Coherent control of spin squeezing

C. K. Law, H. T. Ng, and P. T. Leung
Department of Physics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong, People's Republic of China
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We examine the spin-squeezing phenomena in a collection of interacting spins in the presence of an external field. We show that the combination of the external field and the nonlinear atom-atom interaction leads to a strong reduction of spin fluctuations that can be maintained in an extended period of time. Our results can be applied to achieve spin squeezing in a spinor Bose condensate.

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Squeezed-spin states (SSS) of atoms have reduced quantum fluctuations that are useful in enhancing sensitivity in precision spectroscopy [1]. Since the early work by Kitagawa and Ueda [2], there have been several proposals for generating SSS in different configurations [1,3–7]. In the original proposal, Kitagawa and Ueda [2] have identified two fundamental types of nonlinear spin interactions that lead to spin squeezing,

$$H_1 = 2\kappa J_z^2, \tag{1}$$

$$H_2 = i\kappa (J_+^2 - J_-^2). \tag{2}$$

Here Js' are collective angular momentum operators and κ describes the nonlinear interaction strength. The physical realization of these interactions is still a challenge but recently, several studies have identified that the nonlinear interaction H_1 can naturally be found in a trapped Bose condensate [6–8].

In this paper we examine spin dynamics of a generic-spin model specified by the Hamiltonian ($\hbar = 1$),

$$H_3 = 2\kappa J_z^2 + \Omega J_x. \tag{3}$$

This Hamiltonian generalizes H_1 by including a linear interaction term ΩJ_x. In practice the Hamiltonian H_3 can be applied to a trapped spinor Bose condensate interacting with an external Raman field or a magnetic field. Under the condition that the trap is sufficiently tight, the external motion is essentially frozen. In this situation only the internal spin dynamics of the condensate is relevant, the second-quantized field Hamiltonian can be reduced to the form of H_3. A detailed derivation of H_3 was given by Raghavan for spin-1 condensates [7]. Here we note that Ω (assumed positive) can be controlled by the strength of the external field, and the nonlinear strength κ depends on the scattering lengths between particles and the condensate density.

The main purpose of this paper is to describe two significant effects of the external field on dynamics of spin fluctuations: (i) the applied external field induces a stronger squeezing while preserving the coherent component of the collective spin and (ii) spin squeezing can be maintained in a much longer time interval.

To be definite, we consider a J-spin system that can be regarded as a collection of $2J$ spin 1/2 particles. Following Kitagawa and Ueda's criteria of spin squeezing, we introduce the squeezing parameter

$$\xi_s = \frac{\sqrt{2}\langle(\Delta J_\perp)_{min}\rangle}{J^{1/2}}, \tag{4}$$

where $\langle(\Delta J_\perp)_{min}\rangle$ is the smallest uncertainty of an angular momentum component perpendicular to the mean angular momentum $\langle \mathbf{J}\rangle$. A state is said to be a squeezed-spin state if $\xi_s < 1$.

Let us first examine the exact numerical solutions of the time-dependent Schrödinger equation governed by the Hamiltonian H_3. We consider that the system starts from the lowest eigenvector of J_x, $|J, m_x = -J\rangle$. Such an initial state is favorable in generating spin squeezing because of the twisting effect due to the nonlinear interaction $2\kappa J_z^2$ along the z axis. In Fig. 1 we show the typical behavior of the squeezing parameter as a function of time. In the case of $\Omega = 0$ (i.e., the H_1 model) shown in Fig. 1(a), we see that ξ_s reaches a minimum after a characteristic time. However, such a squeezing can only be maintained in a certain time period. As the time increases, the system is less squeezed and eventually becomes unsqueezed.

The key advantage of the model H_3 ($\Omega \neq 0$) is the maintenance of squeezing in an extended period of time. This is clearly shown in Figs. 1(b)–1(d). We see that ξ_s can be kept below unity for a much longer period of time than that for the $\Omega = 0$ case. Indeed, for the choice of Ω used in Figs. 1(b)–1(d), the system exhibits squeezing most of the time. The interaction strength Ω affects the minimal ξ_s that the system can reach. Figure 1(c) represents a near optimal choice of Ω for $J = 100$. A closer look at the minimal value of ξ_s indicates that it is less than (i.e., more squeezing) that in Fig. 1(a).

Another advantage of the interaction model H_3 is the maintenance of a large coherent (mean) component of the collective spin. This feature is shown in Fig. 2 where the expectation values $\langle J_x\rangle$ are plotted against time for various values of Ω. We remark that when the system starts from $|J, m_x = -J\rangle$, the only nonvanishing spin component is J_x because $\langle J_y\rangle = \langle J_z\rangle = 0$ at all times. In Fig. 2, we see that for a sufficiently large Ω (for example the $\Omega = 25\kappa$ curve), $\langle J_x\rangle$ changes slightly in the course of time. This is in contrast to

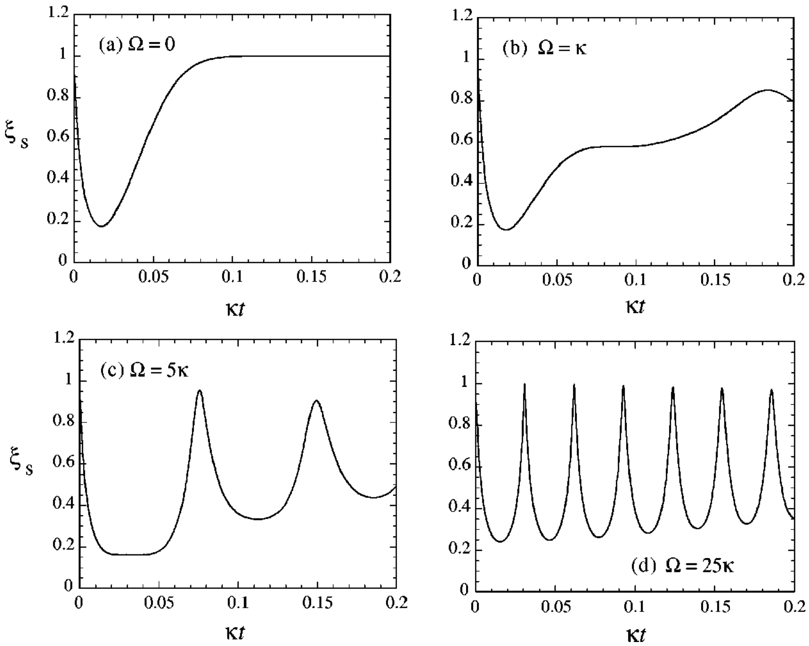

FIG. 1. Typical time dependence of ξ_s for the system starting from the initial state $|J,m_x=-J\rangle$. Here $J=100$ is used.

the $\Omega=0$ case in which $\langle J_x\rangle$ vanishes after some time. Since a strong coherent-spin component is often needed in increasing the sensitivity of precision measurement (such as in Ramesy specroscopy [1]), H_3 is more desirable in this regard.

Although exact analytic solutions of the nonlinear problem are not available, the squeezing behavior produced by H_3 can be understood when Ω is sufficiently larger than κ, i.e., $\Omega \gg \kappa$. First we recall that the system is prepared to start from the lowest eigenstate of J_x, $|J,m_x=-J\rangle$, i.e.,

$$J_x|J,m_x=-J\rangle=-J|J,m_x=-J\rangle. \tag{5}$$

Such a state minimizes the energy assoicated with the interaction ΩJ_x. If $\Omega \gg \kappa$, the external field forces the total spin to remain polarized in the negative x direction because it costs energy to change the direction of the spin vector. This explains why a large coherent component of $\langle J_x\rangle$ can be maintained. However, the unexpected effect of the linear-interaction term is the *enhancement of spin squeezing*.

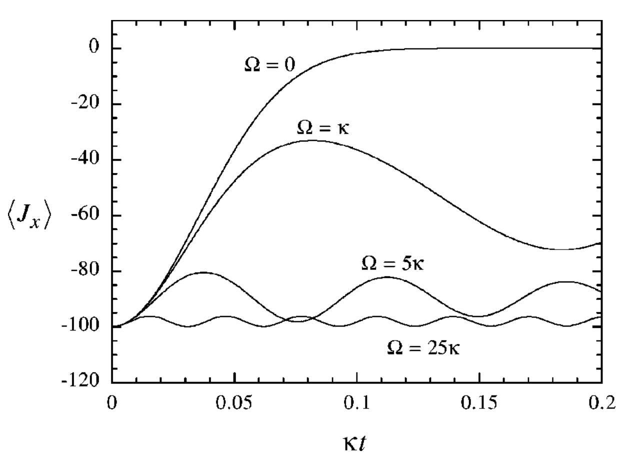

FIG. 2. Time dependence of the expectation value of J_x, same parameters as in Fig. 1.

To see this let us look at the Heisenberg equation of motion of the angular momentum operators in the y and z directions,

$$\dot{J}_z=\Omega J_y, \tag{6}$$

$$\dot{J}_y=-\Omega J_z+2\kappa(J_zJ_x+J_xJ_z). \tag{7}$$

Based on the fact that J_x remains unchanged approximately, it is justified to make an approximation: replacing J_x by $-J$. We call such an approximation a *frozen-spin approximation* [9]. In this way, we have

$$\ddot{J}_z\approx-(\Omega^2+4\kappa\Omega J)J_z, \tag{8}$$

which permits harmonic solutions

$$J_z(t)\approx J_z(0)\cos\omega t+\Omega J_y(0)\sin\omega t/\omega, \tag{9}$$

$$J_y(t)\approx-\omega J_z(0)\sin\omega t/\Omega+J_y(0)\cos\omega t, \tag{10}$$

where the frequency $\omega\equiv\sqrt{\Omega^2+4\kappa\Omega J}$ is defined.

Equations (9) and (10) are operator solutions under the frozen-spin approximation. The time-dependent spin fluctuations are given by

$$\langle[\Delta J_z(t)]^2\rangle\approx\langle J_z^2(0)\rangle\cos^2\omega t+\frac{\Omega^2}{\omega^2}\langle J_y^2(0)\rangle\sin^2\omega t, \tag{11}$$

$$\langle[\Delta J_y(t)]^2\rangle\approx\langle J_y^2(0)\rangle\cos^2\omega t+\frac{\omega^2}{\Omega^2}\langle J_z^2(0)\rangle\sin^2\omega t. \tag{12}$$

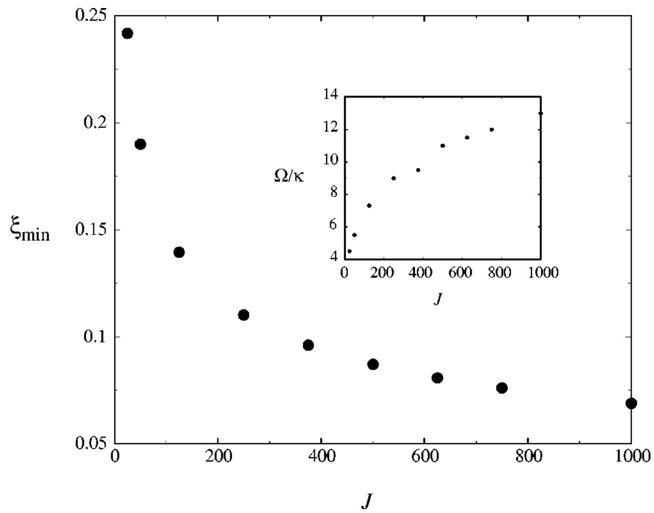

FIG. 3. The minimum value of the squeezing parameter that can be attained for various J. The inset shows the optimal Ω used for each J.

Here the cross terms $\langle J_z(0)J_y(0)\rangle$ and $\langle J_y(0)J_z(0)\rangle$ do not appear because they are identically zero with respect to the initial state (5). Now using the fact that $\omega > \Omega$ and

$$\langle J_z^2(0)\rangle = \langle J_y^2(0)\rangle = J/2, \tag{13}$$

we find that reduced spin fluctuations occurs in the z direction, i.e., $\langle [\Delta J_z(t)]^2\rangle \leqslant J/2$. In other words *the system is always squeezed* except at the times $t = n\pi/\omega$. The strongest squeezing occurs at $t = t^* = (2n+1)\pi/2\omega$ with

$$\langle [\Delta J_z(t)]^2\rangle_{t=t^*} \approx \frac{\Omega^2 J}{2\omega^2}. \tag{14}$$

This corresponds to the squeezing parameter at t^*,

$$\xi_{min} \equiv \xi_s|_{t=t^*} \approx \frac{\Omega}{\omega} < 1. \tag{15}$$

From the definition of ω above, we see that the squeezing parameter ξ_{min} is approximately $(4\kappa J/\Omega)^{-1/2}$ when $\kappa J \gg \Omega$. Therefore the system is less squeezed if Ω is large, but more squeezing can be achieved by increasing the number of particles. In fact, the power law $J^{-1/2}$ dependence decreases faster than that in H_1 (which is $J^{-1/3}$ according to Ref. [2]). Hence a linear-field interaction can substantially enhance the squeezing compared with the H_1 in the large J limit.

We should point out that the frozen-spin approximation becomes less valid when Ω is comparable to κ. Nevertheless, the approximation captures the essential physical picture. We have compared the approximate analytical results with the exact numerical solutions, we found a good agreement in ξ_{min} and the oscillation frequency ω as long as $\Omega \gg \kappa$.

In order to to determine how optimal squeezing depends on Ω and particle number $2J$ beyond the frozen-spin approximation, we have examined the exact numerical solutions for a wide range of J and Ω. In Fig. 3 we show the values of ξ_{min} that can be attained by our model H_3 for different Js. These values of ξ_{min} are attained by using the optimal Ω for the corresponding J (see the inset of Fig. 3). For example, we have found that for $J = 500$, $\Omega \approx 10\kappa$ yields an optimal squeezing parameter $\xi_{min} \approx 0.09$. These numerical findings show that strong squeezing is exhibited in the domain $\kappa J \gg \Omega \gg \kappa$.

In conclusion, we have discovered how an external field can be applied to reduce spin fluctuations, which is expected to play a prominent role in controlling the internal dynamics of Bose condensates. Our results indicate that a cooperation between the nonlinear self-interaction $2\kappa J_z^2$ and the external interaction ΩJ_x can generate spin squeezing in an extended period of time. In our scheme the coherent component of the collect spin can be locked in the x axis, and the reduced fluctuations always appear in the z axis. These important advantages are not found in the original spin-squeezing model H_1.

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