Vacuum-polarization corrections to the hyperfine splitting in heavy ions and to the nuclear magnetic moments

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Complete calculations of the vacuum-polarization corrections to the hyperfine splitting of the 1*s* and 2*s* states in heavy ions are presented. The magnetic-loop Wichmann-Kroll correction is evaluated for the point-dipole model of the nuclear magnetization as well as for the single-particle nuclear model. For the latter case the related correction to the nuclear magnetic moments is also evaluated. The results of the calculations are compared with previous evaluations of this effect.

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I. INTRODUCTION

It is known that the interaction of atomic electrons with the magnetic field of the nucleus results in a splitting of the energy levels called as *hyperfine splitting*. In Ref. [1] it has been proposed to employ an astronomical search of radio lines in the millimeter range, which corresponds to the transition between the hyperfine-structure components of highly charged ions with $Z<30$, for the investigation of the chemical composition of the hot astrophysical plasma. This has initiated accurate calculations of the hyperfine splitting in low- and middle-Z hydrogenlike and lithiumlike ions (see Ref. [2] and references therein).

Successful experiments on the hyperfine splitting in heavy hydrogenlike ions [3–6] stimulated theorists to perform complete αZ-dependence calculations of various contributions to this effect [7–21]. The uncertainty of the theoretical predictions is mainly determined by the uncertainty of the nuclear magnetization distribution correction, the so-called Bohr-Weisskopf (BW) effect [22]. In heavy ions, this uncertainty is even larger than the total QED correction. This fact does not allow for the investigations of QED effects on the hyperfine splitting in heavy hydrogenlike ions. However, as was found in Refs. [17,20], this uncertainty can be almost eliminated in the calculation of the hyperfine splitting in a heavy lithiumlike ion by employing the experimental value of the 1*s* hyperfine splitting in the corresponding hydrogenlike ion. To obtain a high-precision value of the hyperfine splitting in a lithiumlike ion this method has to be combined with accurate calculations of the interelectronic-interaction and QED corrections. The QED correction consists of the self-energy (SE) and vacuum-polarization (VP) contributions. Both contributions were included in the numerical evaluation of the 2*s* hyperfine splitting performed in Refs. [17,20]. However, the VP correction was accounted for without the magnetic-loop Wichmann-Kroll (WK) contribution. An evaluation of the magnetic-loop WK contribution for the 1*s* state performed in Ref. [19] indicated that for high Z it contributes on the level of about of 10% of the total QED correction. This value is much smaller than the uncertainty of the Bohr-Weisskopf effect and, thus, hardly improves the theoretical predictions for the 1*s* hyperfine splitting. However, the calculation of this correction is important for high-precision predictions of the hyperfine splitting in Li-like ions provided it is performed for both 1*s* and 2*s* states. A first estimate of this effect for both states in the case of bismuth [23] showed that its scaling from the 1*s* to 2*s* state is similar to the related scaling for the BW correction and, therefore, hardly affects the final theoretical prediction for the hyperfine splitting in lithiumlike bismuth. However, this correction should be included in calculations aiming on tests of QED effects in hyperfine splitting investigations. In the present paper we perform accurate calculations of this effect for the 1*s* and 2*s* states in heavy ions for the point-dipole model as well as for the single-particle nuclear model chosen to describe the nuclear magnetization. Since the calculation of this correction is closely related to the calculation of the VP correction to the nuclear magnetic moment, we evaluate the last correction as well. For completeness, we also present numerical results for the Uehling and electric-loop WK corrections to the hyperfine splitting of the 1*s* and 2*s* states.

Relativistic units ($\hbar=c=1$) are used in this paper.

II. FORMULATION

The magnetic-dipole hyperfine splitting in a hydrogenlike ion is conveniently written in the form [10]

$$\Delta E_\mu=\frac{\alpha(\alpha Z)^3}{n^3}\frac{\mu}{\mu_N}\frac{m}{m_p}\frac{F(F+1)-I(I+1)-j(j+1)}{2Ij(j+1)(2l+1)}mc^2 \times\{A(\alpha Z)(1-\delta)(1-\varepsilon)+x_{\rm rad}\}, \tag{1}$$

where α is the fine-structure constant, Z is the nuclear charge number, m is the electron mass, m_p is the proton mass, μ is the nuclear magnetic moment, μ_N is nuclear magneton, I is the nuclear spin, j is the total electronic angular momentum, l is the orbital momentum of the electron, F is the total atomic angular momentum, and n is the principal quantum number. $A(\alpha Z)$ denotes a relativistic factor [24–26]:

$$A(\alpha Z)=\frac{n^3(2l+1)\kappa[2\kappa(\gamma+n_r)-N]}{N^4\gamma(4\gamma^2-1)}, \tag{2}$$

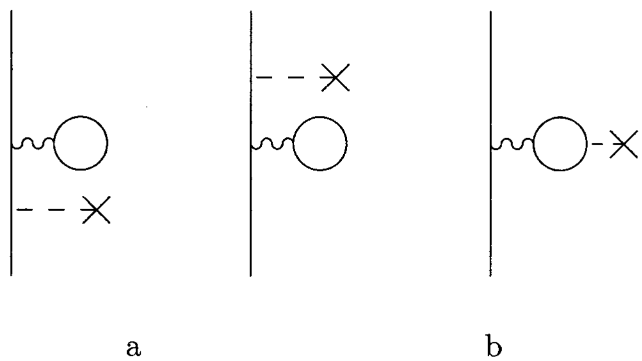

FIG. 1. Feynman diagrams describing the vacuum-polarization corrections to the hyperfine splitting.

where $\kappa=(-1)^{j+l+1/2}(j+1/2)$ is the relativistic angular momentum quantum number, $\gamma=\sqrt{\kappa^2-(\alpha Z)^2}$, $N=\sqrt{n_r^2+2n_r\gamma+\kappa^2}$, and $n_r=n-|\kappa|$ is the radial quantum number. δ denotes the nuclear charge distribution correction, ε is the nuclear magnetization distribution correction (the Bohr-Weiskopf effect), and $x_{\rm rad}$ is the QED correction. In Eq. (1) we neglected the nuclear recoil effect because it turns out to be small for heavy ions.

In the present paper we calculate the VP part of $x_{\rm rad}$. The Feynman diagrams, which determine this contribution, are shown in Fig. 1, where the dashed line ended by a cross denotes the hyperfine interaction. In what follows, we will call the diagrams shown in Fig. 1(a) as the electric-loop diagrams and the diagram shown in Fig. 1(b) as the magnetic-loop diagram.

The expressions for the contributions of these diagrams to the hyperfine splitting can easily be derived using the two-time Green function method [27]. So, for the electric-loop diagrams one finds

$$\Delta E_{\rm VP}^{\rm EL}=2\sum_{N}^{\varepsilon_N\neq\varepsilon_A}\frac{\langle A|U_{\rm hfs}|N\rangle\langle N|U_{\rm VP}|A\rangle}{\varepsilon_A-\varepsilon_N}, \tag{3}$$

where $|A\rangle$ and $|N\rangle$ are the state vectors of the whole (electron plus nucleus) atomic system, $U_{\rm hfs}$ is the hyperfine interaction operator:

$$U_{\rm hfs}(\mathbf{r})=\frac{|e|}{4\pi}\frac{(\boldsymbol{\alpha}\cdot[\boldsymbol{\mu}\times\mathbf{r}])}{r^3}, \tag{4}$$

$\boldsymbol{\mu}$ is the nuclear-magnetic-moment operator, $\boldsymbol{\alpha}$ is a vector incorporating the Dirac matrices, and $U_{\rm VP}$ is the vacuum-polarization potential. The unrenormalized expression for the VP potential is given by

$$U_{\rm VP}(\mathbf{x})=\frac{\alpha}{2\pi i}\int d\mathbf{y}\frac{1}{|\mathbf{x}-\mathbf{y}|}\int_{-\infty}^{\infty}d\omega\,{\rm Tr}[G(\omega,\mathbf{y},\mathbf{y})], \tag{5}$$

where

$$G(\omega,\mathbf{x},\mathbf{y})=\sum_n\frac{\psi_n(\mathbf{x})\psi_n^\dagger(\mathbf{y})}{\omega-\varepsilon_n(1-i0)}$$

is the Dirac-Coulomb Green function. The expression (5) is ultraviolet divergent. A simple way to renormalize it is to divide it into two parts, $U_{\rm VP}=U_{\rm Ue}+U_{\rm WK}$, where $U_{\rm Ue}$ and $U_{\rm WK}$ are the Uehling and Wichmann-Kroll potentials, respectively [28,29]. The renormalized expression for the Uehling potential is well known:

$$U_{\rm Ue}(r)=-\alpha Z\frac{2\alpha}{3\pi}\int_0^\infty dr'\,4\pi r'\rho(r')\int_1^\infty dt\left(1+\frac{1}{2t^2}\right)\frac{\sqrt{t^2-1}}{t^2}\times\frac{\{\exp(-2m|r-r'|t)-\exp[-2m(r+r')t]\}}{4mrt}, \tag{6}$$

where $|e|Z\rho(r)$ is the density of the nuclear charge distribution ($\int\rho(r)d\mathbf{r}=1$). The Wichmann-Kroll potential is calculated by summing up the partial-wave differences between expression (5) and the unrenormalized Uehling term [30–32]. The final formula for this potential is the following:

$$U_{\rm WK}^{\rm EL}(r)=\frac{2\alpha}{\pi}\sum_{\kappa=\pm1}^{\pm\infty}|\kappa|\int_0^\infty d\omega\int_0^\infty dr\,r_1^2\int_0^\infty dr_2\,r_2^2\times\frac{1}{\max(r,r_1)}V_{\rm C}(r_2)\sum_{i,k=1}^{2}{\rm Re}\{F_\kappa^{ik}(i\omega,r_1,r_2)\times[G_\kappa^{ik}(i\omega,r_1,r_2)-F_\kappa^{ik}(i\omega,r_1,r_2)]\}, \tag{7}$$

where G_κ^{ik} and F_κ^{ik} are the radial components of the partial-wave expansion of the bound and free-electron Green functions, respectively (see, e.g., Refs. [31,33]) and $V_{\rm C}$ is the Coulomb potential of the nucleus.

For the correction to the hyperfine splitting due to the magnetic loop [Fig. 1(b)] one obtains

$$\Delta E_{\rm VP}^{\rm ML}=\langle A|U_{\rm hfs\text{-}VP}^{\rm mag}|A\rangle, \tag{8}$$

where

$$U_{\rm hfs\text{-}VP}^{\rm ML}(\mathbf{x})=\frac{\alpha}{2\pi i}\int_{-\infty}^{\infty}d\omega\int d\mathbf{y}\int d\mathbf{z}\times\frac{\boldsymbol{\alpha}}{|\mathbf{x}-\mathbf{y}|}{\rm Tr}[\boldsymbol{\alpha}G(\omega,\mathbf{y},\mathbf{z})U_{\rm hfs}(\mathbf{z})G(\omega,\mathbf{z},\mathbf{y})]. \tag{9}$$

The scalar product is implicit in Eq. (9). This contribution is also ultraviolet divergent. It can be renormalized using the same scheme as for the electric loop. The magnetic-loop Uehling term is given by

$$U_{\rm hfs\text{-}VP}^{\rm ML\text{-}Ue}(\mathbf{r})=U_{\rm hfs}(\mathbf{r})\frac{2}{3}\frac{\alpha}{\pi}\int_1^\infty dt\frac{\sqrt{t^2-1}}{t^2}\left(1+\frac{1}{2t^2}\right)\times(1+2mrt)\exp(-2mrt). \tag{10}$$

The Wichmann-Kroll contribution is calculated by summing up the partial differences between Eq. (9) and the related expression with the bound-electron Green function replaced by the free-electron one. However, this contribution remains divergent if the point-dipole approximation is used to describe the nuclear magnetization. Using an extended nuclear magnetization model results in a finite contribution. Since the magnetic WK interaction also contributes to the nuclear magnetic moment (the corresponding Uehling contribution is equal to zero), the related correction must be accounted for to determine the bare value of the nuclear magnetic moment [34]. This implies that the values of the nuclear magnetic moments, presented, e.g, in Ref. [35] (see also Ref. [36]), must be replaced by the corrected values:

$$\mu \to \mu_{\rm bare} = \mu - \Delta\mu, \tag{11}$$

where $\Delta\mu$ is the correction to the nuclear magnetic moment due to the magnetic-loop WK effect. This correction is conveniently expressed in terms of a dimensionless parameter ϵ defined as

$$\Delta\mu = \epsilon\mu. \tag{12}$$

The parameter ϵ is calculated by

$$\begin{aligned}\epsilon = \frac{1}{2\pi i}\frac{|e|}{2\mu}\int d\mathbf{x}\int d\mathbf{y}\int_{-\infty}^{\infty} d\omega \{ &\mathrm{Tr}\{[\mathbf{x}\times\boldsymbol{\alpha}]_z G(\omega,\mathbf{x},\mathbf{y}) \\ &\times \langle II|U_{\rm hfs}(\mathbf{y})|II\rangle G(\omega,\mathbf{y},\mathbf{x})\} - \mathrm{Tr}\{[\mathbf{x}\times\boldsymbol{\alpha}]_z F(\omega,\mathbf{x},\mathbf{y}) \\ &\times \langle II|U_{\rm hfs}(\mathbf{y})|II\rangle F(\omega,\mathbf{y},\mathbf{x})\}\},\end{aligned} \tag{13}$$

where the hyperfine interaction operator $U_{\rm hfs}$ is averaged with respect to the nuclear wave function with $M_I = I$. Thus to deduce the magnetic-loop WK contribution to the hyperfine splitting we have to subtract the value $\epsilon\langle A|U_{\rm hfs}|A\rangle$ from the WK part of Eq. (9). This subtraction allows us to calculate the magnetic-loop WK correction even for the case of the point-dipole model by summing up the related partial-wave differences.

To take into account the nuclear magnetization distribution effect we used the nuclear single-particle model. In this model the nuclear magnetization is ascribed to an odd nucleon. The wave function of the odd nucleon is calculated by solving the Schrödinger equation with the Woods-Saxon potential:

$$U_{\rm WS} = V(r) + V_{\rm SO}(r) + V_{\rm Coul}(r), \tag{14}$$

where

$$V(r) = -V_0 f(r), \tag{15}$$

$$V_{\rm SO}(r) = \phi_{\rm SO}(r)(\mathbf{s}\cdot\mathbf{l}), \tag{16}$$

$$V_{\rm Coul}(r) = \begin{cases} \alpha(Z-1)(3-r^2/R_0^2)/(2R_0), & r \le R_0 \\ \alpha(Z-1)/r, & r > R_0, \end{cases} \tag{17}$$

$$\phi_{\rm SO}(r) = \frac{\lambda}{2}\left(\frac{\hbar}{m_p c}\right)^2 \frac{V_0}{r}\frac{df_{\rm SO}(r)}{dr}, \tag{18}$$

$$f(r) = \{1+\exp[(r-R_0)/a]\}^{-1}, \tag{19}$$

$$f_{\rm SO}(r) = \{1+\exp[(r-R_{\rm SO})/a]\}^{-1}, \tag{20}$$

and V_0 is the depth of the central nuclear potential, R_0 is its radius, a is its diffusivity, λ is a positive dimensionless parameter of the nuclear spin-orbit interaction, and $R_{\rm SO}$ is its radius. The diffusivity of the spin-orbit interaction is taken to be the same as for the central nuclear potential. In the neutron case, the term $V_{\rm Coul}$ should be omitted. The potential parameters were chosen to yield the nuclear binding energies in the lead region [37].

To include the nuclear magnetization distribution effect in the calculation of the hyperfine splitting within the single-particle nuclear model and, in particular, in the calculation of the magnetic-loop WK correction, one has to adopt the following replacement [15,20]:

$$U_{\rm hfs}(\mathbf{r}) \to F(r) U_{\rm hfs}(\mathbf{r}), \tag{21}$$

where

$$F(r) = \begin{cases} F_+(r), & I = L + \frac{1}{2} \\ F_-(r), & I = L - \frac{1}{2}, \end{cases} \tag{22}$$

$$\begin{aligned} F_+(r) = \frac{\mu_N}{\mu}\Bigg\{ &\int_0^r dr'\, r'^2 u^2(r')\left[\frac{1}{2}g_S + \left(I - \frac{1}{2} + \frac{2I+1}{4(I+1)}\frac{m_p}{\hbar^2}\phi_{\rm SO}(r')r'^2\right)g_L\right] \\ &+ \int_r^\infty dr'\, r'^2 u^2(r')\left(\frac{r}{r'}\right)^3\left[-\frac{2I-1}{8(I+1)}g_S + \left(I - \frac{1}{2} + \frac{2I+1}{4(I+1)}\frac{m_p}{\hbar^2}\phi_{\rm SO}(r')r'^2\right)g_L\right]\Bigg\}, \end{aligned} \tag{23}$$

$$\begin{aligned} F_-(r) = \frac{\mu_N}{\mu}\Bigg\{ &\int_0^r dr' r'^2 u^2(r')\left[-\frac{I}{2(I+1)}g_S + \left(\frac{I(2I+3)}{2(I+1)} - \frac{2I+1}{4(I+1)}\frac{m_p}{\hbar^2}\phi_{\rm SO}(r')r'^2\right)g_L\right] + \int_r^\infty dr' r'^2 u^2(r')\left(\frac{r}{r'}\right)^3 \\ &\times\left[\frac{2I+3}{8(I+1)}g_S + \left(\frac{I(2I+3)}{2(I+1)} - \frac{2I+1}{4(I+1)}\frac{m_p}{\hbar^2}\phi_{\rm SO}(r')r'^2\right)g_L\right]\Bigg\}. \end{aligned} \tag{24}$$

Here $u(r)$ is the radial part of the wave function of the odd nucleon, g_L is equal to 1 for the proton and to 0 for the neutron, g_S is chosen to yield the experimental value of μ within the single-particle nuclear model according to the formulas

$$\frac{\mu}{\mu_N}=\frac{1}{2}g_S+\left[I-\frac{1}{2}+\frac{2I+1}{4(I+1)}\frac{m_p}{\hbar^2}\times\left(\int_0^\infty dr\, r^2|u(r)|^2\phi_{\rm SO}(r)r^2\right)\right]g_L \qquad (25)$$

for $I=L+1/2$ and

$$\frac{\mu}{\mu_N}=-\frac{I}{2(I+1)}g_S+\left[\frac{I(2I+3)}{2(I+1)}-\frac{2I+1}{4(I+1)}\frac{m_p}{\hbar^2}\times\left(\int_0^\infty dr\, r^2|u(r)|^2\phi_{\rm SO}(r)r^2\right)\right]g_L \qquad (26)$$

for $I=L-1/2$.

III. CALCULATION

The calculation of the electric-loop VP corrections to the hyperfine splitting caused no major problem. For the determination of the reduced Green function and the one-electron wave functions, we employed the B-spline method for the Dirac equation [38]. To calculate the vacuum-polarization potential we used the same subroutines as in our previous calculations in Refs. [39,40]. The calculations were performed for the extended nucleus case. Except for the Green function in formula (7), the Fermi model for the nuclear charge distribution was employed. To calculate the bound-electron Green function in Eq. (7) we used the homogeneously-charged-sphere model for the nuclear charge distribution. The radial parts of the Green function for this model were evaluated using the method developed by Mohr in Ref. [41].

The calculation of the magnetic-loop Uehling correction was performed for the Fermi model of the nuclear charge distribution. The calculation of the magnetic-loop Wichmann-Kroll correction required some additional procedures to improve the convergence of the partial-wave-difference series. As in Ref. [39], we divided the total magnetic-loop WK contribution into two parts, each containing only odd or even powers of the nucleus charge number Z. According to the Furry theorem, only the part containing even powers of Z yields nonzero contribution. It means that the product $G_\kappa U_{\rm hfs} G_\kappa$ can be replaced by $G_\kappa^{\rm odd} U_{\rm hfs} G_\kappa^{\rm odd} + G_\kappa^{\rm even} U_{\rm hfs} G_\kappa^{\rm even}$. The elements of the Green function containing only odd or only even powers of Z were determined analytically for the spherical shell model chosen to describe the nuclear charge distribution (see the Appendix). The reason for choosing this model is that the radial parts of the Green function can be written in a simple analytical form [30,31]. The uncertainty due to a deviation of this simple model from a more realistic model is much smaller than the uncertainty due to the nuclear magnetization distribution effect. The calculation of the magnetic-loop WK correction was performed for two models of the nuclear magnetization distribution: the point-dipole model and the single-particle nuclear model. For the latter case the WK correction to the nuclear magnetic moments was also derived.

The numerical integration over energy and radial variables have been done using Gauss-Legendre quadratures. The knots of integration were chosen to provide the relative accuracy of the final result not worse than 10^{-5}. In the calculations of the electric-loop Wichmann-Kroll corrections [Fig. 1(a)], as in Refs. [31,32,39,40], the infinite partial-wave summation was terminated at $|\kappa|=5$ and the remainder of the sum was evaluated using a polynomial fitting in $1/|\kappa|$. For the magnetic-loop Wichmann-Kroll correction [Fig. 1(b)], we found that the partial-wave series decreases with increasing κ as $1/|\kappa|^\rho$, where ρ varies from $\rho=2.81$ for $Z=49$ to $\rho=2.93$ for $Z=83$. In the calculation of this correction we terminated the partial-wave summation at $|\kappa|=10$ and evaluated the remainder of the sum using a polynomial fitting in $1/|\kappa|$.

The numerical results for the VP corrections to the nuclear magnetic moments and to the hyperfine splitting of the $1s$ and $2s$ states are presented in Table I. The values of the individual Uehling and WK corrections to the hyperfine splitting of the $1s$ and $2s$ states are listed in Tables II and III, respectively. The values $x_{\rm VP}^{\rm WK,mag,0}$ and $x_{\rm VP}^{\rm WK,mag,s\text{-}p}$, which are presented in the last columns of the tables, determine the magnetic-loop Wichmann-Kroll corrections calculated for the point-dipole and single-particle models of the nuclear magnetization, respectively. We estimate the total relative numerical error to be less than 10^{-4}. However, we note that this uncertainty does not include an error due to a deviation of the nuclear models used in the calculation from a more realistic model.

In Ref. [19], similar calculations were performed for the $1s$ state, where a spherically symmetric distribution was assumed for the nuclear megnetization: $\mathbf{M}(r)=\boldsymbol{\mu}w(r)$. The density function was given by $w(r)=k_n r^n$ for the interior of the nucleus and by $w(r)=0$ for the region outside the nucleus. The model parameter n was chosen to be 0 and 2 while the parameter k_n was defined by the normalization condition. In the case of bismuth a more elaborated model developed in Ref. [12] was also used. Comparing our results for the total VP corrections to the $1s$ hyperfine splitting with the related results of Ref. [19], we found that they are in a very good agreement with each other. For instance, for $Z=83$ the result of Ref. [19] for the point-dipole model is $x_{\rm VP}^{(1s)}=0.011\,54$ while our corresponding result is $x_{\rm VP}^{(1s)}=0.011\,53$.[1] Concerning the correction to the nuclear magnetic moment, in the case of bismuth in Ref. [19] the values $\epsilon=1.419\times10^{-3}$, 1.388×10^{-3}, and 1.483×10^{-3} were ob-

[1]Note that in Table I we list the results with the magnetic-loop Wichmann-Kroll contribution calculated in the framework of the single-particle nuclear model, while for comparison with Ref. [19], we use the point-dipole results.

TABLE I. Vacuum polarization corrections to the nuclear magnetic moments and to the hyperfine splitting of the $1s$ and $2s$ states.

Ion	Nucleon state	$\langle r^2\rangle^{1/2}$	g_S	ϵ	$x_{\mathrm{VP}}^{(1s)}$	$x_{\mathrm{VP}}^{(2s)}$
$^{113}\mathrm{In}^{48+}$	$1g_{9/2}$	4.598	3.674	4.98×10^{-4}	3.038×10^{-3}	3.285×10^{-3}
$^{121}\mathrm{Sb}^{50+}$	$2d_{5/2}$	4.681	3.045	5.38×10^{-4}	3.270×10^{-3}	3.568×10^{-3}
$^{123}\mathrm{Sb}^{50+}$	$1g_{7/2}$	4.689	4.207	5.72×10^{-4}	3.266×10^{-3}	3.564×10^{-3}
$^{127}\mathrm{I}^{52+}$	$2d_{5/2}$	4.749	1.948	5.84×10^{-4}	3.521×10^{-3}	3.878×10^{-3}
$^{133}\mathrm{Cs}^{54+}$	$1g_{7/2}$	4.804	4.143	6.64×10^{-4}	3.789×10^{-3}	4.215×10^{-3}
$^{139}\mathrm{La}^{56+}$	$1g_{7/2}$	4.850	3.637	7.08×10^{-4}	4.085×10^{-3}	4.592×10^{-3}
$^{141}\mathrm{Pr}^{58+}$	$2d_{5/2}$	4.892	4.878	7.19×10^{-4}	4.410×10^{-3}	5.013×10^{-3}
$^{151}\mathrm{Eu}^{62+}$	$2d_{5/2}$	5.044	3.275	8.25×10^{-4}	5.129×10^{-3}	5.968×10^{-3}
$^{159}\mathrm{Tb}^{64+}$	$2d_{3/2}$	5.099	-0.2034	8.94×10^{-4}	5.536×10^{-3}	6.522×10^{-3}
$^{165}\mathrm{Ho}^{66+}$	$1f_{7/2}$	5.190	2.904	9.39×10^{-4}	5.978×10^{-3}	7.134×10^{-3}
$^{175}\mathrm{Lu}^{70+}$	$1g_{7/2}$	5.370	5.104	1.13×10^{-3}	6.963×10^{-3}	8.543×10^{-3}
$^{181}\mathrm{Ta}^{72+}$	$1g_{7/2}$	5.480	4.757	1.18×10^{-3}	7.530×10^{-3}	9.374×10^{-3}
$^{185}\mathrm{Re}^{74+}$	$2d_{5/2}$	5.351	2.714	1.19×10^{-3}	8.239×10^{-3}	1.042×10^{-2}
$^{203}\mathrm{Tl}^{80+}$	$3s_{1/2}$	5.463	3.469	1.39×10^{-3}	1.063×10^{-2}	1.412×10^{-2}
$^{205}\mathrm{Tl}^{80+}$	$3s_{1/2}$	5.470	3.502	1.39×10^{-3}	1.063×10^{-2}	1.411×10^{-2}
$^{207}\mathrm{Pb}^{81+}$	$3p_{1/2}$	5.513	-3.556	1.27×10^{-3}	1.115×10^{-2}	1.495×10^{-2}
$^{209}\mathrm{Bi}^{82+}$	$1h_{9/2}$	5.533	2.801	1.52×10^{-3}	1.156×10^{-2}	1.563×10^{-2}

tained for three different models of the nuclear magnetization distribution. The last value, which is based on the model developed in Ref. [12], is close to our result, which amounts to $\epsilon=1.52\times10^{-3}$.

As one can deduce from Tables II and III, the VP correction to the hyperfine splitting is mainly determined by the Uehling term. Analytical calculations of this term for a pointlike nucleus were performed in Ref. [42]. These analytical results may serve as a good approximation for the VP correction to the hyperfine splitting in low- and middle-Z systems.

IV. HYPERFINE SPLITTING PREDICTIONS

As one can deduce from Tables II and III, the magnetic-loop WK correction to the hyperfine splitting contributes on the level of about 10% of the total VP correction. This value is much smaller than the uncertainty of the Bohr-Weisskopf correction and, thus, hardly affects the theoretical predictions for the hyperfine splitting in hydrogenlike and lithiumlike ions presented in Refs. [15,20]. However, the calculation of this correction is important for high-precision theoretical predictions of the hyperfine splitting in Li-like ions based on the

TABLE II. Various components of the vacuum-polarization correction to the hyperfine splitting of the $1s$ state.

Ion	$x_{\mathrm{VP}}^{\mathrm{Ue,el}}$	$x_{\mathrm{VP}}^{\mathrm{WK,el}}$	$x_{\mathrm{VP}}^{\mathrm{Ue,mag}}$	$x_{\mathrm{VP}}^{\mathrm{WK,mag,0}}$	$x_{\mathrm{VP}}^{\mathrm{WK,mag},s-p}$
$^{113}\mathrm{In}^{48+}$	2.037×10^{-3}	-2.316×10^{-5}	1.119×10^{-3}	-9.820×10^{-5}	-9.483×10^{-5}
$^{121}\mathrm{Sb}^{50+}$	2.216×10^{-3}	-2.701×10^{-5}	1.191×10^{-3}	-1.139×10^{-4}	-1.098×10^{-4}
$^{123}\mathrm{Sb}^{50+}$	2.215×10^{-3}	-2.701×10^{-5}	1.191×10^{-3}	-1.138×10^{-4}	-1.128×10^{-4}
$^{127}\mathrm{I}^{52+}$	2.412×10^{-3}	-3.142×10^{-5}	1.268×10^{-3}	-1.317×10^{-4}	-1.273×10^{-4}
$^{133}\mathrm{Cs}^{54+}$	2.627×10^{-3}	-3.645×10^{-5}	1.350×10^{-3}	-1.520×10^{-4}	-1.505×10^{-4}
$^{139}\mathrm{La}^{56+}$	2.863×10^{-3}	-4.219×10^{-5}	1.437×10^{-3}	-1.750×10^{-4}	-1.726×10^{-4}
$^{141}\mathrm{Pr}^{58+}$	3.122×10^{-3}	-4.873×10^{-5}	1.531×10^{-3}	-2.012×10^{-4}	-1.932×10^{-4}
$^{151}\mathrm{Eu}^{62+}$	3.712×10^{-3}	-6.454×10^{-5}	1.734×10^{-3}	-2.632×10^{-4}	-2.531×10^{-4}
$^{159}\mathrm{Tb}^{64+}$	4.054×10^{-3}	-7.415×10^{-5}	1.848×10^{-3}	-3.014×10^{-4}	-2.921×10^{-4}
$^{165}\mathrm{Ho}^{66+}$	4.425×10^{-3}	-8.499×10^{-5}	1.968×10^{-3}	-3.435×10^{-4}	-3.307×10^{-4}
$^{175}\mathrm{Lu}^{70+}$	5.284×10^{-3}	-1.113×10^{-4}	2.235×10^{-3}	-4.445×10^{-4}	-4.439×10^{-4}
$^{181}\mathrm{Ta}^{72+}$	5.776×10^{-3}	-1.272×10^{-4}	2.382×10^{-3}	-5.039×10^{-4}	-5.012×10^{-4}
$^{185}\mathrm{Re}^{74+}$	6.381×10^{-3}	-1.466×10^{-4}	2.561×10^{-3}	-5.797×10^{-4}	-5.559×10^{-4}
$^{203}\mathrm{Tl}^{80+}$	8.511×10^{-3}	-2.202×10^{-4}	3.149×10^{-3}	-8.545×10^{-4}	-8.107×10^{-4}
$^{205}\mathrm{Tl}^{80+}$	8.508×10^{-3}	-2.201×10^{-4}	3.148×10^{-3}	-8.541×10^{-4}	-8.103×10^{-4}
$^{207}\mathrm{Pb}^{81+}$	8.929×10^{-3}	-2.354×10^{-4}	3.259×10^{-3}	-9.093×10^{-4}	-7.979×10^{-4}
$^{209}\mathrm{Bi}^{82+}$	9.378×10^{-3}	-2.518×10^{-4}	3.377×10^{-3}	-9.701×10^{-4}	-9.396×10^{-4}

TABLE III. Various components of the vacuum-polarization correction to the hyperfine splitting of the $2s$ state.

Ion	$x_{\rm VP}^{\rm Ue,el}$	$x_{\rm VP}^{\rm WK,el}$	$x_{\rm VP}^{\rm Ue,mag}$	$x_{\rm VP}^{\rm WK,mag,0}$	$x_{\rm VP}^{{\rm WK,mag},s-p}$
$^{113}In^{48+}$	2.146×10^{-3}	-2.292×10^{-5}	1.269×10^{-3}	-1.113×10^{-4}	-1.075×10^{-4}
$^{121}Sb^{50+}$	2.356×10^{-3}	-2.696×10^{-5}	1.364×10^{-3}	-1.303×10^{-4}	-1.257×10^{-4}
$^{123}Sb^{50+}$	2.356×10^{-3}	-2.696×10^{-5}	1.364×10^{-3}	-1.302×10^{-4}	-1.291×10^{-4}
$^{127}I^{52+}$	2.591×10^{-3}	-3.163×10^{-5}	1.466×10^{-3}	-1.521×10^{-4}	-1.471×10^{-4}
$^{133}Cs^{54+}$	2.852×10^{-3}	-3.716×10^{-5}	1.576×10^{-3}	-1.774×10^{-4}	-1.756×10^{-4}
$^{139}La^{56+}$	3.143×10^{-3}	-4.335×10^{-5}	1.696×10^{-3}	-2.064×10^{-4}	-2.036×10^{-4}
$^{141}Pr^{58+}$	3.468×10^{-3}	-5.061×10^{-5}	1.826×10^{-3}	-2.399×10^{-4}	-2.303×10^{-4}
$^{151}Eu^{62+}$	4.229×10^{-3}	-6.859×10^{-5}	2.116×10^{-3}	-3.211×10^{-4}	-3.088×10^{-4}
$^{159}Tb^{64+}$	4.681×10^{-3}	-7.985×10^{-5}	2.281×10^{-3}	-3.720×10^{-4}	-3.606×10^{-4}
$^{165}Ho^{66+}$	5.182×10^{-3}	-9.301×10^{-5}	2.459×10^{-3}	-4.293×10^{-4}	-4.133×10^{-4}
$^{175}Lu^{70+}$	6.373×10^{-3}	-1.252×10^{-4}	2.865×10^{-3}	-5.700×10^{-4}	-5.693×10^{-4}
$^{181}Ta^{72+}$	7.076×10^{-3}	-1.453×10^{-4}	3.095×10^{-3}	-6.550×10^{-4}	-6.515×10^{-4}
$^{185}Re^{74+}$	7.949×10^{-3}	-1.703×10^{-4}	3.372×10^{-3}	-7.642×10^{-4}	-7.328×10^{-4}
$^{203}Tl^{80+}$	1.117×10^{-2}	-2.698×10^{-4}	4.332×10^{-3}	-1.177×10^{-3}	-1.117×10^{-3}
$^{205}Tl^{80+}$	1.117×10^{-2}	-2.698×10^{-4}	4.331×10^{-3}	-1.177×10^{-3}	-1.116×10^{-3}
$^{207}Pb^{81+}$	1.183×10^{-2}	-2.911×10^{-4}	4.517×10^{-3}	-1.262×10^{-3}	-1.108×10^{-3}
$^{209}Bi^{82+}$	1.254×10^{-2}	-3.141×10^{-4}	4.717×10^{-3}	-1.357×10^{-3}	-1.315×10^{-3}

experimental values of the hyperfine splitting in the corresponding H-like ions [17,20]. This method is based on the fact that, with a high accuracy, the ratio of the BW correction for the $2s$ state to the one for the $1s$ state does not depend on the nuclear structure. This ratio is only a function of the electronic structure,

$$\frac{\varepsilon^{(2s)}}{\varepsilon^{(1s)}}=f(\alpha Z), \tag{27}$$

and, therefore, can be calculated to a rather high accuracy. For instance, in the case of bismuth: $f(\alpha Z)=1.078$. The Bohr-Weisskopf correction for the $1s$ state can be derived using the experimental value of the hyperfine splitting for the $1s$ state:

$$\varepsilon^{(1s)}=\frac{\Delta E_{\rm Dirac}^{(1s)}+\Delta E_{\rm QED}^{(1s)}-\Delta E_{\rm exp}^{(1s)}}{\Delta E_{\rm Dirac}^{(1s)}}, \tag{28}$$

where $\Delta E_{\rm Dirac}^{(1s)}$ is the relativistic value of the $1s$ hyperfine splitting including the nuclear charge distribution correction, $\Delta E_{\rm QED}^{(1s)}$ is the $1s$ QED correction, and $\Delta E_{\rm exp}^{(1s)}$ is the experimental value of the $1s$ hyperfine splitting. Then the Bohr-Weisskopf correction for the $2s$ state is calculated by Eq. (27). To obtain high-precision theoretical predictions for the hyperfine splitting in lithiumlike ions, accurate calculations of the interelectronic-interaction corrections are also needed. Corresponding calculations were performed to first order in $1/Z$ in Ref. [43] and to second and higher order in $1/Z$ in Ref. [44]. In Table IV we present total theoretical results for the hyperfine splitting in lithiumlike ions based on the experimental results for the hyperfine splitting in the corresponding hydrogenlike ions. In the case of bismuth the individual contributions and the total value of the hyperfine splitting coincide with our previous data presented in Ref. [23]. Our hyperfine splitting prediction for bismuth, 0.7971(2) eV, is also in excellent agreement with a recent result, 0.797 15(13) eV, obtained by Sapirstein and Cheng [45]. Both theoretical values are in agreement with the related experimental result, which amounts to 0.820(26) eV [46]. Referring to other ions, the hyperfine splitting values for Ho, Pb, and two isotopes of Re almost coincide with the related results of Ref. [44], al-

TABLE IV. The individual contributions to the ground-state hyperfine splitting in lithiumlike ions, in eV. The Bohr-Weisskopf corrections are deduced from the experimental values of the $1s$ hyperfine splitting taken from Refs. [3–6]. The nuclear magnetic moments are taken from Refs. [35,36].

Ion	μ/μ_N	Dirac value	BW effect	One-electron QED	Interel. interaction	Interel. int.-QED	Total theory
$^{165}Ho^{64+}$	4.177(5)	0.3267(4)	−0.0069(4)	−0.0016	−0.0134(1)	0.00007(3)	0.3050(1)
$^{185}Re^{72+}$	3.1871(3)	0.4314(3)	−0.0105(4)	−0.0024	−0.0160(1)	0.00009(4)	0.4026(3)
$^{187}Re^{72+}$	3.2197(5)	0.4358(3)	−0.0109(4)	−0.0024	−0.0162(1)	0.00009(4)	0.4065(3)
$^{207}Pb^{79+}$	0.59258	0.2061(1)	−0.0091(1)	−0.0012	−0.0070	0.00004(2)	0.1888(1)
$^{209}Bi^{80+}$	4.1106(2)	0.8447(2)	−0.0134(2)	−0.0051	−0.0292(1)	0.00018(9)	0.7971(2)

though some individual contributions are different. This is caused by the fact that the magnetic-loop WK correction, which was omitted in Ref. [44] for both 1s and 2s states, affects the hyperfine splitting in lithiumlike ions in a manner similar to hydrogenlike ions.

We note that the uncertainty of the total hyperfine splitting values given in Table IV is not equal to a quadratic sum of the uncertainties of the individual contributions. This is caused by the fact that the total hyperfine splitting value, found as described above, is very stable in respect to possible variations of the nuclear charge radius and the nuclear magnetic moment. This behavior of the hyperfine splitting predictions was explained in detail in Ref. [20]. The uncertainties of the hyperfine splitting values presented in Table IV are mainly determined by the experimental uncertainties of the 1s hyperfine splitting in the corresponding hydrogenlike ions. It follows that the accuracy of the theoretical predictions for the hyperfine splitting in Li-like ions can be improved significantly by increasing the experimental accuracy for the corresponding H-like ions. This implies excellent perspectives for testing QED effects in a combination of hyperfine splitting values for the hydrogenlike and lithiumlike ions.

V. CONCLUSION

In this paper we calculated the magnetic-loop WK corrections to the hyperfine splitting of the 1s and 2s states and the corresponding corrections to the nuclear magnetic moments. We included these corrections in the calculation of the hyperfine splitting values of heavy Li-like ions based on the experimental values of the hyperfine splitting in the corresponding H-like ions. We found that the magnetic-loop WK correction affects the 1s and 2s hyperfine splitting in a similar manner and, thus, hardly improves the total theoretical predictions for the hyperfine splitting in lithiumlike ions. However, the accurate values of this correction for both 1s and 2s states obtained in our paper will be needed for testing QED effects when the hyperfine splitting in H- and Li-like ions is measured to a higher accuracy.

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APPENDIX

The general form of the Green function of the Dirac equation for a spherically symmetric potential is well known (see, e.g., Ref. [31]). For a fixed relativistic angular quantum number κ, the radial components of the Green function for $x<y$ are given by[2]

$$G^{11}(\omega,x,y)=g_0(x)g_\infty(y)/W, \tag{A1}$$

$$G^{12}(\omega,x,y)=g_0(x)f_\infty(y)/W, \tag{A2}$$

$$G^{21}(\omega,x,y)=f_0(x)g_\infty(y)/W, \tag{A3}$$

$$G^{22}(\omega,x,y)=f_0(x)f_\infty(y)/W, \tag{A4}$$

where g and f are the upper and lower components of the solutions of the radial Dirac equation and the subscripts 0 and ∞ label the solutions regular at the origin and at infinity, respectively. W denotes the Wronskian,

$$W=\{g_0(r)f_\infty(r)-f_0(r)g_\infty(r)\}r^2. \tag{A5}$$

For $x>y$ the radial components of the Green function are determined using the following symmetry condition: $G^{ik}(y,x)=G^{ki}(x,y)$.

In the case of a homogeneously charged spherical shell, the functions g_0, g_∞, f_0, and f_∞ can be expressed analytically [30]:

$$g_0(r)=\Theta(R-r)u_1+\Theta(r-R)(aU_1+bV_1), \tag{A6}$$

$$f_0(r)=\Theta(R-r)u_2+\Theta(r-R)(aU_2+bV_2), \tag{A7}$$

$$g_\infty(r)=\Theta(R-r)(cu_1+dv_1)+\Theta(r-R)V_1, \tag{A8}$$

$$f_\infty(r)=\Theta(R-r)(cu_2+dv_2)+\Theta(r-R)V_2, \tag{A9}$$

where R is the radius of the spherical shell,

$$u_1=(1+\omega')j_{|\kappa+1/2|-1/2}(ic'r), \tag{A10}$$

$$u_2=ic'\,\mathrm{sgn}(\kappa)j_{|\kappa-1/2|-1/2}(ic'r), \tag{A11}$$

$$v_1=-c'h^{(1)}_{|\kappa+1/2|-1/2}(ic'r), \tag{A12}$$

$$v_2=i(\omega'-1)\mathrm{sgn}(\kappa)h^{(1)}_{|\kappa-1/2|-1/2}(ic'r), \tag{A13}$$

$$U_1=\frac{1+\omega}{r^{3/2}}\bigg[(\lambda-\nu)M_{\nu-1/2,\lambda}(2cr) -\left(\kappa-\frac{\alpha Z}{c}\right)M_{\nu+1/2,\lambda}(2cr)\bigg], \tag{A14}$$

$$U_2=\frac{c}{r^{3/2}}\bigg[(\lambda-\nu)M_{\nu-1/2,\lambda}(2cr)+\left(\kappa-\frac{\alpha Z}{c}\right)M_{\nu+1/2,\lambda}(2cr)\bigg], \tag{A15}$$

[2]Note, that we use $G=(\omega-H)^{-1}$ instead of $G=(H-\omega)^{-1}$ in Ref. [31].

$$V_1=\frac{1+\omega}{r^{3/2}}\left[\left(\kappa+\frac{\alpha Z}{c}\right)W_{\nu-1/2,\lambda}(2cr)+W_{\nu+1/2,\lambda}(2cr)\right], \tag{A16}$$

$$V_2=\frac{c}{r^{3/2}}\left[\left(\kappa+\frac{\alpha Z}{c}\right)W_{\nu-1/2,\lambda}(2cr)-W_{\nu+1/2,\lambda}(2cr)\right], \tag{A17}$$

j_n, $h_n^{(1)}$, $M_{\alpha,\beta}$, and $W_{\alpha,\beta}$ are the spherical Bessel and Hankel functions of first kind and the Whittaker functions, respectively; $\omega'=\omega+\alpha Z/R$, $c'=\sqrt{1-\omega'}$, $\mathrm{Re}(c')>0$, $\lambda=\sqrt{\kappa^2-(\alpha Z)^2}$, $c=\sqrt{1-\omega^2}$, $\mathrm{Re}(c)>0$, and $\nu=\alpha Z\omega/c$.

The coefficients a, b, c, and d can be found from the condition of continuity at $r=R$:

$$a=\frac{(u_2V_1-u_1V_2)}{W_1},$$

$$b=\frac{(u_1U_2-u_2U_1)}{W_1},$$

$$c=\frac{(v_1V_2-v_2V_1)}{W_2},$$

$$d=\frac{(u_2V_1-u_1V_2)}{W_2}, \tag{A18}$$

with

$$W_1=U_2V_1-U_1V_2=4(1+\omega)\frac{c^2}{R^2}\frac{\Gamma(2\lambda+1)}{\Gamma(\lambda-\nu)}, \tag{A19}$$

$$W_2=u_2v_1-u_1v_2=R^{-2}. \tag{A20}$$

The functions in Eq. (A18) must be evaluated at $r=R$. The Wronskian (A5) is equal to

$$W=W(\omega)=R^2(u_2V_1-u_1V_2). \tag{A21}$$

To divide the Green function into two parts, each containing only even or only odd powers of the nuclear charge number Z, we note that in the case of $\omega=i\varepsilon$, where ε is real, we can divide the functions defined by (A10)–(A17) into the ''even'' or ''odd'' parts as follows:

$$u_1^{\mathrm{even}}=i\,\mathrm{Im}[j_{|\kappa+1/2|-1/2}(ic'r)]+\mathrm{Re}[\omega' j_{|\kappa+1/2|-1/2}(ic'r)], \tag{A22}$$

$$u_1^{\mathrm{odd}}=\mathrm{Re}[j_{|\kappa+1/2|-1/2}(ic'r)]+i\,\mathrm{Im}[\omega' j_{|\kappa+1/2|-1/2}(ic'r)], \tag{A23}$$

$$u_2^{\mathrm{even}}=i\,\mathrm{sgn}(\kappa)\mathrm{Re}[c' j_{|\kappa-1/2|-1/2}(ic'r)], \tag{A24}$$

$$u_2^{\mathrm{odd}}=-\mathrm{sgn}(\kappa)\mathrm{Im}[c' j_{|\kappa-1/2|-1/2}(ic'r)], \tag{A25}$$

$$v_1^{\mathrm{even}}=-i\,\mathrm{Im}[c' h^{(1)}_{|\kappa+1/2|-1/2}(ic'r)], \tag{A26}$$

$$v_1^{\mathrm{odd}}=-\mathrm{Re}[c' h^{(1)}_{|\kappa+1/2|-1/2}(ic'r)], \tag{A27}$$

$$v_2^{\mathrm{even}}=-\mathrm{sgn}(\kappa)\{\mathrm{Im}[\omega' h^{(1)}_{|\kappa-1/2|-1/2}(ic'r)]+i\,\mathrm{Re}[h^{(1)}_{|\kappa-1/2|-1/2}(ic'r)]\}, \tag{A28}$$

$$v_2^{\mathrm{odd}}=\mathrm{sgn}(\kappa)\{i\,\mathrm{Re}[\omega' h^{(1)}_{|\kappa-1/2|-1/2}(ic'r)]+\mathrm{Im}[h^{(1)}_{|\kappa-1/2|-1/2}(ic'r)]\}, \tag{A29}$$

$$U_1^{\mathrm{even}}=\frac{1+\omega}{r^{3/2}}\left\{\mathrm{Re}[(\lambda-\nu)M_{\nu-1/2,\lambda}(2cr)-\kappa M_{\nu+1/2,\lambda}(2cr)]+i\frac{\alpha Z}{c}\mathrm{Im}[M_{\nu+1/2,\lambda}(2cr)]\right\}, \tag{A30}$$

$$U_1^{\mathrm{odd}}=\frac{1+\omega}{r^{3/2}}\left\{i\,\mathrm{Im}[(\lambda-\nu)M_{\nu-1/2,\lambda}(2cr)-\kappa M_{\nu+1/2,\lambda}(2cr)]+\frac{\alpha Z}{c}\mathrm{Re}[M_{\nu+1/2,\lambda}(2cr)]\right\}, \tag{A31}$$

$$U_2^{\mathrm{even}}=\frac{c}{r^{3/2}}\left\{\mathrm{Re}[(\lambda-\nu)M_{\nu-1/2,\lambda}(2cr)+\kappa M_{\nu+1/2,\lambda}(2cr)]-i\frac{\alpha Z}{c}\mathrm{Im}[M_{\nu+1/2,\lambda}(2cr)]\right\}, \tag{A32}$$

$$U_2^{\mathrm{odd}}=\frac{c}{r^{3/2}}\left\{i\,\mathrm{Im}[(\lambda-\nu)M_{\nu-1/2,\lambda}(2cr)+\kappa M_{\nu+1/2,\lambda}(2cr)]-\frac{\alpha Z}{c}\mathrm{Re}[M_{\nu+1/2,\lambda}(2cr)]\right\}, \tag{A33}$$

$$V_1^{\mathrm{even}}=\frac{1+\omega}{r^{3/2}}\left\{\mathrm{Re}[\kappa W_{\nu-1/2,\lambda}(2cr)+W_{\nu+1/2,\lambda}(2cr)]+i\frac{\alpha Z}{c}\mathrm{Im}[W_{\nu-1/2,\lambda}(2cr)]\right\}, \tag{A34}$$

$$V_1^{\mathrm{odd}}=\frac{1+\omega}{r^{3/2}}\left\{i\,\mathrm{Im}[\kappa W_{\nu-1/2,\lambda}(2cr)+W_{\nu+1/2,\lambda}(2cr)]+\frac{\alpha Z}{c}\mathrm{Re}[W_{\nu-1/2,\lambda}(2cr)]\right\}, \tag{A35}$$

$$V_2^{\mathrm{even}}=\frac{c}{r^{3/2}}\left\{\mathrm{Re}[\kappa W_{\nu-1/2,\lambda}(2cr)-W_{\nu+1/2,\lambda}(2cr)]+i\frac{\alpha Z}{c}\mathrm{Im}[W_{\nu-1/2,\lambda}(2cr)]\right\}, \tag{A36}$$

$$V_2^{\mathrm{odd}}=\frac{c}{r^{3/2}}\left\{i\,\mathrm{Im}[\kappa W_{\nu-1/2,\lambda}(2cr)-W_{\nu+1/2,\lambda}(2cr)]+\frac{\alpha Z}{c}\mathrm{Re}[W_{\nu-1/2,\lambda}(2cr)]\right\},$$

$$W_1^{\text{even}}=4(1+\omega)\frac{c^2}{R^2}\Gamma(2\lambda+1)\text{Re}\left(\frac{1}{\Gamma(\lambda-\nu)}\right), \quad (A37)$$

$$W_1^{\text{odd}}=4i(1+\omega)\frac{c^2}{R^2}\Gamma(2\lambda+1)\text{Im}\left(\frac{1}{\Gamma(\lambda-\nu)}\right), \quad (A38)$$

$$W_2^{\text{even}}=\frac{1}{R^2}, \quad (A39)$$

$$W_2^{\text{odd}}=0. \quad (A40)$$

Now using Eqs. (A22)–(A40) and the simple rules

$$(AB)^{\text{even}}=A^{\text{even}}B^{\text{even}}+A^{\text{odd}}B^{\text{odd}}, \quad (A41)$$

$$(AB)^{\text{odd}}=A^{\text{even}}B^{\text{odd}}+A^{\text{odd}}B^{\text{even}}, \quad (A42)$$

$$\left(\frac{1}{A}\right)^{\text{even}}=\frac{A^{\text{even}}}{(A^{\text{even}})^2-(A^{\text{odd}})^2}, \quad (A43)$$

$$\left(\frac{1}{A}\right)^{\text{odd}}=\frac{-A^{\text{odd}}}{(A^{\text{even}})^2-(A^{\text{odd}})^2}, \quad (A44)$$

it is easy to find the parts of the Green function that contain only even or only odd powers of Z.

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