation operators a and $a^\dagger$, while the collective atomic operators are defined by

$$S_{\pm,3}=\sum_{j=1}^{A}\sigma_{\pm,3}^{j}, \tag{2.4}$$

and obey the commutation relations

$$[S_3,S_\pm]=\pm S_\pm, \quad [S_+,S_-]=2S_3. \tag{2.5}$$

Since all the atoms have the same coupling constant, we need to consider only symmetrical atomic states. Then, let us introduce the atomic Dicke states as [2]

$$|M\rangle_{\rm a}=\sqrt{\frac{M!(A-M)!}{A!}}\sum_{p}|+\rangle_{j_1}\ldots|+\rangle_{j_M}\times|-\rangle_{j_{M+1}}\ldots|-\rangle_{j_A}, \tag{2.6}$$

where $|\pm\rangle_j$ are the eigenstates of the jth atom and the sum runs over all possible manners of choosing M indistinguishable atoms from the group of A atoms. In the space spanned by these Dicke states, the action of the collective atomic operators is

$$S_+|M\rangle_{\rm a}=\sqrt{(M+1)(A-M)}|M+1\rangle_{\rm a},$$

$$S_-|M\rangle_{\rm a}=\sqrt{M(A-M+1)}|M-1\rangle_{\rm a}, \tag{2.7}$$

$$S_3|M\rangle_{\rm a}=(M-A/2)|M\rangle_{\rm a},$$

where the label M $(0\leqslant M\leqslant A)$ denotes the number of excited atoms and $-A/2$ represents the bottom energy level of the atomic system. Therefore, the collective atomic operators form a $(A+1)$-dimensional representation of the algebra su(2) corresponding to a spin $A/2$. The case of a single resonant atom $(A=1)$ corresponds to the well-known Jaynes-Cummings model.

For simplicity, we shall restrict henceforth our attention to the case of exact resonance between the atomic and the field frequency $\omega_{\rm a}=\omega_{\rm f}\equiv\omega$. Since the field mode is described in the usual Fock space $|n\rangle_{\rm f}$, the natural bare basis for the total system is $|n,M\rangle\equiv|n\rangle_{\rm f}\otimes|M\rangle_{\rm a}$. However, it is straightforward to check that

$$[H_0,H_{\rm int}]=0, \tag{2.8}$$

so both are constants of motion. The Hamiltonian H_0 (or, equivalently, the excitation number N) determines the total energy stored by the radiation field and the atomic system, which is conserved by the interaction. This means that the appearance of M excited atoms requires the annihilation of M photons. This allows us to factor out $\exp(-iH_0t)$ from the evolution operator and drop it altogether. Hence, we can relabel the total basis as

$$|N-M,M\rangle\equiv|N-M\rangle_{\rm f}\otimes|M\rangle_{\rm a}. \tag{2.9}$$

In such a basis, the interaction Hamiltonian, for a fixed value of N, is represented by the tridiagonal matrix

$$H_{\rm int}^{(N)}=g\begin{pmatrix}0 & h_0 & 0 & . & \ldots\\ h_0 & 0 & h_1 & 0 & \ldots\\ 0 & h_1 & 0 & h_2 & \ldots\\ \vdots & \vdots & \vdots & \ddots & \vdots\end{pmatrix}, \tag{2.10}$$

with

$$h_M=\sqrt{(M+1)(N-M)(A-M)}. \tag{2.11}$$

The dimension of this matrix depends on whether $A>N$ or $A<N$, which are situations essentially different and must be handled separately.

Let us assume that $A>N$ and initially all the atoms are unexcited. Then, $M=0$ and the conservation of the number of excitations implies that only the states (2.9) with $0\leqslant M\leqslant N$ take part in the dynamics. Thus, the dimension of the subspace is $N+1$.

On the contrary, when $A<N$, the number of initial photons is greater than the number of atoms and only the states (2.9) with $0\leqslant M\leqslant A$ are involved in the evolution. The dimension is now $A+1$

It is easy to check that, due to the properties of the tridiagonal matrices, the eigenvalues are distributed symmetrically with respect to zero, with one eigenvalue equal to zero if there are an odd number of them [12].

To find the state evolution, we shall need the following matrix elements of the evolution operator

$$C_{M'M}^{N}(t)=\langle N-M',M'|\exp[-iH_{\rm int}^{(N)}t]|N-M,M\rangle, \tag{2.12}$$

which can be written as

$$C^N_{M'M}(t)=\sum_{J=0}^{\mathcal{D}} U_{MJ}U^\dagger_{M'J}\exp[-i\varepsilon^{(N)}_J t], \qquad (2.13)$$

where U is the unitary matrix that diagonalizes the Hamiltonian and $\varepsilon^{(N)}_I$ are the corresponding eigenvalues. In what follows, we shall use the convention of denoting the dimension of the Hamiltonian matrix $H^{(N)}_{\rm int}$ by $\mathcal{D}+1$, that is,

$$\mathcal{D}=\min(N,A). \qquad (2.14)$$

Now, let us assume that the initial field is taken to be in a coherent state $|\alpha\rangle_{\rm f}$ and that the atomic state is initially prepared in an atomic coherent state $|\zeta\rangle_{\rm a}$ [13,14]; i.e.,

$$|\Psi(0)\rangle=|\alpha\rangle_{\rm f}\otimes|\zeta\rangle_{\rm a}, \qquad (2.15)$$

where

$$|\alpha\rangle_{\rm f}=\sum_n Q_n|n\rangle_{\rm f}, \qquad (2.16)$$

Q_n being the Poissonian weighting factor of the coherent state (with zero phase) with mean number of photons $\overline{n}$

$$Q_n=\sqrt{e^{-\overline{n}}\frac{\overline{n}^n}{n!}}; \qquad (2.17)$$

and

$$|\zeta\rangle_{\rm a}=\frac{1}{(1+|\zeta|^2)^{A/2}}\sum_{M=0}^{A}\sqrt{\frac{A!}{M!(A-M)!}}\zeta^M|M\rangle_{\rm a} \equiv\sum_{M=0}^{A}A_M|M\rangle_{\rm a}, \qquad (2.18)$$

where the parameter ζ is normally rewritten in terms of the spherical angles as

$$\zeta=-\tan(\vartheta/2)e^{-i\varphi}. \qquad (2.19)$$

In other words, the initial state can be rewritten, taking into account Eq. (2.9), as

$$|\Psi(0)\rangle=\sum_{N,M}Q_{N-M}A_M|N-M,M\rangle. \qquad (2.20)$$

With this initial condition, the resulting state can be recast as

$$|\Psi(t)\rangle=\exp(-iH_{\rm int}t)|\Psi(0)\rangle =\sum_{N=0}^{\infty}\sum_{M',M=0}^{\mathcal{D}}Q_{N-M}A_M C^N_{M'M}(t)|N-M',M'\rangle. \qquad (2.21)$$

If the initial state is not of the same form, but has a decomposition with different amplitudes A_M or Q_n, Eq. (2.21) is still valid when the appropriate coefficients are taken.

III. RELATIVE-PHASE OPERATOR FOR THE DICKE MODEL

In the spirit of our previous work on the relative phase for the Jaynes-Cummings model [15], we shall describe the atom-field relative phase in terms of a polar decomposition of the complex amplitudes. To this end, let us introduce the operators

$$X_+=aS_+, \quad X_-=a^\dagger S_-, \quad X_3=S_3. \qquad (3.1)$$

These operators maintain the first commutation relation of su(2) in Eq. (2.5), $[X_3,X_\pm]=\pm X_\pm$, but the second one is modified in the following way:

$$[X_-,X_+]=P(X_3), \qquad (3.2)$$

where $P(X_3)$ represents a second-order polynomial function of the operator X_3. This is a typical example of the so-called polynomial deformations of the algebra su(2). Without embarking us in mathematical subtleties, the essential point for our purposes here is that one can develop a theory in a very close analogy with the standard su(2) algebra. In particular, it is clear that the state $|N,0\rangle$ plays the role of a *vacuum state*, since

$$X_-|N,0\rangle=0. \qquad (3.3)$$

Then, we can construct invariant subspaces, as in the usual theory of angular momentum, by

$$|N-M,M\rangle=\frac{1}{\mathcal{N}}X_+^M|N,0\rangle, \qquad (3.4)$$

where $\mathcal{N}$ is a normalization constant. One can check that

$$X_+^{\mathcal{D}+1}|N,0\rangle=0, \qquad (3.5)$$

confirming that the number of accessible states is $\mathcal{D}+1$.

In consequence, the whole space of the system can be split as the direct sum $\mathcal{H}=\oplus_{N=0}^{\infty}\mathcal{H}_N$ of subspaces invariant under the action of the operators (X_+,X_-,X_3) and each one of them having a fixed number of excitations. These independent subspaces do not overlap in the evolution, in such a way that if the initial state belongs to one of them, it will remain in that subspace for all the evolution.

In each one of these invariant subspaces the operator X_3 is diagonal, while X_+ and X_- are ladder operators represented by finite-dimensional matrices. This suggests to introduce a polar decomposition in the form

$$X_+=\sqrt{X_+X_-}E, \quad X_-=E^\dagger\sqrt{X_+X_-}, \qquad (3.6)$$

where the *radial* operator $\sqrt{X_+X_-}$ is diagonal in the basis $|N-M,M\rangle$:

$$\sqrt{X_+X_-}|N-M,M\rangle = \sqrt{M(N-M+1)(A-M+1)}|N-M,M\rangle, \tag{3.7}$$

and

$$[X_3,E]=E. \tag{3.8}$$

We can guarantee now that the operator $E=\exp(i\Phi)$, representing the exponential of the relative phase, is unitary and commutes with the excitation number

$$EE^\dagger=E^\dagger E=I,$$

$$[E,N]=0. \tag{3.9}$$

Thus, we may rather study its restriction to each invariant subspace $\mathcal{H}_N$, we shall denote by $E^{(N)}$. It is easy to check that the action of the operator $E^{(N)}$ in each subspace is given by

$$E^{(N)}|N-M,M\rangle=|N-(M+1),M+1\rangle,$$

$$E^{(N)\dagger}|N-M,M\rangle=|N-(M-1),M-1\rangle. \tag{3.10}$$

Obviously, the action of $E^{(N)}$ and $E^{(N)\dagger}$ becomes undefined on the marginal states $|N-\mathcal{D},\mathcal{D}\rangle$ and $|N,0\rangle$. Therefore, it is necessary to add some conventions for closing the actions of these operators on the subspace $\mathcal{H}_N$. By analogy once again with the usual su(2) algebra, we shall use standard cyclic conditions and impose (up to global phase factors)

$$E^{(N)}|N-\mathcal{D},\mathcal{D}\rangle=|N,0\rangle,$$

$$E^{(N)\dagger}|N,0\rangle=|N-\mathcal{D},\mathcal{D}\rangle. \tag{3.11}$$

With these conditions, the operator $E^{(N)}$ can be expressed as

$$E^{(N)}=\sum_{M=0}^{\mathcal{D}}|N-(M+1),M+1\rangle \times\langle N-M,M|+e^{i(\mathcal{D}+1)\phi^{(N)}}|N,0\rangle\langle N-\mathcal{D},\mathcal{D}|, \tag{3.12}$$

$\phi^{(N)}$ being an arbitrary phase. Note that the crucial extra term in this equation, which establishes the unitarity of $E^{(N)}$, is precisely based on the finite number of states. Therefore, in each invariant subspace $\mathcal{H}_N$ there are $\mathcal{D}+1$ orthonormal states satisfying

$$E^{(N)}|\phi_r^{(N)}\rangle=e^{i\phi_r^{(N)}}|\phi_r^{(N)}\rangle, \tag{3.13}$$

with $r=0,\ldots,\mathcal{D}$. These states can be expressed as

$$|\phi_r^{(N)}\rangle=\frac{1}{\sqrt{\mathcal{D}+1}}\sum_{M=0}^{\mathcal{D}}e^{iM\phi_r^{(N)}}|N-M,M\rangle, \tag{3.14}$$

and, by taking the same 2π window in each subspace, we have

$$\phi_r^{(N)}=\phi_0+\frac{2\pi r}{\mathcal{D}+1}, \tag{3.15}$$

and ϕ_0 is a fiducial or reference phase shift that can be arbitrarily chosen. The expression for E on the whole space is

$$E=\sum_{N=0}^{\infty}E^{(N)}=\sum_{N=0}^{\infty}\sum_{r=0}^{\mathcal{D}}|\phi_r^{(N)}\rangle e^{i\phi_r^{(N)}}\langle\phi_r^{(N)}|, \tag{3.16}$$

and, since E is unitary, it defines a Hermitian relative-phase operator

$$\Phi=\sum_{N=0}^{\infty}\Phi^{(N)}=\sum_{N=0}^{\infty}\sum_{r=0}^{\mathcal{D}}|\phi_r^{(N)}\rangle\phi_r^{(N)}\langle\phi_r^{(N)}|, \tag{3.17}$$

that, obviously, has discrete eigenvalues. In the limit $\mathcal{D}\gg1$, this spectrum becomes dense, as it might be expected. But, on the opposite limit, one may be surprised to find that the state $|0,0\rangle$ is a relative-phase eigenstate (with arbitrary eigenvalue ϕ_0). While this may provide a convincing argument that the theory is unreasonable, we think that is not the case. The value of ϕ_0 will not lead to any contradictions, because any choice will lead to a consistent theory. Our choice of this parameter says nothing about nature, it only makes a statement about our individual preference [16].

Note as well, that the relative-phase eigenstates are maximally entangled states. This has the consequence that the relative-phase operator has no classical correspondence in the general case, not even for highly excited states.

IV. ATOM-FIELD RELATIVE PHASE IN TERMS OF ABSOLUTE PHASES

In the previous section, we have shown a clear way of obtaining *ab initio* the atom-field relative phase. In spite of this, one could still insist in describing this variable in terms of the absolute phases of each subsystem. One must start then from previous descriptions of the field and atomic phases and manage them until getting the probability distribution for their difference. The goal of this section is to show that this way of proceeding leads naturally to a positive operator-valued measure (POVM) [17,18], and how such a POVM is precisely generated by the eigenstates of the relative-phase operator.

To this end, we shall adopt the elegant axiomatic approach developed by Leonhardt *et al.* [19] to describe the phase properties of both subsystems we are dealing with. To make the discussion as self contained as possible, we first briefly recall the essential ingredients of the general formalism.

When dealing with generic angle-action variables, one imposes that the complex exponential of the angle (denoted by E) and the action variable (denoted by L_z) satisfy [compare with Eq. (3.8)]

$$[L_z,E]=E. \tag{4.1}$$

If E were unitary, its action on the basis of eigenstates of L_z (denoted by $|m\rangle$) will be as a ladder operator

$$E|m\rangle = |m+1\rangle. \tag{4.2}$$

The eigenstates of E (denoted by $|\theta\rangle$) provide then an adequate description of the quantum angle [20]. However, realistic measurements always involve extra noise beyond that due to the intrinsic quantum fluctuations described by canonical conjugation and it is essential to extend the canonical formalism by including fuzzy or unsharp generalizations of the ideal description provided by E [21]. To this end we shall use POVM's, that are a set of linear operators $\Delta(\theta)$ furnishing the correct probabilities in any measurement process through the fundamental postulate that

$$P(\theta) = \mathrm{Tr}[\rho\Delta(\theta)]. \tag{4.3}$$

The reality, positiveness, and normalization of $P(\theta)$ impose

$$\Delta^\dagger(\theta) = \Delta(\theta), \quad \Delta(\theta) \geqslant 0, \quad \int_{2\pi} d\theta \Delta(\theta) = I, \tag{4.4}$$

but, in general, $\Delta(\theta)$ are not orthogonal projectors like in the standard measurements described by self-adjoint operators.

In addition to these basic statistical conditions, some other requirements must be imposed to ensure that $\Delta(\theta)$ provides a meaningful description of the angle as a canonically conjugate variable with respect to L_z (even in the sense of a weak Weyl relation [22]). Then, we require

$$e^{i\theta' L_z}\Delta(\theta)e^{-i\theta' L_z} = \Delta(\theta+\theta'), \tag{4.5}$$

which reflects nothing but the basic feature that an angle shifter is an angle-distribution shifter. This condition restricts the form of the POVM to

$$\Delta(\theta) = \frac{1}{2\pi}\sum_{n,m=0}^{\infty} b_{n,m} e^{i(m-n)\theta}|m\rangle\langle n|. \tag{4.6}$$

We must take also into account that a shift in L_z should not change the angle distribution. A shift in L_z is expressed by the operator E, since it shifts the distribution of L_z by one step. Therefore, we require as well

$$E\Delta(\theta)E^\dagger = \Delta(\theta), \tag{4.7}$$

which, loosely speaking, is the physical translation of the fact that the angle should be complementary to the action variable. This implies the invariance

$$b_{n,m} = b_{n-m}, \tag{4.8}$$

that allows us to recast Eq. (4.6) as

$$\Delta(\theta) = \frac{1}{2\pi}\sum_{\nu}^{\infty} b_{-\nu} e^{-i\nu\theta} E^\nu, \tag{4.9}$$

while conditions (4.4) read now as

$$|b_\nu| \leqslant 1, \quad b_\nu^* = b_\nu. \tag{4.10}$$

Expressing E in terms of its eigenvectors $|\theta\rangle$, we finally arrive at the general form of a POVM describing the angle variable and fulfilling the natural requirements (4.5) and (4.7):

$$\Delta(\theta) = \int_{2\pi} d\theta' B(\theta')|\theta+\theta'\rangle\langle\theta+\theta'|, \tag{4.11}$$

where

$$B(\theta) = \frac{1}{2\pi}\sum_{\nu=0}^{\infty} b_\nu e^{i\nu\theta}. \tag{4.12}$$

This convolution shows that this effectively represents a noisy measurement, the function $B(\theta)$ giving the resolution provided by this POVM [20].

Let us now focus on the phase properties of our two subsystems. Concerning the field phase, the question has attracted the attention of physicists for almost as long as quantum mechanics has existed as a physical theory (for recent reviews, see Ref. [23]). Nowadays, it seems indisputable that an operator representing the phase of a single-mode field in a infinite-dimensional Hilbert space cannot exist [24] and the proper way to face upto the problem involves the use of a relative-state formalism. In spite of this serious drawback, a variety of solutions have been proposed to circumvent the difficulties. Virtually all of them can be formulated within the POVM formalism discussed before, but with the role of L_z being played by the number operator $a^\dagger a$. Then, a number shifter is expressed by the Susskind-Glogower phase operator [25] (note that we are not concerned about the problems of E as a phase operator here, we only use the number-shifter property) and the phase states are

$$|\theta_{\mathrm{f}}\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} e^{in\theta_{\mathrm{f}}}|n\rangle_{\mathrm{f}}. \tag{4.13}$$

On the other hand, the difficulties with hermiticity that phase operators encounter in the case of a single-mode field disappear for a two-level system. In general, for the group SU(2) it is possible, by working in the standard $(A+1)$-dimensional Hilbert space associated with the usual angular-momentum operators, to find a true phase operator from a polar decomposition of the amplitudes $S_\pm$ [26,27]. The procedure is quite similar to that followed in Sec. III for the relative phase and, perhaps, the most striking consequence of this approach is that the atomic-dipole phase can take only $A+1$ different values, due to the dimension of the atomic-state space.

Because of this particular behavior, one may think it is rather preferable to describe the dipole phase by a POVM taking continuous values in a 2π interval, even though this cannot lead to a standard operator description. To this end, it suffices to note that the general properties (4.5) and (4.7) still hold, but the role of L_z is played now by S_3, and the ''Susskind-Glogower'' eigenstates of the shifter are now

$$|\theta_a\rangle = \frac{1}{\sqrt{2\pi}} \sum_{M=0}^{A} e^{iM\theta_a} |M\rangle_a . \tag{4.14}$$

The joint probability distribution for atomic and field phases can be defined in a very natural way as

$$P(\theta_a, \theta_f) = \mathrm{Tr}[\rho \Delta(\theta_a, \theta_f)], \tag{4.15}$$

with

$$\Delta(\theta_a, \theta_f) = \Delta^a(\theta_a) \otimes \Delta^f(\theta_f). \tag{4.16}$$

Our remaining task is to consistently derive a POVM for the relative phase $\phi = \theta_a - \theta_f$ from these expressions. This goal can be achieved in many ways [28]: for example, one can perform a change of variables to express $\Delta(\theta_a, \theta_f)$ in terms of the phase sum and phase difference and then remove the phase-sum dependence by simple integration [20]. Another possibility is to directly define the probability distribution for the relative phase as

$$P(\phi) = \int_{2\pi} d\theta P(\theta, \theta + \phi) = \mathrm{Tr}[\rho \Delta(\phi)], \tag{4.17}$$

where

$$\Delta(\phi) = \int_{2\pi} d\theta \Delta(\theta, \theta + \phi). \tag{4.18}$$

For our purposes here, it is sufficient to note that we must get the same values for any periodic function of the relative phase whether we use the variable ϕ or (θ_a, θ_f). In consequence, we can impose that

$$\int_{2\pi} d\phi P(\phi) e^{i\nu\phi} = \int_{2\pi} d\theta_a d\theta_f P(\theta_a, \theta_f) e^{i\nu(\theta_a - \theta_f)}. \tag{4.19}$$

To proceed with this, we need to take a definite choice for the corresponding POVM's. Concerning the field phase, we recall that the Pegg-Barnett formalism and many others embodying the concept of phase as an observable canonically conjugate to photon number, lead to the POVM induced by the Susskind-Glogower phase states, namely [15]

$$\Delta^f(\theta_f) = |\theta_f\rangle\langle\theta_f|. \tag{4.20}$$

Motivated by this choice, we can use, for the atomic phase, the finite-dimensional translation of this POVM; i.e.,

$$\Delta^a(\theta_a) = |\theta_a\rangle\langle\theta_a|, \tag{4.21}$$

with $|\theta_a\rangle$ given in Eq. (4.14). A simple calculation shows then that

$$\Delta(\phi) = \sum_{N=0}^{\infty} |\phi^{(N)}\rangle\langle\phi^{(N)}|, \tag{4.22}$$

where

$$|\phi^{(N)}\rangle = \frac{1}{2\pi} \sum_{M=0}^{\mathcal{D}} e^{iM\phi} |N-M, M\rangle. \tag{4.23}$$

Therefore, we conclude that the POVM generated by the eigenstates of our relative phase operator is just the same induced by other absolute-phase approaches (such as Susskind-Glogower or Pegg-Barnett), when cast to the appropriate 2π range. This is, in our view, another confirmation that the theory proposed works correctly.

V. RELATIVE-PHASE DISTRIBUTION FUNCTION

For any state, the information one can reap using a measurement of some observable is given by the statistical distribution of the measurement outcomes. For the relative phase, it seems natural to define the probability distribution function of a state described by the density matrix ρ as

$$P(N, \phi_r, t) = \langle \phi_r^{(N)} | \rho(t) | \phi_r^{(N)} \rangle. \tag{5.1}$$

However, for *physical states* [29] (i.e., states for which finite moments of the number operator are bounded) this expression will converge to a simpler form involving a continuous probability density that we shall write as

$$P(N, \phi, t) = \langle \phi^{(N)} | \rho(t) | \phi^{(N)} \rangle, \tag{5.2}$$

where the vectors $|\phi^{(N)}\rangle$ defined in Eq. (4.23) lie in the subspace $\mathcal{H}_N$ with total number of excitations N. In fact, this expression can be interpreted as a joint probability distribution for the relative phase and the total number of excitations. From it, we can derive the distribution for the relative phase as

$$P(\phi, t) = \sum_{N=0}^{\infty} P(N, \phi, t), \tag{5.3}$$

while

$$P(N, t) = \int_{2\pi} d\phi P(N, \phi, t), \tag{5.4}$$

can be viewed as the probability distribution of having N excitations in the system. These factorizations are an obvious consequence of the fact that the relative phase and the excitation number are compatible.

For the general initial state of Eq. (2.20) and the evolution given by Eq. (2.21) we have

$$P(N, \phi, t) = |\langle \phi^{(N)} | \Psi(t) \rangle|^2, \tag{5.5}$$

which, through direct calculation, gives

$$P(N, \phi, t) = \frac{1}{2\pi} \left| \sum_{M', M=0}^{\mathcal{D}} Q_{N-M} A_M C^N_{M'M}(t) e^{iM'\phi} \right|^2, \tag{5.6}$$

and then we arrive at the total relative-phase probability distribution:

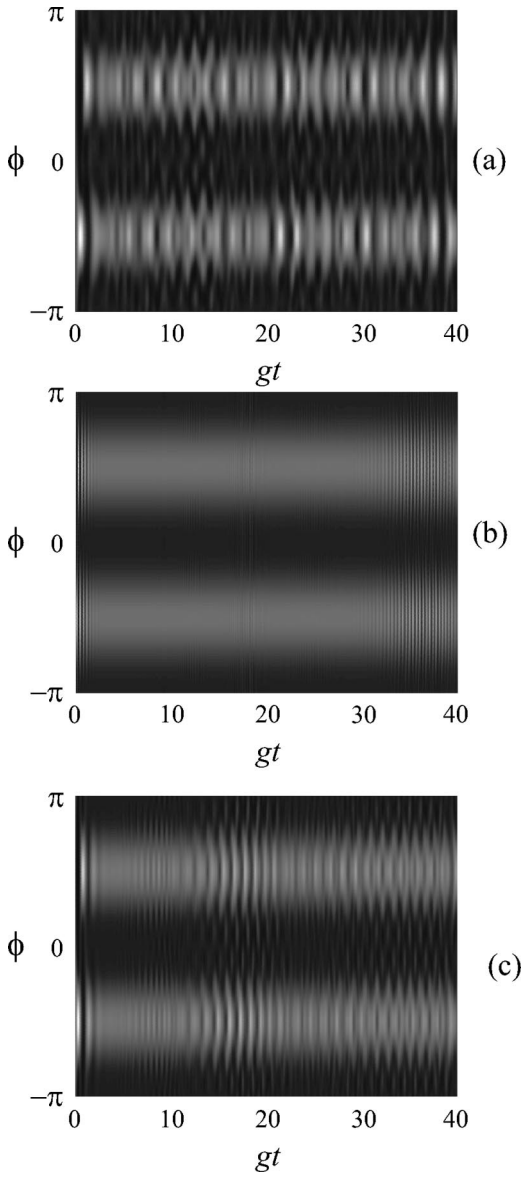

FIG. 1. Gray-level contour plot of the probability distribution $P(\phi,t)$ as a function of ϕ and the rescaled time gt for the case of $A=5$ atoms initially unexcited and the field in a coherent state with the following values of the mean number of photons: (a) $\bar{n}=1$ (weak field), (b) $\bar{n}=50$ (strong field), and $\bar{n}=5$ (intermediate field).

$$P(\phi,t)=\frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{M',M=0}^{\mathcal{D}}Q_{N-M}A_M C^N_{M'M}(t)e^{iM'\phi}\right|^2. \tag{5.7}$$

This is our basic and compact result that we use to analyze the evolution of the phase properties of the Dicke model.

In Fig. 1 we have numerically evaluated this distribution $P(\phi,t)$ as a function of ϕ and the rescaled a dimensional time gt, for the case when all the atoms are initially unexcited and the field is in a coherent state with various values of the mean number of photons $\bar{n}$. In all the cases, when $\tau=0$, we have that $C^N_{M'M}(0)=\delta_{M'M}$ and therefore

$$P(\phi,t=0)=\frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{M=0}^{\mathcal{D}}Q_{N-M}A_M e^{iM\phi}\right|^2. \tag{5.8}$$

In particular, when all the atoms are initially unexcited only the coefficient A_0 survives and the previous expression reduces to

$$P(\phi,t=0)=\frac{1}{2\pi}. \tag{5.9}$$

This flat distribution reflects the fact that the random phase of the dipole in such states induces a uniform distribution centered at ϕ_0. In this respect, it is interesting to notice that classically, the Lorentz model at resonance predicts for the relative phase values of $\pm\pi/2$. It turns out that this is also a possible choice to fix the reference phase ϕ_0 in the quantum description. For simplicity, in all the graphics we have chosen ϕ_0 as the origin 0.

Two quite different behaviors are evident from these graphics. The first occurs in the weak-field region [30–32], when the number of excitations in the system is much smaller than the number of the atoms, $N\ll A$. If, for simplicity, we assume that all the atoms are unexcited and the average number of photons in the initially coherent field is small, say $\bar{n}\sim 1$, then we can retain only the dominant terms in Eq. (5.7), getting

$$P(\phi,t)\simeq\frac{1}{2\pi}\{1+\bar{n}[|C^1_{00}|^2+|C^1_{01}|^2+2\,\mathrm{Re}(C^1_{00}C^1_{01}{}^*e^{i\phi})]\}e^{-\bar{n}}. \tag{5.10}$$

We see that, due to the periodic temporal dependence of the terms $C^N_{M'M}(t)$, this distribution is oscillatory for all times, which is corroborated numerically in Fig. 1(a).

The second (and perhaps more interesting) case corresponds to the strong-field region [33–35], when the initial number of photons is much larger than the number of atoms $A\ll N$. Then, following the ideas of Ref. [33] one can show that the coefficients $C^N_{M'M}(t)$ can be approximated, up to order $A/\sqrt{\bar{n}}$, by

$$C^N_{M'M}(t)\simeq d^A_{M'M}(-\Omega_N t), \tag{5.11}$$

where

$$\Omega_N=2g\sqrt{N-A/2+1/2}, \tag{5.12}$$

and $d^A_{M'M}$ are Wigner d functions [36], which are defined as the matrix elements for finite rotations by operators from SU(2) group representations

$$d^A_{M'M}(\vartheta)=d^A_{MM'}(\vartheta)=\langle M'|e^{i\vartheta S_x}|M\rangle, \tag{5.13}$$

where $M,M'=0,1,\ldots,A$. The point now is that essentially only one subspace of dimension $A+1$ dominates the dynamics. Moreover, a simple calculation using the explicit form of these d functions, gives

$$P(\phi,t)=\frac{1}{2\pi}\sum_{N=0}^{\infty}Q_N^2 \times\left|\sum_{M=0}^{A}\sqrt{\frac{A!}{(A-M)!M!}}\tan^M(\Omega_N t/2)e^{iM(\phi-\pi/2)}\right|^2 \times\cos^{2A}(\Omega_N t/2), \tag{5.14}$$

where we have assumed that all the atoms are initially unexcited. When $A\gg 1$ and when oscillations are well resolved, one can perform an expansion of the square root getting

$$P(\phi,t)=\sqrt{\frac{A}{2\pi}}\sum_N \Phi_N(t)\exp\left[-\frac{A}{2}(\phi-\pi/2+\delta_N)^2\right], \tag{5.15}$$

where $\Phi_N(t)$ is a function of time of complicated structure that accounts for the collapses and revivals and that is of little interest for our purposes here, and $\delta_N=\arg[\tan(\Omega_N t/2)]$. Now, it is clear that, since δ_N takes only the values 0 and π, the previous Gaussian distributions tend to have two peaks at $\phi=\pm\pi/2$, in agreement with the classical expectations. The presence of collapses and revivals are evident in Fig. 1(b), which confirms previous numerical and analytical evidence. The well-known nearly time-independent behavior in the time windows between collapse and revival is also clear. As we can see, the distribution tends to be randomized in the evolution, although keeping these two peaks at $\pm\pi/2$.

In the intermediate region, when $N\sim A$, the behavior is more complex, as shown in Fig. 1(c), and no analytical approximations are available.

For the particular case of the Jaynes-Cummings model, one can diagonalize exactly the Hamiltonian in each subspace $\mathcal{H}_N$, obtaining the well-known dressed states [37], that turn to be trapping states [38]; i.e., the atomic population $\langle S_3(t)\rangle$ remains constant in spite of the existence of both the radiation field and atomic transitions [39]. These states play a fundamental role in that model, so it seems interesting to analyze the corresponding problem for the case of the Dicke model. In the strong-field limit one can make the replacement $a\to\alpha=\sqrt{\bar{n}}e^{i\theta}$ and the interaction Dicke Hamiltonian becomes proportional to the operator

$$H_{\rm cl}=(e^{i\theta}S_+ + e^{-i\theta}S_-), \tag{5.16}$$

where the phase of the classical field has been chosen to coincide with the phase of the initial coherent state of the field. The semiclassical atomic states are defined now as eigenstates of $H_{\rm cl}$ taking this phase as zero:

$$2S_x|\underline{P}\rangle_{\rm a}=\Lambda_P|\underline{P}\rangle_{\rm a}, \tag{5.17}$$

with $\Lambda_P=A-2P$ and $P=0,1,\ldots,A$.

Following Ref. [33], we shall call *factorized states* those states for which the initial field is taken to be in a strong coherent state $|\alpha\rangle_{\rm f}$ and the atomic system is initially prepared in a semiclassical atomic state $|\underline{P}\rangle_{\rm a}$. For such states, the total wave function of the system can be approximately written as a product of its field and atomic parts

$$|\Psi(t)\rangle\simeq|\underline{P}(t)\rangle_{\rm a}\otimes|\alpha(t)\rangle_{\rm f}, \tag{5.18}$$

with

$$|\underline{P}(t)\rangle_{\rm a}=\exp\left[-i\frac{\Lambda_P(S_3+A/2)}{2\sqrt{\bar{n}}-A/2+1/2}gt\right]|\underline{P}\rangle_{\rm a},$$

$$|\alpha(t)\rangle_{\rm f}=\exp[-i\Lambda_P\sqrt{a^\dagger a-A/2+1/2}gt]|\alpha\rangle_{\rm f}, \tag{5.19}$$

and one can verify that they are also (approximately) trapping states. For these states, one can find after a simple calculation,

$$P(\phi,t)=\frac{1}{A+1}\left|\sum_M {}_{\rm a}\langle M|\underline{P}\rangle_{\rm a}e^{iM\phi}\right|^2. \tag{5.20}$$

The probability distribution is time independent due to the factorization (5.18). From the arguments in Ref. [33], one infers that this factorization holds up to times $gt\sim\sqrt{\bar{n}}$ (which can be very long times, in this limit) and with an accuracy in the coefficients of the order of $A/\sqrt{\bar{n}}$.

Moreover, using the properties of the semiclassical atomic states and assuming $A\gg 1$, one can replace the sum by an integral, obtaining finally

$$P(\phi,t)\simeq\sqrt{\frac{A}{2\pi}}e^{-A\phi^2/2}; \tag{5.21}$$

i.e., a Gaussian independent of time. In Fig. 2 we have plotted the probability distribution obtained from a numerical computation of Eq. (5.7), showing this quite remarkable behavior, except for the presence of very small (almost inappreciable) oscillations superimposed.

To gain more physical insight into these behaviors, in Fig. 3 we have plotted the evolution of the mean value of $\langle\sin\Phi\rangle$ for various values of N, confirming the previous physical discussion.

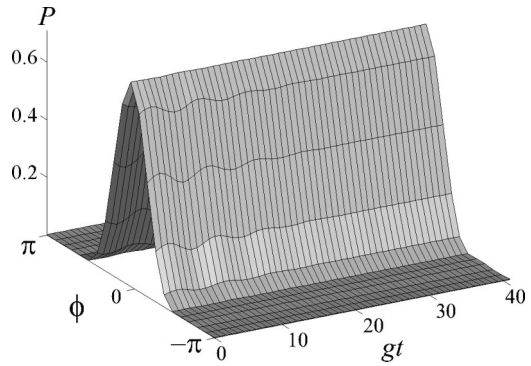

FIG. 2. Probability distribution function $P(\phi,t)$ as a function of ϕ and the rescaled time gt for the case of a factorized state with $A=3$. The atomic coherent state has $\vartheta=\pi/2$ and $\varphi=0$ and the field state has $\bar{n}=20$.

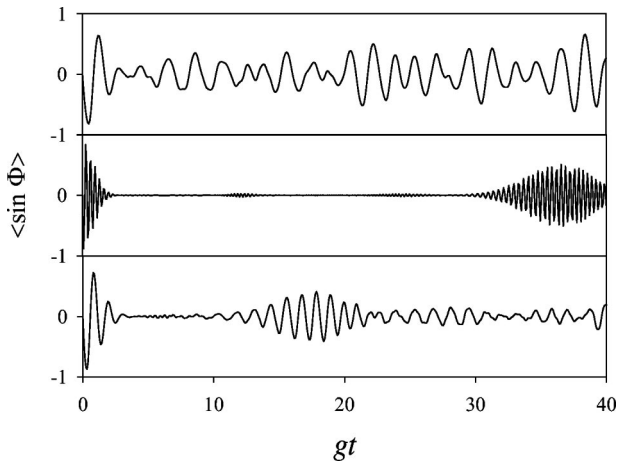

FIG. 3. Plots of $\langle \sin \phi \rangle$ versus gt for the same values of $\bar{n}$ as in Fig. 1 (from top to bottom).

To conclude, let us consider the Dicke model in the large-detuning limit; which is usually known as the dispersive limit. More specifically, we are in the case when

$$\Delta \gg g\sqrt{\bar{n}+1}A. \tag{5.22}$$

Then, following the procedure developed in Ref. [40], the interaction Hamiltonian in Eq. (2.2) can be replaced by the effective Hamiltonian

$$H_{\mathrm{eff}} = \Delta S_3 + \lambda(2a^{\dagger}a+1)S_3 + \lambda(C - S_3^2), \tag{5.23}$$

where

$$C = \frac{A}{2}\left(\frac{A}{2}+1\right), \quad \lambda = \frac{g^2}{\Delta}. \tag{5.24}$$

The obvious advantage of this Hamiltonian is that it is diagonal and allows for a compact analytical expression for the coefficients $C^N_{M'M}(t)$ as

$$C^N_{M'M}(t) = \delta_{M'M} \exp(-it\{\Delta(M-A/2) + \lambda[2(N-M)+1] \times (M-A/2) + \lambda[C-(M-A/2)^2]\}). \tag{5.25}$$

When the atoms are initially unexcited or excited (or, more generally, when $A_M = \delta_{MK}$) and for any initial state of field, we have

$$P(\phi,t) = \frac{1}{2\pi}, \tag{5.26}$$

for all the times.

For an arbitrary initial state of the atomic system and the field, we get

$$P(\phi,t) = \frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{M=0}^{A} Q_{N-M}A_M e^{-if_M^N t}e^{iM\phi}\right|^2, \tag{5.27}$$

with

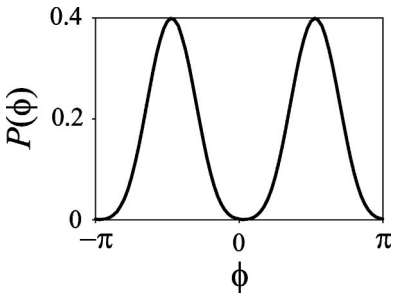

FIG. 4. Probability distribution function $P(\phi,t)$ as a function of ϕ for the time $\lambda t = \pi/6$ for the case of an atomic coherent state ($A=5$) with $\vartheta = \pi/2$ and $\varphi = 0$ and a field coherent state with $\bar{n} = 10$. The presence of the two humps corroborate the presence of a catlike state.

$$f_M^N = 2NM\lambda + [\Delta + \lambda(2A+1)]M - 3\lambda M^2. \tag{5.28}$$

Since Eq. (5.23) is quadratic in the population inversion operator S_3 and is, therefore, analogous to the Hamiltonian quadratic in the number operator of a single-mode field propagating through a Kerr medium, one could expect Eq. [41] that the evolution of coherent atomic states in the dispersive limit of the Dicke model leads to the generation of Schrödinger cat states. This superposition reaches the most pure form for initial number field states (in particular, the vacuum state minimizes the atomic entropy [42]).

The situation with the relative-phase distribution is quite different. It is easy to see, for example, that if the field is prepared initially in a number state $|k\rangle$ then $Q_n = \delta_{kn}$ and the relative phase distribution is flat. Nevertheless, for initial atomic and field coherent states the relative-phase distribution splits for some special times into several humps. These catlike states, according to Eq. (5.28), appear at times $\tau = \lambda t = \pi/6 \ (\mathrm{mod}\, 2\pi)$. To confirm this behavior analytically, we expand Eq. (5.27) when initially we have strong coherent states for both field and atoms, with $\bar{n} \gg A \gg 1$. By replacing once again the sum by integrals, one easily gets

$$P(\phi, t=\pi/6\lambda) = \sqrt{\frac{A}{8\pi}}\{e^{-[\phi-\pi\phi_{\bar{n}}/(3\lambda)]^2 A/2} + e^{-[\phi+\pi-\pi\phi_{\bar{n}}/(3\lambda)]^2 A/2}\}, \tag{5.29}$$

where

$$\phi_{\bar{n}} = 2\bar{n}\lambda + A + \lambda(2A+1), \tag{5.30}$$

and all the phases must be understood $\mathrm{mod}(2\pi)$. The two separated Gaussians indicates the presence of two humps and, therefore, the presence of catlike states. To further confirm this, in Fig. 4 we have computed numerically the distribution function $P(\phi, t=\pi/6\lambda)$ at the times predicted by the theory. The graphic clearly demonstrates the presence of the two-component state, according to our previous considerations.

VI. CONCLUSION

In this paper, we have investigated an appropriate operator for the quantum description of the relative phase in the Dicke model. We have used a proper polar decomposition of the corresponding field amplitudes, much in the spirit of our previous work on the subject. This polar decomposition has been justified on physical grounds, as well as using the theory of polynomial deformations of su(2).

The eigenvalue spectrum of this operator is discrete, as it happens, for the polar decomposition corresponding to two field modes. From these eigenstates we have obtained the probability distribution for the relative phase and we have studied its time evolution. For the weak-field region, the behavior is essentially oscillatory, while for the strong-field region, the relative phase tends to be randomized in the evolution, although showing collapses and revivals. For both limiting regions, we have developed analytical approximations that have allowed us an easy physical interpretation of some remarkable phenomena.

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