# Oscillations of rotating trapped Bose-Einstein condensates

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The tensor-virial method is applied for a study of oscillation modes of uniformly rotating Bose-Einstein condensed gases, whose rigid-body rotation is supported by an vortex array. The second-order virial equations are derived in the hydrodynamic regime for an arbitrary external harmonic trapping potential assuming that the condensate is a superfluid at zero temperature. The axisymmetric equilibrium shape of the condensate is determined as a function of the deformation of the trap; its domain of stability is bounded by the constraint $\Omega<1$ on the rotation rate (measured in units of the trap frequency $\omega_0$). The oscillations of the axisymmetric condensate are stable with respect to the transverse-shear and toroidal modes of oscillations, corresponding to the $l=2$, $|m|=1,2$ surface deformations. The eigenfrequencies of the modes are real and represent undamped oscillations. The condensate is also stable against quasiradial pulsation modes ($l=2$, $m=0$), and its oscillations are undamped, if the superflow is assumed incompressible. In the compressible case we find that for a polytropic equation of state, the quasiradial oscillations are unstable when $\gamma(3-\Omega^2)<1-3\Omega^2$, and are stable otherwise. Thus, a dilute Bose gas, whose equation of state is polytropic with $\gamma=2$ to leading order in the diluteness parameter, is stable irrespective of the rotation rate. In nonaxisymmetric traps, the equilibrium constrains the (dimensionless) deformation in the plane orthogonal to the rotation to the domain $A_2>\Omega^2$ with $\Omega<1$. The second-harmonic-oscillation modes in nonaxisymmetric traps separate into two classes that have even or odd parity with respect to the direction of the rotation axis. Numerical solutions show that these modes are stable in the parameter domain where equilibrium figures exist.

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## I. INTRODUCTION

After the experimental realization of Bose-Einstein condensation in vapors of alkali atoms [1–3], the understanding of their behavior under rotation became the main focus of theoretical and experimental work. Experimental observation of the vortex states in a two-component system [4], and the realization of the vortex states in a stirred one-component Bose-Einstein condensate [5] confirmed the expectations that sufficiently large condensates exhibit bulk superfluid properties by supporting their rotation above the critical angular frequency by the Feynman-Onsager vortex state [6–10].

The theoretical approaches to the Bose-Einstein condensed gases commonly distinguish between the two regimes of the strong and the weak interparticle interactions, which correspond to the conditions $Na/d\gg 1$ and $Na/d\ll 1$, respectively, where $N$ is the number of particles in the gas, $a$ is the scattering length, and $d=\sqrt{\hbar/m\omega_0}$ is the oscillator length defined in terms of the oscillator frequency $\omega_0$ and the boson mass $m$ (for a review see, e.g., Refs. [8,11].) In the weak-coupling regime, where the scale of the variation of the condensate wave function is of the same order as the size of the cloud, quantum effects play an essential role and the system behaves much like an atomic nucleus (see, e.g., Ref. [12]). In the strong-coupling regime and for sufficiently large condensate, the condensate is well described in terms of hydrodynamics. Under these conditions the time-dependent Gross-Pitaevskii (GP) equation [13] for the (complex) condensate wave function reduces to two (real) hydrodynamic equations for the density and velocity of the condensate at zero temperature. One consequence of the assumption that the system is sufficiently large is that the quantum pressure term can be ignored compared with the interaction terms. Another is that the Thomas-Fermi approximation [6] is valid and the equilibrium kinetic-energy density is negligible compared to the potential-energy density. This approximation also ensures that the thermodynamic quantities like the pressure or the chemical potential of the gas are well defined at any local point of the gas cloud. Finally, the coherence length of the condensate $\xi=1/\sqrt{8\pi na}$, with $n$ being the number density of condensate particles, becomes sufficiently small in this limit to allow for a description of a vortex core as a singularity.

The superfluid hydrodynamic approach has been used in the studies of the collective excitations of the condensate clouds [14,15], the scissors mode oscillations [16], the moment of inertia of a rotating condensate [17], mainly in the irrotational regime; see however Ref. [18]. The critical angular velocity of condensate $\Omega_{c1}$, at which the creation of a vortex is energetically favorable, scales, in units of the trap frequency, as $\Omega_{c1}\propto(d^2/R^2)\ln(R/\xi)$, where $R$ is the size of the cloud; the numerical coefficient depends on the details of the geometry of the trap [8]. For a sufficiently large condensate the ground state would correspond to coarse-grained rigid-body rotation supported by a vortex lattice. In this case, the irrotational constraint on the average superfluid velocity $\boldsymbol{\nabla}\times\mathbf{v}=\mathbf{0}$ is replaced by the rigid-body rotation condition $\boldsymbol{\nabla}\times\mathbf{v}=2\boldsymbol{\Omega}$. The purpose of this paper is the study of the oscillations of the trapped Bose condensates under rigid-body rotation. We shall not discuss the modes related to the vortex structure itself (e.g., Tkachenko modes). We shall confine ourselves to the zero-temperature limit, in which case the effects of mutual friction due to the interaction of the thermal excitations with the vortex state, and dissipation due to the viscosity of the thermal component both can be neglected.

 

Below, we shall apply the tensor-virial method originally developed for the study of equilibrium and stability of rotating liquid masses bound by self-gravitation [19]. The tensor-virial method transforms the local hydrodynamical equations into global virial equations that contain the full information on the structure and stability of a system as a whole. The method is especially useful for studying perturbations of incompressible uniform ellipsoids from equilibrium, in which case the each perturbed virial equation yields (in the absence of viscous dissipation) a different set of normal modes. The extension to compressible flows is straightforward in the case of adiabatic perturbations. Moreover, when the equilibrium is sustained by an external confining potential, fluid perturbations do not backreact (change) the confining potential itself, and the tensor virial method can be extended to nonuniform compressible flows. The description is particularly useful for gases with polytropic equations of state $p \propto \rho^{\gamma}$, where $p$ is the pressure, $\rho$ is the density, and $\gamma$ is the adiabatic index. In that case, we shall see that all of the modes of the gas can be found from the tensor-virial equations. (We illustrate this only for the second-harmonic modes of oscillation in this paper, but the extension to higher-order modes is straightforward, and will be discussed elsewhere.) For an interacting Bose gas the pressure is a nonanalytic function of the diluteness parameter $na^3$. However the equation of state to the leading order in the small parameter $na^3 \ll 1$ can be written in a polytropic form: $p = K\rho^2$, with $K = 2\pi\hbar^2 a/m^2$. Since the zero-temperature equations of motion of a trapped rotating condensate turn out formally identical to the corresponding equations of motion of a ordinary (nonsuperfluid) liquid, our results might be of significance in a broader context. On the other hand, present results can serve as a starting point for an extension to finite temperatures where, in addition to the superfluid, the fluid of normal quasiparticle excitations plays a role.

The paper is organized as follows. In Sec. II we derive the first- and second-order virial equations for trapped Bose condensed gases in the hydrodynamic regime. The equilibrium shape of rotating condensates and the second harmonic modes of oscillations in axisymmetric traps are discussed in Sec. III. In Sec. IV the second harmonic modes of oscillations in nonaxisymmetric traps are discussed. Our results are summarized in Sec. V. The Appendix gives a brief derivation of the hydrodynamic equations from the GP equation.

## II. VIRIAL EQUATIONS FOR A TRAPPED CONDENSATE

The Euler equation for a condensate in a harmonic, arbitrary deformed trap, can be written as

$$\rho\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right) u_i = -\frac{\partial p}{\partial x_i} - \frac{\rho}{2}\frac{\partial \phi_{\rm tr}}{\partial x_i} + \frac{\rho}{2}\frac{\partial |\mathbf{\Omega}\times\mathbf{x}|^2}{\partial x_i} + 2\rho\epsilon_{ilm}u_l\Omega_m, \tag{1}$$

where the Latin subscripts denote coordinate directions; $\rho$, $p$, and $\mathbf{u}$ are the density, pressure, and velocity of condensate (summation over repeated indexes is assumed). The external harmonic trapping potential is

$$\phi_{\rm tr}(\mathbf{x}) = \omega_0^2 \sum_{i=1}^{3} A_i x_i^2, \tag{2}$$

where $A_i$ are the dimensionless deformation parameters, and $\omega_0$ is the frequency of the harmonic oscillator in the trap. Equation (1) is written in a frame rotating with angular velocity $\mathbf{\Omega}$ relative to some inertial coordinate reference system. Its derivation from the GP equation for the condensate wave function is given in the Appendix. Our fundamental assumption is that the condensate undergoes a rigid-body rotation, which is supported by an array of vortices. In this case, as explained in the Appendix, the equations of motion are formally identical to those of a ordinary compressible (nonsuperfluid) liquid.

Starting from Eq. (1) we can construct a set of virial equations of various order. Suppose that the condensate occupies a volume $V$. Taking the zeroth moment of Eq. (1) amounts to integrating over $V$; doing so, we obtain

$$\frac{d}{dt}\left(\int_V d^3x\, \rho u_i\right) = 2\epsilon_{ilm}\Omega_m \int_V d^3x\, \rho u_l + (\Omega^2\delta_{ij} - \Omega_i\Omega_j)\int_V d^3x\, \rho x_j - \int_V d^3x \frac{\rho}{2}\frac{\partial \phi_{\rm tr}}{\partial x_i}. \tag{3}$$

The first-order virial equation describes the center-of-mass motion of the condensate, and is trivial as these motions correspond to a uniform translation of the system as a whole.

Taking the first moment of Eq. (1) results in the second-order virial equation

$$\frac{d}{dt}\left(\int_V d^3x\, \rho x_j u_i\right) = 2\epsilon_{ilm}\Omega_m\left(\int_V d^3x\, \rho x_j u_l\right) + (\Omega^2 - \omega_0^2 A_i) I_{ij} - \Omega_i\Omega_k I_{kj} + 2\mathcal{T}_{ij} + \delta_{ij}\Pi, \tag{4}$$

where

$$I_{ij} \equiv \int_V d^3x\, \rho x_i x_j, \quad \Pi \equiv \int_V d^3x\, p, \quad \mathcal{T}_{ij} \equiv \frac{1}{2}\int_V d^3x\, \rho u_i u_j. \tag{5}$$

Consider the variation of the second-order virial equation under the influence of perturbations. The Eulerian variations of various terms in Eq. (4) are straightforward [20] and we find

$$\delta\frac{d}{dt}\int_V d^3x\, \rho u_i x_j = (\Omega^2 - \omega_0^2 A_i) V_{ij} - \Omega_i\Omega_k V_{kj} + 2\epsilon_{ilm}\Omega_m \delta\int_V d^3x\, \rho u_l x_j + 2\delta\mathcal{T}_{ij} + \delta_{ij}\delta\Pi, \tag{6}$$

where for a Lagrangian displacement $\boldsymbol{\xi}$:

$$V_{ij}\equiv\delta I_{ij}=\int_V d^3x\,\rho(\xi_i x_j+\xi_j x_i). \tag{7}$$

The tensors $V_{ij}$ are manifestly symmetric in their indexes, as is evident from their definition. It is useful to define also their nonsymmetric parts as

$$V_{i;j}=\int_V d^3x\,\rho\xi_i x_j, \tag{8}$$

such that $V_{ij}=V_{i;j}+V_{j;i}$. When there are no condensate motions in the unperturbed state in the rotating frame ($\mathbf{u}=0$), the second-order virial equation becomes

$$\frac{d^2V_{i;j}}{dt^2}=2\epsilon_{ilm}\Omega_m\frac{dV_{l;j}}{dt}+(\Omega^2-\omega_0^2A_i)V_{ij}-\Omega_i\Omega_kV_{kj}+\delta_{ij}\delta\Pi. \tag{9}$$

The tensor virial [Eq. (9)] can be used for the study of small amplitude oscillations of a trapped condensate. For time-dependent Lagrangian displacements of the form

$$\boldsymbol{\xi}(x_i,t)=\boldsymbol{\xi}(x_i)e^{\lambda t}, \tag{10}$$

Eq. (9) becomes

$$\lambda^2V_{i;j}-2\Omega\epsilon_{ilm}\Omega_m\lambda V_{l;j}=(\Omega^2-\omega_0^2A_i)V_{ij}-\Omega_i\Omega_kV_{kj}+\delta_{ij}\delta\Pi. \tag{11}$$

Equation (11) contains all the second-harmonic modes of the rotating condensate in an arbitrary harmonic trap.

## III. EQUILIBRIUM AND OSCILLATIONS IN AXISYMMETRIC TRAPS

### A. Equilibrium shape

Next, we specialize Eq. (4) to the case of axisymmetric traps, $A_1=A_2\neq A_3$ and assume, without loss of generality, $A_1=1$. The assumption of the axisymmetry is, in practice, an approximation as a perfectly axisymmetric trap will not exert a torque on the condensate. Our assumption is that the deviations from the axisymmetry, which drive the rotation of the condensate, are small compared with the deformation of the trap $|A_1-A_2|/A_1\ll 1$. In principle, once the condensate is brought to rotation the axisymmetry can be restored.

Consider a equilibrium state in which the condensate rotates uniformly with the rotation vector along the $x_3$ axis of Cartesian system of coordinates. Then Eq. (4) reduces to

$$\Omega^2(I_{ij}-\delta_{i3}I_{3j})-\omega_0^2A_iI_{ij}=-\delta_{ij}\Pi. \tag{12}$$

In axisymmetric traps with $a_3$ denoting the ellipsoidal semiaxis along the rotation, the two remaining semiaxis of the ellipsoid, $a_1$ and $a_2$, are equal. The diagonal component of Eq. (12) provides the relation between the rotation frequency, the deformation parameters along the $x_3$ direction (both are fixed in an experiment) and the ratio of the semiaxis of the resulting spheroidal figure:

$$\frac{a_3^2}{a_1^2}=\frac{1}{A_3}\left(1-\frac{\Omega^2}{\omega_0^2}\right). \tag{13}$$

Depending on whether $a_3\leqslant a_1$ or $a_3\geqslant a_1$ the condensate shape is either oblate or prolate. Note that the rotation frequency is bounded from above, $\Omega/\omega_0<1$, and for $A_3>1$ the equilibrium figures are always oblate. A striking consequence of Eq. (13), when $A_3=1-\Omega^2/\omega_0^2$, is an equilibrium figure that is a rotating sphere [21]. Although we had to assume small deviations from axisymmetry to make the condensate rotate, this assumption is a prerequisite for producing rotation, but not an intrinsic property of the rotating condensate.

The equilibrium density profile of the condensate, for a polytropic equation of state, can be obtained by a direct integration of the unperturbed limit of Eq. (1):

$$\rho(\mathbf{x})=\rho_0\left\{1-\frac{\gamma-1}{2K\gamma\rho_0^{\gamma-1}}[(1-\Omega^2)x_1^2+(A_2-\Omega^2)x_2^2+A_3x_3^2]\right\}^{1/\gamma-1}\theta(\cdots), \tag{14}$$

where $\rho_0$ is the central density of the cloud; the dots in the argument of the $\theta$ function (which insures that the condensate density is positive) stand for the expression in the curly brackets. Here we have retained the parameter $A_2\neq A_1$ to incorporate the case of nonaxisymmetric traps, which is treated in Sec. III B. The effect of the rotation is the stretching of the profile of the condensate in the equatorial plane due to the centrifugal potential. According to Eq. (14), the density is constant on concentric ellipsoids in the unperturbed rotating background. Note that the normalization of the wave function of the condensate implies

$$\int_V d^3x\rho(\mathbf{x})=M, \tag{15}$$

where $M$ is the total mass in the cloud.

### B. Second-harmonic modes of oscillation

Next consider perturbations from the equilibrium state of uniform rotation, with the spin vector along the $x_3$ axis. Surface deformations related to various modes can be classified by corresponding terms of the expansion in surface harmonics labeled by indexes $l$ and $m$. Second-order harmonic deformations correspond to $l=2$ with five distinct values of $m$, $-2\leqslant m\leqslant 2$. The 18 equations represented by Eq. (11) separate into two independent subsets that are odd and even with respect to the index 3. The corresponding oscillation modes can be treated separately.

### *1. Transverse-shear modes*

These modes correspond to the surface deformations with $|m|=1$ and represent relative shearing of the northern and southern hemispheres of the spheroid. The components of Eq. (11), which are odd in index 3, are

$$\lambda^2 V_{3;1} = -A_3 V_{31}, \tag{16}$$

$$\lambda^2 V_{3;2} = -A_3 V_{32}, \tag{17}$$

$$\lambda^2 V_{1;3} - 2\Omega\lambda V_{2;3} = -V_{13} + \Omega^2 V_{13}, \tag{18}$$

$$\lambda^2 V_{2;3} + 2\Omega\lambda V_{1;3} = -V_{23} + \Omega^2 V_{23}. \tag{19}$$

We sum Eqs. (16), (18) and (17), (19), respectively, and use the symmetry properties of $V_{ij}$ combined with Eqs. (17) and (19). This results in the relations

$$\lambda(\lambda^2 + A_3 + 1 - \Omega^2)V_{13} - 2\Omega(\lambda^2 + A_3)V_{23} = 0, \tag{20}$$

$$\lambda(\lambda^2 + A_3 + 1 - \Omega^2)V_{23} - 2\Omega(\lambda^2 + A_3)V_{23} = 0. \tag{21}$$

The characteristic equation can be factorized by substituting $\lambda = i\sigma$ and we find

$$\sigma^3 - 2\Omega\sigma^2 - (1 + A_3 - \Omega^2)\sigma + 2\Omega A_3 = 0. \tag{22}$$

The roots are given by

$$\sigma_1 = \frac{2\Omega}{3} + (s_+ + s_-), \quad \sigma_{2,3} = \frac{2\Omega}{3} - \frac{1}{2}(s_+ + s_-) \pm \frac{i\sqrt{3}}{2}(s_+ - s_-), \tag{23}$$

where

$$s_\pm^3 = \frac{\Omega}{3}\left(1 - 2A_3 - \frac{\Omega^2}{9}\right) \mp \frac{1}{9}\left[\left(1 + A_3 + \frac{\Omega^2}{3}\right)^3 - \Omega^2\left(1 - 2A_3 - \frac{\Omega^2}{9}\right)^2\right]^{1/2}. \tag{24}$$

Three complementary modes follow from Eqs. (23) and (24) via the replacement $\Omega \to -\Omega$. It can be verified that the condition

$$\left[\frac{\Omega}{3}\left(1 - 2A_3 - \frac{\Omega^2}{9}\right)\right]^2 - \left[\frac{1}{3}\left(1 + A_3 + \frac{\Omega^2}{3}\right)\right]^3 \leqslant 0 \tag{25}$$

is satisfied for any $A_3$ and $\Omega < 1$, therefore all three roots are real. The real frequencies of the transverse-shear modes are shown in Fig. 1 as a function of $A_3$. Note that in the zero-temperature limit, to which present analysis is restricted, these modes are purely real, i.e., represent undamped oscillations.

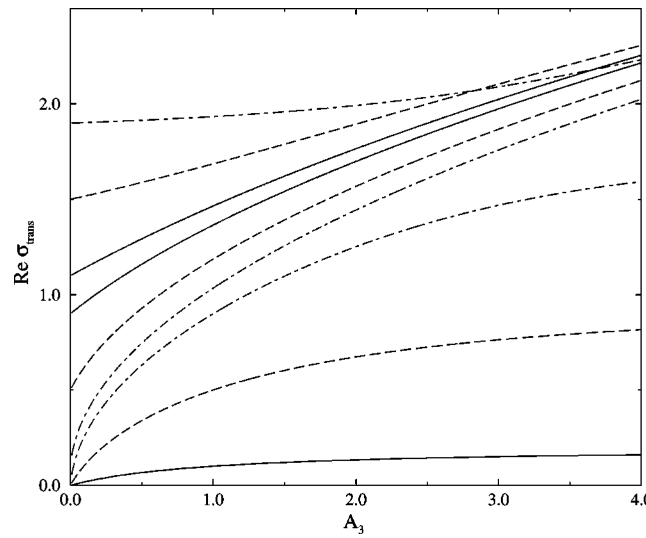


FIG. 1. The three real frequencies of the transverse-shear modes in axisymmetric traps as a function of deformation parameter $A_3$ for three values of $\Omega = 0.1$ (*solid line*), 0.5 (*dashed line*), 0.9 (*dashed-dotted line*); here the spin frequency is measured in units of $\omega_0$.

### *2. Toroidal modes*

These modes correspond to $|m|=2$ and the motions in this case are confined to the planes parallel to the equatorial plane. The components of Eq. (11), which are even in index 3, are

$$\lambda^2 V_{3;3} = \delta\Pi - A_3 V_{33}, \tag{26}$$

$$\lambda^2 V_{1;1} - 2\Omega\lambda V_{2;1} = \delta\Pi + (\Omega^2 - 1)V_{11}, \tag{27}$$

$$\lambda^2 V_{2;2} + 2\Omega\lambda V_{1;2} = \delta\Pi + (\Omega^2 - 1)V_{22}, \tag{28}$$

$$\lambda^2 V_{1;2} - 2\Omega\lambda V_{2;2} = (\Omega^2 - 1)V_{12}, \tag{29}$$

$$\lambda^2 V_{2;1} + 2\Omega\lambda V_{1;1} = (\Omega^2 - 1)V_{21}. \tag{30}$$

We add Eqs. (29) and (30), and subtract Eqs. (27) and (28) to find the following coupled equations:

$$[\lambda^2 + 2(1 - \Omega^2)](V_{11} - V_{22}) - 4\Omega\lambda V_{12} = 0, \tag{31}$$

$$[\lambda^2 + 2(1 - \Omega^2)]V_{12} + \Omega\lambda(V_{11} - V_{22}) = 0. \tag{32}$$

The characteristic equation for the toroidal modes is

$$[\lambda^2 - 2(\Omega^2 - 1)]^2 + 4\lambda^2\Omega^2 = 0, \tag{33}$$

which is factorized by writing $\lambda = i\sigma$. Two solutions are then

$$\sigma_{1,2} = \Omega \pm \sqrt{2 - \Omega^2}. \tag{34}$$

There are two complementary modes that are found by substituting $-\Omega$ for $\Omega$. Since the rotation frequency is bounded $(\Omega < 1)$ the toroidal modes are stable independent of the magnitude of the deformation in the equatorial plane. For the same reason these modes are undamped in the zero-temperature limit considered here.

#### *3. Pulsation modes*

To find the pulsation modes, which correspond to $m=0$, we first add the Eqs. (27) and (28), and then use Eq. (26) to eliminate $\delta\Pi$ in the result. In this manner we find that

$$(\lambda^2/2-\Omega^2+1)(V_{11}+V_{22})+2\Omega\lambda(V_{1;2}-V_{2;1}) -(\lambda^2+2A_3)V_{33}=0. \tag{35}$$

Subtracting Eqs. (30) and (29) (and discarding the $\lambda=0$ root) one finds

$$\lambda(V_{1;2}-V_{2;1})-\Omega(V_{11}+V_{22})=0. \tag{36}$$

Equations (35) and (36) can be further combined to a single equation:

$$(\lambda^2+2\Omega^2+2)(V_{11}+V_{22})-2(\lambda^2+2A_3)V_{33}=0. \tag{37}$$

It is instructive first to consider the case where the superfluid is incompressible. Then Eq. (37) should be supplemented by the divergence free condition

$$\frac{V_{11}+V_{22}}{a_1^2}+\frac{V_{33}}{a_3^2}=0. \tag{38}$$

Again, writing $\lambda=i\sigma$, we find for the square of frequency of the pulsation mode in the incompressible limit

$$\sigma_0^2=\left(\frac{1}{2}+\frac{a_3^2}{a_1^2}\right)^{-1}\left(1+\Omega^2+2\frac{a_3^2}{a_1^2}A_3\right). \tag{39}$$

As $\sigma_0^2$ is always positive, these modes correspond to undamped stable oscillations.

For compressible fluids we need the variation of the pressure tensor, which for adiabatic perturbations can be written as

$$\delta\Pi=(\gamma-1)\int d^3x\,\xi_i\boldsymbol{\nabla}_i p, \tag{40}$$

where we assumed a polytropic equation of state $p=K\rho^\gamma$; the polytropic index for a Bose gas to leading order in the parameter $\rho a^3$ is equal 2. To evaluate the gradient of the pressure we turn to the Euler Eq. (1) in the unperturbed limit,

$$\frac{\partial h}{\partial x_i}\equiv\frac{1}{\rho}\frac{\partial p}{\partial x_i}=-\frac{1}{2}\frac{\partial\phi_{\rm tr}}{\partial x_i}+\frac{1}{2}\frac{\partial|\boldsymbol{\Omega}\times\mathbf{x}|^2}{\partial x_i}, \tag{41}$$

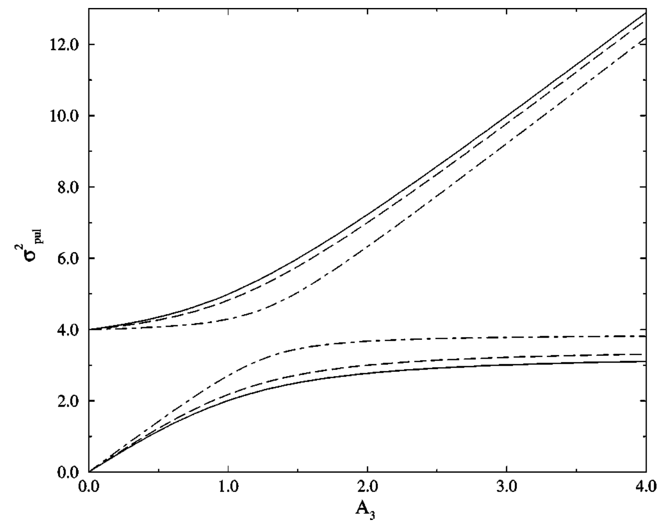


FIG. 2. The square of the pulsation modes in axisymmetric traps for $\gamma=2$. Conventions are the same as in Fig. 1.

where the enthalpy $h$ is defined by $dh=dp/\rho$ and is simply $h=\gamma K\rho^{\gamma-1}/(\gamma-1)$ for $p=K\rho^\gamma$. Substituting the explicit expression for the trapping potential, we find that the gradient of the enthalpy is a linear function of the coordinates and, hence, the variation of the pressure tensor can be expressed in terms of virials $V_{ii}$:

$$\delta\Pi=-\frac{(\gamma-1)}{2}[(V_{11}+V_{22})(1-\Omega^2)+A_3V_{33}]. \tag{42}$$

Any of the equations, which are even in the index 3, now can be used to close the system of equations for $V_{33}$ and the virial combination $V_{11}+V_{22}$. Substituting Eq. (42) for $\delta\Pi$ in, e.g, Eq. (26), one finds

$$[\lambda^2+(\gamma+1)A_3]V_{33}+(\gamma-1)(1-\Omega^2)(V_{11}+V_{22})=0. \tag{43}$$

Equations (37) and (43) completely determine the unknown virials; the characteristic equation for the pulsation modes, which is quadratic in $\lambda^2$, is

$$\lambda^4+\lambda^2[\gamma(A_3+2-2\Omega^2)+A_3+4\Omega^2]+2A_3[\gamma(3-\Omega^2) +3\Omega^2-1]=0. \tag{44}$$

On substituting $\lambda=i\sigma$, the solution of the resulting quadratic equation becomes

$$\sigma_\pm^2=\tfrac{1}{2}[\gamma(A_3+2-2\Omega^2)+A_3+4\Omega^2]\pm\tfrac{1}{2}\sqrt{[\gamma(A_3+2-2\Omega^2)+A_3+4\Omega^2]^2-8A_3[\gamma(3-\Omega^2)+3\Omega^2-1]}. \tag{45}$$

It is easy to see that there are only unstable modes if $\gamma<(1-3\Omega^2)/(3-\Omega^2)\leqslant 1/3$; otherwise all modes are stable. In particular, all modes are stable for $\gamma=2$. Note, too, that there are twice as many solutions as were found in the incompressible case; Eq. (39) can be recovered from Eq. (45) by taking the $\gamma\to\infty$ limit of $\sigma_-^2$. The origin of the additional root $\sigma_+^2$, may be traced to the $\lambda^2$ dependence of Eq. (43), which only reduces to the incompressibility condition, Eq. (38), in the $\gamma\to\infty$ limit if it is assumed that $|\lambda^2| = |\sigma^2| \ll \gamma$.

The square of the pulsation modes as a function of the $A_3$ is plotted in Fig. 2 for $\gamma=2$, the polytropic index of a dilute Bose gas.

## IV. EQUILIBRIUM AND OSCILLATIONS IN NONAXISYMMETRIC TRAPS

### A. Equilibrium shape

When the symmetry with respect to the rotation axis is broken, the equilibrium constraint [Eq. (13)] needs to be supplemented by a relation fixing the semiaxis ratios in the plane orthogonal to the spin axis. The diagonal components of Eq. (12) provide the triangle relations that determine the nonaxisymmetric equilibrium figure

$$a_1^2(\Omega^2-\omega_0^2A_1)=a_2^2(\Omega^2-\omega_0^2A_2)=-a_3\omega_0^2A_3. \quad (46)$$

These simultaneous constraints can be written in an equivalent form

$$\frac{a_3^2}{a_1^2}=\frac{1}{A_3}\left(1-\frac{\Omega^2}{\omega_0^2}\right), \quad \frac{a_3^2}{a_2^2}=\frac{1}{A_3}\left(A_2-\frac{\Omega^2}{\omega_0^2}\right). \quad (47)$$

Note that in addition to the upper bound on the rotation frequency set by the first relation (as in the case of the axisymmetric traps), the second relations places a lower bound on the deformation in the plane orthogonal to the spin axis: $A_2\geq\Omega^2/\omega_0^2$. Given the experimentally controlled values of $\Omega$, $A_3$ and $A_2$, relation (47) determine, in a unique manner, the semiaxis ratios of the resulting figure.

### B. Second-harmonic modes of oscillation

The nonaxisymmetric modes can be found from Eq. (11) in a manner similar to the axisymmetric modes; however, now the degeneracy in indexes 1 and 2 should be relaxed. The oscillation modes separate into two noncombining groups, which have even or odd parity with respect to the index 3. Below, we shall treat these modes separately.

#### *1. Odd modes*

Among the four components of Eq. (11), which are odd in index 3, three are identical to Eqs. (16), (17), and (18) under nonaxisymmetric conditions; the component which is modifed reads

$$\lambda^2V_{2;3}+2\Omega\lambda V_{1;3}=-A_2V_{23}+\Omega^2V_{23}. \quad (48)$$

Summing Eqs. (16), (18), and (17), (48), we arrive at

$$\lambda(\lambda^2+A_3+1-\Omega^2)V_{13}-2\Omega(\lambda^2+A_3)V_{23}=0, \quad (49)$$

$$\lambda(\lambda^2+A_3+A_2-\Omega^2)V_{23}-2\Omega(\lambda^2+A_3)V_{23}=0. \quad (50)$$

The sixth order characteristic equation derived from this algebraic system is

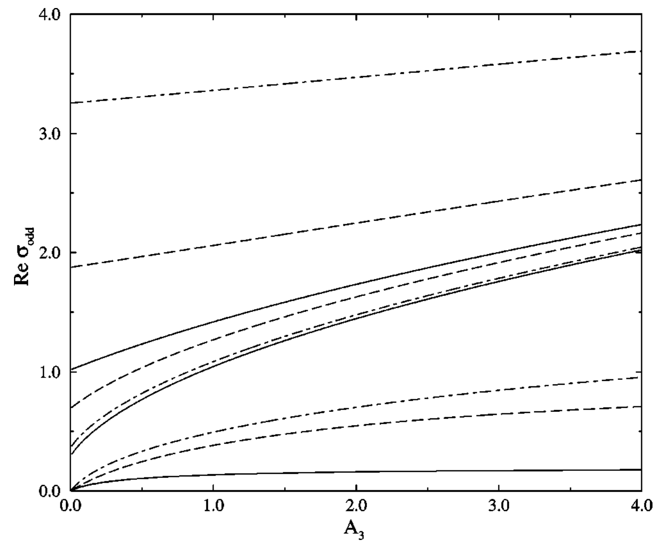


FIG. 3. The real parts of the odd-parity modes in nonaxisymmetric traps for the fixed ratio $A_2/\Omega^2=0.1$. Conventions are the same as in Fig. 1.

$$\lambda^6+[1+A_2+2(A_3+\Omega^2)]\lambda^4+[A_2+A_3+A_2A_3+A_3^2-(1+A_2-6A_3)\Omega^2+\Omega^4]\lambda^2+4A_3^2\Omega^2=0. \quad (51)$$

The corresponding modes appear as conjugate pairs, i.e., there are only 3 distinct modes. The real parts of these modes (the modes are purely real) are show in Fig. 3 as a function of $A_3$ for several values of $\Omega$ and fixed ratio $\Omega^2/A_2=0.1$.

#### *2. Even modes*

The even-parity components of Eq. (11) which are given by the Eqs. (26), (27), and (29), remain unchanged when axisymmetry is relaxed; the remainder equations read

$$\lambda^2V_{2;2}+2\Omega\lambda V_{1;2}=\delta\Pi+(\Omega^2-A_2)V_{22}, \quad (52)$$

$$\lambda^2V_{2;1}+2\Omega\lambda V_{1;1}=(\Omega^2-A_2)V_{21}. \quad (53)$$

In the incompressible limit these equations should be supplemented by condition (38) (with an obvious modification of the second term). In the compressible case the variations of the pressure become

$$\delta\Pi=-\frac{(\gamma-1)}{2}[(1-\Omega^2)V_{11}+(A_2-\Omega^2)V_{22}+A_3V_{33}]. \quad (54)$$

Using this relation in Eq. (26) we find

$$[\lambda^2+(\gamma+1)A_3]V_{33}+(\gamma-1)[(1-\Omega^2)V_{11}+(A_2-\Omega^2)V_{22}]=0. \quad (55)$$

Equations (26), (27), (29) and (52) and (53) can be manipulated to the following set:

$$[\lambda^2+2(1-\Omega^2)]V_{11}-[\lambda^2+2(A_2-\Omega^2)]V_{22}-4\lambda\Omega V_{12}=0, \quad (56)$$

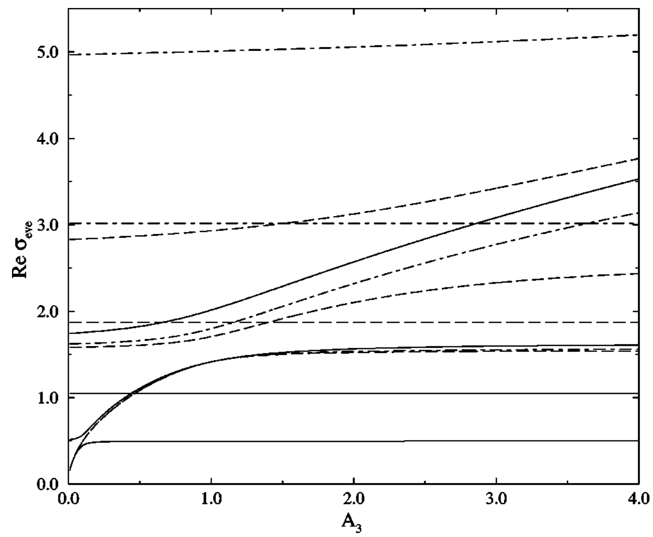


FIG. 4. The real parts of the even-parity modes in nonaxisymmetric traps for the fixed ratio $A_2/\Omega^2=0.1$ and $\gamma=2$. Conventions are the same as in Fig. 1.

$$[\lambda^2+2(1-\Omega^2)]V_{11}+[\lambda^2+2(A_2-\Omega^2)]V_{22} -2(\lambda^2+2A_3)V_{33}+4\lambda\Omega(V_{1;2}-V_{2;1})=0, \quad (57)$$

$$(\lambda^2+1+A_2-2\Omega^2)V_{12}+\Omega\lambda(V_{11}-V_{22})=0, \quad (58)$$

$$\lambda^2(V_{1;2}-V_{2;1})-\Omega\lambda(V_{11}+V_{22})+(1-A_2)V_{12}=0, \quad (59)$$

which should be supplemented with Eq. (55). The corresponding characteristic equation is of order 8 and has been solved numerically. The results for the real parts of the modes (which are purely real) are shown in Fig. 4 as a function of $A_3$ for several values of $\Omega$ for fixed ratio $\Omega^2/A_2=0.1$.

## V. CONCLUSIONS

We have analyzed the hydrodynamic oscillations of Bose-condensed atomic clouds at zero temperature in the Thomas-Fermi approximation. The equilibrium shape of the cloud in an axisymmetric trap representing either a prolate or oblate spheroid of revolution, which, for a particular choice of the rotation rate and trap potential, degenerates into a rotating sphere. The rotation frequency of the condensate is bounded from above by the characteristic frequency of the harmonic oscillator in a given trap ($\Omega^2/\omega_0^2\leqslant 1$). We have also analyzed nonaxisymmetric, triaxial ellipsoidal figures, which admit equilibrium solutions under additional constraints on the deformation in the plane orthogonal to the rotation axis $A_2\geqslant\Omega^2/\omega_0^2$.

Small amplitude oscillations have been derived for linear perturbations from the rigidly rotating equilibrium background state. The oscillations in axisymmetric traps, which are related to the transverse-shear and toroidal modes are found to be stable for all values of the trap deformation and its rotation frequency. These modes represent undamped oscillations (all eigenfrequencies are real) in the absence of dissipation, i.e., they are dynamically stable. A dynamical instability against the quasiradial pulsation mode can arise when $\gamma\leqslant(1-3\Omega^2)/(3-\Omega^2)\leqslant 1/3$. Otherwise the system is stable, including the limit of the incompressible superfluid ($\gamma\to\infty$), independent of the shape of equilibria. For a Bose gas with $\gamma=2$, the above stability condition is satisfied; hence, these modes represent stable oscillations independent of the rotation rate and deformation. Numerical results for the oscillations in nonaxisymmetrical traps show that, both, the even and odd partity second-harmonic modes of oscillations of triaxial clouds are stable for the adiabatic index $\gamma=2$ relevant to the interacting dilute Bose gas.

Our results for axisymmetric traps (in which case analytical results are available) reduce to those of Ref. [15] when the rotation frequency is set to zero and $\gamma=2$. Indeed, from Eqs. (22), (34), and (45), which describe the oscillations with $m=\pm 1$, $m=\pm 2$, and $m=0$, respectively, for $l=2$, we find

$$\sigma(l=2,m=\pm 1)=\pm\sqrt{1+A_3}, \quad (60)$$

$$\sigma(l=2,m=\pm 2)=\pm\sqrt{2}, \quad (61)$$

$$\sigma(l=2,m=0)=2+\frac{3A_3}{2}\pm\frac{1}{2}\sqrt{16-16A_3+9A_3^2}, \quad (62)$$

which coincide with Eqs. (22), (23), and (24) of Ref. [15] on making the appropriate changes in the parameters describing the trapping potential.

It is instructive also to compare the results above to the classical analysis of the equilibrium and stability of the self-gravitating fluids [19] in view of speculations [22] that intense off-resonant laser beams can give rise to a gravitational-type potential between the condensate particles leading to self-bound configurations. Self-gravitating systems, in particular the axisymmetric figures, are stable against transverse-shear and pulsation modes in the incompressible limit; the same is found in the above model of trapped rotating condensates. However, in the compressible case, the self-gravitating fluids are unstable against the pulsation modes whenever the adiabatic index $\gamma\leqslant 4/3$ (the precise value of the critical $\gamma$ depends on the rotation rate) and are stable otherwise. In the present case, the system is unstable when $\gamma<(1-3\Omega^2)/(3-\Omega^2)$, where the rotation frequency covers the range $\Omega_{c1}\leqslant\Omega\leqslant 1$. Another major difference between the two systems is their stability against the toroidal modes. The axisymmetric self-gravitating fluids are unstable dynamically (i.e., in the absence of dissipation) beyond the point $a_3^2/a_1^2=0.05$, as the deformation is increased. At a smaller deformation $a_3^2/a_1^2=0.19$, these oscillations become neutral, which is a prerequisite of the onset of secular (i.e., driven by the viscosity) instability. In contrast, the toroidal modes of the trapped condensates are always dynamically stable and there are no neutral points within the allowed parameter space where equilibrium figures exist.

The present model can be extended for a study of the higher-order ($l>2$) harmonic oscillations by constructing higher-order virial equations as well as to finite temperatures, in which case the viscosity of thermal excitations, hence secular instabilities, and mutual friction between the vortex lattice state and the excitations play a role.[1]

## ACKNOWLEDGMENTS

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## APPENDIX: HYDRODYNAMIC EQUATIONS OF MOTION FOR ROTATING CONDENSATE

As is well-known, dilute Bose gases can be described in the mean-field approximation in terms of the GP theory. The latter theory (in analogy to the phenomenological theories of superconductivity and of superfluid $He^4$ near the λ point), is based on a Ginzburg-Landau type functional for the wave function $\psi$ of the coherent state, whose variation provides the equation of motion for $\psi$. In the stationary case this functional for a dilute Bose gas has the well-known form

$$E(\psi)=\int\left[\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{x})|^2+\frac{m}{2}\phi_{\rm tr}(\mathbf{x})|\psi(\mathbf{x})|^2+\frac{1}{2}U_0|\psi(\mathbf{x})|^4\right]d^3x, \tag{A1}$$

where $U_0\equiv 4\pi\hbar^2 a/m$. The GP equation is obtained by taking the functional derivative with respect to $\psi^*$, subject to the constraint that the particle number $N$ is constant. The extremum condition $\delta(E-\mu N)/\delta\psi^*=0$ gives

$$-\frac{\hbar^2}{2m}\Delta\psi(\mathbf{x})+\frac{1}{2}[m\phi_{\rm tr}(\mathbf{x})+U_0|\psi(\mathbf{x})|^2]\psi(\mathbf{x})=\mu\psi(\mathbf{x}), \tag{A2}$$

where the Lagrangian multiplier $\mu$ has the meaning of the chemical potential of particles. The time-dependent generalization of Eq. (A2) follows on the assumption that the temporal variations of $\psi$ should be described by a first-order equation which, by analogy with the quantum mechanics, is written as

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi(\mathbf{x},t)+\left[\frac{m}{2}\phi_{\rm tr}(\mathbf{x})+U_0|\psi(\mathbf{x},t)|^2\right]\psi(\mathbf{x},t). \tag{A3}$$

On writing $\psi(\mathbf{x},t)=\eta(\mathbf{x},t)e^{i\varphi(\mathbf{x},t)}$, the superfluid density and velocity can be expressed as

$$\rho(\mathbf{x},t)=m|\psi(\mathbf{x},t)|^2=m\,\eta(\mathbf{x},t)^2,\quad \mathbf{v}(\mathbf{x},t)=\frac{\hbar}{m}\nabla\varphi(\mathbf{x},t). \tag{A4}$$

The real part of Eq. (A3) after deviding by $m\eta$, applying a gradient, and multiplying the result by $\rho$ becomes

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}+\rho\nabla\left(-\frac{\hbar^2}{2m^2}\frac{\Delta\eta}{\eta}+\frac{1}{2}\phi_{\rm tr}+\frac{1}{m}U_0\eta^2\right)=\mathbf{0}, \tag{A5}$$

which is the Euler equation for the condensate in the zero-temperature, inviscid limit. It differs from the analogous equation for the ordinary fluids by the ‘‘quantum pressure’’ term $\propto\Delta\eta/\eta$. Note that any constant term can be added to the second bracket, for example, the chemical potential in the ground state.

The imaginary part of Eq. (A3), on multiplying by $2m\eta/\hbar$, leads to the mass conservation equation

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\rho\mathbf{v}=0. \tag{A6}$$

When gradients of the $\psi$ are small, the quantum pressure term in Eq. (A5) can be dropped and it reduces to

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}=-\nabla p-\frac{\rho}{2}\nabla\phi_{\rm tr}, \tag{A7}$$

where $p\equiv(U_0/2m)\rho^2=(2\pi\hbar^2a/m^2)\rho^2$. Note that *formally* Eq. (A7) is identical to the Euler equation in the ordinary hydrodynamics; the distinctive feature of the superfluid is that the superflow is irrotational $\nabla\times\mathbf{v}=\mathbf{0}$ in general (the special case of the rotating superfluid, when the analogy becomes perfect, is discussed below).

Equation (A7) can be derived also starting from the momentum-conservation equation:

$$\frac{i\hbar}{2}\frac{\partial}{\partial t}[\psi(\mathbf{x},t)\nabla\psi^*(\mathbf{x},t)-\psi^*(\mathbf{x},t)\nabla\psi(\mathbf{x},t)]+\frac{\partial}{\partial x_k}\Pi_{ik}=\rho\frac{1}{2}\nabla\phi_{\rm tr}, \tag{A8}$$

where

$$\Pi_{ik}=\frac{\hbar^2}{4m^2}\left[\frac{\partial\psi}{\partial x_i}\frac{\partial\psi^*}{\partial x_k}-\psi\frac{\partial^2\psi^*}{\partial x_i\partial x_k}+\text{c.c.}\right]+p\,\delta_{ik}, \tag{A9}$$

where $p$ is the pressure, and c.c. stands for complex conjugate. The right-hand side of Eq. (A8) is the external force per unit volume. On writing $\psi(\mathbf{x},t)=\eta(\mathbf{x},t)e^{i\varphi(\mathbf{x},t)}$, and using the relations (A4), Eq. (A8) becomes

$$\frac{\partial}{\partial t}\rho v_i+\frac{\partial}{\partial x_k}(\rho v_i v_k+p\,\delta_{ik})=\rho\frac{1}{2}\nabla_i\phi_{\rm tr}. \tag{A10}$$

[1]Such a program for self-gravitating superfluids has been carried out recently in Ref. [23].

The time derivative of $\rho$ can be eliminated in terms of Eq. (A6). If we use for the pressure of the dilute Bose gas the relation

$$p=\frac{2\pi\hbar^2 a\rho^2}{m^2}\left[1+\frac{64}{5}\left(\frac{\rho a^3}{\pi m}\right)^{1/2}\right] \tag{A11}$$

and keep the leading order term in the diluteness parameter, we arrive again at Eq. (A7).

If the condensate is rotating at a angular velocity, which is larger than the critical one $\Omega_{c1}$, its energy is minimized via a creation of vortices; then the curl of the last relation in Eq. (A4) is nonzero, rather the phase of the superfluid order parameter changes by $2\pi$ around a path that encircles vortex lines

$$\nabla\times\mathbf{v}=\frac{\hbar}{m}\nabla\times\nabla\varphi(\mathbf{x})=\frac{2\pi\hbar}{m}\boldsymbol{\nu}\sum_j \delta^{(2)}(\mathbf{x}-\mathbf{x}_j), \tag{A12}$$

where $\boldsymbol{\nu}$ is a unit vector along a vortex line, $\mathbf{x}_j$ is the radius vector of a vortex line in the plane orthogonal to the vector $\boldsymbol{\nu}$, and $\delta^{(2)}$ is a two-dimensional delta function in this plane. If $\Omega \gg \Omega_{c1}$ the macroscopic hydrodynamic equations involve only *course-grained* quantities, which are averages over a large number of vortices (i.e., over scales much larger than the size of a single vortex). The right-hand side of Eq. (A12) then becomes proportional to the density of the vortex lines $n_v$, since the continuum limit of vortex distribution implies $\Sigma_j\delta^{(2)}(\mathbf{x}-\mathbf{x}_j)=n_v$. The left-hand side of Eq. (A12) in the course-grained limit gives $2\Omega$, since the energy is minimized by a superflow that mimics a rigid-body rotation (this minimization is carried out, e.g., in Ref. [24]). Writing Eq. (A7) in the frame rotating uniformly with the angular velocity $\boldsymbol{\Omega}$ amounts to adding to the right-hand side of this equation the centrifugal potential $|\boldsymbol{\Omega}\times\mathbf{x}|^2/2$ and the Coriolis acceleration $2\mathbf{u}\times\boldsymbol{\Omega}$ (here $\mathbf{u}$ is the superfluid velocity in the rotating frame). With this substitution we recover Eq. (1). Note that the analogy to the Euler equation for a uniformly rotating ordinary fluid now is complete.


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[20] For *intrinsic* quantities that characterize the thermodynamic state of the fluid (e.g., pressure and density) the Lagrangian and Eulerian variations do not vanish in general. If we use the relation between the Lagrangian and Eulerian variations $\Delta=\delta+\xi_j(\partial/\partial x_j)$ (here $\xi_j$ is a Lagrangian displacement), the conservation of the mass of any fluid element implies that $\Delta\rho=-\rho\,\mathrm{div}\boldsymbol{\xi}$, $\delta\rho=-\mathrm{div}(\rho\boldsymbol{\xi})$. The first variations of the integrals of the various functions $Q$, that characterize the fluid flows, are given by $\delta J=\int_V(\Delta Q+Q\,\mathrm{div}\boldsymbol{\xi})d\mathbf{x}$. For $Q\equiv\rho$ we see that the first Eulerian variation of the density integral vanishes, $\delta\int_V\rho\, d\mathbf{x}=\int_V(\Delta\rho+\rho\,\mathrm{div}\boldsymbol{\xi})d\mathbf{x}=0$, as a consequence of the conservation of fluid mass. Then, for any *extrinsic attribute* of the fluid $F$, which characterizes its dynamics (e.g., velocity) we have $\delta\int_V\rho F\, d\mathbf{x}=\int_V\rho\xi_j(\partial F/\partial x_j)d\mathbf{x}$, on taking into account that for such dynamical variables $\delta F=0$. The latter rule should be applied for calculations of the Eulerian variations of the virial equations. A detailed account of the rules for constructing the variations of the virial equations is given in Ref. [19], Chap. 2, Secs. 13–15.

[21] It is worthwhile to note that Kerr's solution to the equations of the general theory of relativity, which describes a rotating black hole, is the only other known example of an equilibrium figure that is a rotating sphere. In the present case, however, the equilibrium spherical figure is maintained due to the exact cancelation of the effect of the centrifugal potential by the external trapping potential.

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