Radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)m$ to the Lamb shift revisited

Michael I. Eides*
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506
and Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188350, Russia

Howard Grotch†
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506

Valery A. Shelyuto‡
D. I. Mendeleev Institute of Metrology, St. Petersburg 198005, Russia
(Received 8 January 2001; published 16 April 2001)

The results and main steps of an analytic calculation of radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)m$ to the Lamb shift in hydrogen are presented. The calculations are performed in the infrared safe Yennie gauge. The discrepancy between two previous numerical calculations of these corrections existing in the literature is resolved. Our result eliminates the largest source of the theoretical uncertainty in the magnitude of the deuterium-hydrogen isotope shift.

DOI: 10.1103/PhysRevA.63.052509 PACS number(s): 31.30.Jv, 12.20.−m, 36.20.Kd

I. INTRODUCTION

The spectacular experimental progress achieved in recent years in precise measurements of the energy levels in light hydrogenlike atoms was matched by equally impressive theoretical developments (see, e.g., the review in [1], and references therein). Still, there are a number of unsolved theoretical problems, not least among them the magnitude of radiative-recoil corrections to the Lamb shift of order $\alpha(Z\alpha)^5(m/M)m$. To the best of our knowledge this is the first nontrivial radiative-recoil correction, and it is generated by the radiative insertions in the exchanged photon lines and in the electron line. The correction generated by the one-loop polarization insertions in the exchanged photon lines was independently calculated analytically in [2] and [3], with results that are in excellent agreement. Radiative-recoil corrections generated by radiative insertions in the electron line were obtained numerically in [4–6] and in [3]. The results of these calculations do not agree. In this paper we present an analytic calculation of the radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)m$, and resolve the discrepancy between different numerical results.

Calculation of the contributions of order $\alpha(Z\alpha)^5m$ to the Lamb shift is greatly facilitated by the applicability of the scattering approximation. These corrections are generated by the diagrams with at least two photon exchanges in Fig. 1. Naively, one might expect that diagrams with a larger number of exchanges are also relevant. However, this is not the case. For high-exchanged momenta expansion in $Z\alpha$ is valid, and addition of any extra exchanged photons always produces extra powers of $Z\alpha$. Hence, in the high-momentum region only diagrams with two exchanged photons are relevant. Contributions in the low-momentum region are suppressed because the infrared behavior of any radiatively corrected Feynman diagram (or more accurately any gauge-invariant sum of Feynman diagrams) is milder than the behavior of the skeleton diagram. Hence, unlike the leading contribution to the Lamb shift, diagrams with higher number of photon exchanges do not contribute to the corrections of order $\alpha(Z\alpha)^5m$, and it is sufficient to calculate only the contributions of the diagrams in Fig. 1 in the scattering approximation (for more detail, see, e.g., [1]).

In the scattering approximation the contribution to the energy shift generated by the diagrams in Fig. 1 is given by the integral

$$\Delta E=-\frac{(Z\alpha)^5}{\pi n^3}m_r^3\int\frac{d^4k}{i\pi^2k^4}\frac{1}{4}\mathrm{Tr}[(1+\gamma_0)L_{\mu\nu}]\frac{1}{4}\times\mathrm{Tr}[(1+\gamma_0)H_{\mu\nu}]\delta_{l0},\tag{1}$$

where m and m_r are the electron and reduced masses, k is the momentum of the exchanged photon, $L_{\mu\nu}$ and $H_{\mu\nu}$ are the electron and the proton factors, and the Kronecker symbol δ_{l0} reminds us that the radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)m$ are different from zero only for the S states.

The electron factor is equal to the sum of the self-energy, vertex, and spanning photon insertions in the electron line

$$L_{\mu\nu}=L^{\Sigma}_{\mu\nu}+2L^{\Lambda}_{\mu\nu}+L^{\Xi}_{\mu\nu},\tag{2}$$

and the heavy line or proton factor is given by the expression

$$H_{\mu\nu}=\gamma_\mu\frac{\hat{P}+\hat{k}+M}{k^2+2Mk_0+i0}\gamma_\nu+\gamma_\nu\frac{\hat{P}-\hat{k}+M}{k^2-2Mk_0+i0}\gamma_\mu,\tag{3}$$

where $P=(M,\mathbf{0})$ is the momentum of the proton.

The expression for the energy shift in Eq. (1) contains both recoil and nonrecoil contributions of order $\alpha(Z\alpha)^5m$.

*Email address: eides@pa.uky.edu, eides@thd.pnpi.spb.ru
†Email address: asdean@pop.uky.edu
‡Email address: shelyuto@vniim.ru

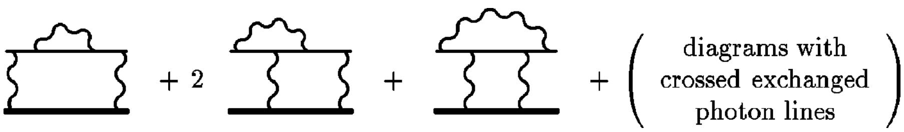

FIG. 1. Electron-line radiative-recoil corrections.

The nonrecoil correction of this order is well known from the early days of quantum electrodynamics, and in order to simplify further calculations we would like to obtain an expression for the energy shift free of such contributions. Let us note, to this end, that the characteristic integration momenta in Eq. (1) are of order of the electron mass, the lower momenta are suppressed by the radiative insertions in electron line, and the momenta of order of the heavy mass are suppressed by the high power of the integration momenta in the denominators. Suppression of the high-integration momenta means that the radiative-recoil correction of order $\alpha(Z\alpha)^5(m/M)m$ does not contain logarithms of the mass ratio $\ln(m/M)$ that could originate only from the wide-integration region between the electron and proton masses $m \ll k \ll M$. Then we can remove all nonrecoil contributions and extract the first order in the mass-ratio contribution simply by differentiating the integral in Eq. (1) with respect to the heavy mass M, and letting this mass go to infinity afterwards. The integral in Eq.(1) contains the heavy mass only in the heavy-particle factor, and to extract the term linear in the mass ratio we make the substitution in the integrand

$$\frac{1}{4}\mathrm{Tr}[(1+\gamma_0)H_{\mu\nu}] \to -M\frac{\partial}{\partial M}\left(\frac{1}{4}\mathrm{Tr}\left\{(1+\gamma_0)\left[\gamma_\mu \frac{\hat{P}-\hat{k}+M}{k^2-2Mk_0+i0}\gamma_\nu+\gamma_\nu\frac{\hat{P}+\hat{k}+M}{k^2+2Mk_0+i0}\gamma_\mu\right]\right\}\right)$$

$$\to -\frac{1}{M}\{k^2 g_{\mu 0}g_{\nu 0}-k_0(g_{\mu 0}k_\nu+g_{\nu 0}k_\mu)+k_0^2 g_{\mu\nu}\}\frac{k_0^2+(k^4/4M^2)}{[k_0^2-(k^4/4M^2)]^2}. \tag{4}$$

Due to the explicit factor $1/M$ before the braces, it is sufficient to consider the last factor in this expression only in limit $k/M \to 0$ (we note that the characteristic integration momenta are much less than the proton mass $k \ll M$). Then the coefficients before the tensor structures in the last line in Eq. (4) simplify dramatically.

We will calculate the Feynman integrals in the polar coordinates in the four-dimensional euclidean space, where the vector k is parametrized in the form $k_0 = k\cos\theta$ and $|\mathbf{k}| = k\sin\theta$. It is easy to see that the integrals with the coefficient before the first tensor structure in the braces in Eq. (4) should be calculated as principal-value integrals over the polar angle

$$k^2 \lim_{k/M\to 0}\frac{k_0^2+(k^4/4M^2)}{[k_0^2-(k^4/4M^2)]^2} \to \lim_{\varepsilon\to 0}\frac{\cos^2\theta-\varepsilon^2}{(\cos^2\theta+\varepsilon^2)^2} = \mathrm{P}\left(\frac{1}{\cos^2\theta}\right), \tag{5}$$

where $\varepsilon = k/2M$ and P is the principal-value symbol.

The second term in the curly brackets in Eq. (4) is odd in k_0, and will give nonzero contribution to the integral over k only when multiplied by another term from the electron factor odd in k_0. But this means that the electron factor would effectively supply an extra power of k_0 in the numerator, and we can safely take the limit $\varepsilon \to 0$ in the coefficient before the second tensor structure in Eq. (4). The factor before the $g_{\mu\nu}$ term in Eq. (4) admits a smooth limit for $\varepsilon \to 0$. Then the heavy particle factor turns into

$$\frac{1}{4}\mathrm{Tr}[(1+\gamma_0)H_{\mu\nu}] \to -\frac{1}{M}\left\{k^2 g_{\mu 0}g_{\nu 0}\,\mathrm{P}\left(\frac{1}{k_0^2}\right)-(g_{\mu 0}k_\nu + g_{\nu 0}k_\mu)\frac{1}{k_0}+g_{\mu\nu}\right\}, \tag{6}$$

and the general expression for all radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)m$ acquires the form

$$\Delta E = \frac{(Z\alpha)^5}{\pi n^3}\frac{m_r^3}{M}\int\frac{d^4k}{i\pi^2k^4}\frac{1}{4}\mathrm{Tr}\{(1+\gamma_0)[L^{\Sigma}_{\mu\nu}+2L^{\Lambda}_{\mu\nu}+L^{\Xi}_{\mu\nu}]\} \left\{k^2 g_{\mu 0}g_{\nu 0}\,\mathrm{P}\left(\frac{1}{k_0^2}\right)-(g_{\mu 0}k_\nu+g_{\nu 0}k_\mu)\frac{1}{k_0}+g_{\mu\nu}\right\}\delta_{l0}. \tag{7}$$

This expression is much more convenient for calculations than Eq. (1) because it depends on the heavy mass only through the explicit factor m_r^3/M before the integral.

The expression for the energy shift in Eq. (7) is linearly infrared divergent like $1/\gamma$ where γ is an auxiliary infrared cutoff in integration over k. This linear infrared divergence is the price we have to pay for the simplicity of the scattering approximation. The point is that the expressions for the energy shifts in Eq. (1) and Eq. (7) contain not only corrections of relative order $(Z\alpha)^5$ but also the corrections of the previous order in $Z\alpha$. If we would not ignore small virtualities of the external electron lines and the external wave functions the naive-infrared divergence would be regularized at the characteristic atomic scale $\gamma\sim mZ\alpha$. We are not interested in the contributions of the previous order in $Z\alpha$, and will simply throw away linearly infrared divergent contributions in our calculations. The remaining infrared-finite contribution is just the radiative-recoil correction of relative order $\alpha(Z\alpha)^5$. This strategy works because the radiative-recoil corrections under consideration do not contain logarithms of $Z\alpha$. However, individual diagrams could contain logarithms of the infrared cutoff that should cancel in the final result. This cancellation serves as an additional test of the correctness of all calculations. In the scattering approximation, the diagrams in Fig. 1 form a complete gauge-invariant set, and we can use an arbitrary gauge for their calculation. In order to improve the low-momentum behavior of individual diagrams we use the Yennie gauge for the radiative photons.

II. MASS OPERATOR CONTRIBUTION

Let us consider first the contribution to the radiative-recoil correction of order $\alpha(Z\alpha)^5(m/M)m$ generated by the diagrams with the self-energy insertions in the electron line in Fig. 1. The renormalized mass operator in the Yennie gauge has the form (see, e.g., [7])

$$\Sigma^R(p)=\frac{\alpha}{4\pi}(\hat{p}-m)^2\int_0^1 dx\,\frac{-3\hat{p}x}{m^2x+(m^2-p^2)(1-x)}. \quad (8)$$

According to Eq. (7) the respective contribution to the Lamb shift may be written as

$$\Delta E_\Sigma=-\frac{3}{4}\,\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\,\frac{m_r^3}{M}\int_0^1 dx\int\frac{d^4k}{i\pi^2}\,\frac{x}{k^4\Delta_1}\,\frac{1}{4}\mathrm{Tr}[(1+\gamma_0)\gamma_\mu(\hat{p}-\hat{k})\gamma_\nu]\left\{k^2g_{\mu0}g_{\nu0}\,\mathrm{P}\left(\frac{1}{k_0^2}\right)-(g_{\mu0}k_\nu+g_{\nu0}k_\mu)\frac{1}{k_0}+g_{\mu\nu}\right\}$$

$$=-\frac{3}{4}\,\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\,\frac{m_r^3}{M}\int_0^1 dx\int\frac{d^4k}{i\pi^2}\,\frac{x}{k^4\Delta_1}\left[-4m+2k_0+k^2\frac{1}{k_0}+mk^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right)\right], \quad (9)$$

where

$$\Delta_1=m^2x+2pk(1-x)-k^2(1-x)$$
$$\equiv(1-x)(-k^2+2mk_0+a_1^2),$$

and $a_1^2=m^2x/(1-x)$. In the formulas below we will often use dimensionless momenta measured in terms of the electron mass $k\to mk$, and a shorthand notation $C=[\alpha(Z\alpha)^5/(\pi^2n^3)](m/M)(m_r/m)^3m$ for the common normalization factor.

As an example of our calculations let us consider the evaluation of the contribution to the energy shift generated by the last most infrared singular term in the square brackets in Eq. (9)

$$\Delta E'_\Sigma=-\frac{3C}{4}\int_0^1 dx\int\frac{d^4k}{i\pi^2}\,\frac{x}{k^2(-k^2+2k_0+a_1^2)}\mathrm{P}\left(\frac{1}{k_0^2}\right)$$
$$=-\frac{3C}{4}\int_0^1 dx a_1^2\int_\gamma^\infty\frac{dk^2}{k^2}\,\frac{2}{\pi}\int_0^\pi d\theta$$
$$\times\frac{\sin^2\theta(k^2+a_1^2)}{(k^2+a_1^2)^2+4k^2\cos^2\theta}\mathrm{P}\left(\frac{1}{\cos^2\theta}\right). \quad (10)$$

The integration in the last line goes over the four-dimensional euclidean space, and we have introduced an auxiliary infrared cutoff γ. Using the identity

$$\frac{k^2+a_1^2}{(k^2+a_1^2)^2+4k^2\cos^2\theta}$$
$$=\frac{1}{k^2+a_1^2}-\frac{4k^2\cos^2\theta}{(k^2+a_1^2)[(k^2+a_1^2)^2+4k^2\cos^2\theta]}, \quad (11)$$

we represent the integral in Eq. (10) in the form

$$\Delta E'_\Sigma=-\frac{3C}{4}\int_0^1 dx a_1^2\int_\gamma^\infty\frac{dk^2}{k^2}\,\frac{2}{\pi}$$
$$\times\int_0^\pi d\theta\sin^2\theta\left\{\frac{1}{k^2+a_1^2}\mathrm{P}\left(\frac{1}{\cos^2\theta}\right)\right.$$
$$\left.-\frac{4k^2}{(k^2+a_1^2)[(k^2+a_1^2)^2+4k^2\cos^2\theta]}\right\}, \quad (12)$$

where the principal-value integral contains only a trivial dependence on the angles.

All principal-value integrals we need in this work, in particular the integrals

$$\frac{2}{\pi}\int_0^\pi d\theta\sin^2\theta\mathrm{P}\left(\frac{1}{\cos^2\theta}\right)=-2,$$

$$\frac{2}{\pi}\int_0^\pi d\theta\sin^4\theta\mathrm{P}\left(\frac{1}{\cos^2\theta}\right)=-3,$$

may be obtained from the basic integral

$$\int_0^{\pi} d\theta \, \mathrm{P}\left(\frac{1}{\cos^2\theta}\right) = 0, \tag{13}$$

with the help of algebraic transformations.

Now we can easily complete calculation of the integral in Eq. (10),

$$\Delta E'_{\Sigma} = \frac{3C}{4}\int_0^1 dx\left\{2\ln\frac{a_1^2}{\gamma^2} - 2\ln\frac{1+a_1^2}{a_1^2} + \frac{4}{a}\arctan\frac{1}{a_1}\right\}$$
$$= C\left[3\ln\frac{1}{\gamma} + \frac{3\pi^2}{8}\right].$$

Other integrals in Eq. (9) are calculated in the same way, and we obtain the total self-energy contribution to the energy shift in the form

$$\delta E_{\Sigma} = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\left[9\ln\frac{1}{\gamma}\right]. \tag{14}$$

III. VERTEX CONTRIBUTION

We calculate the contribution to the Lamb shift generated by the vertex insertion in the electron line in Fig. 1 with the help of the compact expression for the electron-photon vertex in the Yennie gauge used in our earlier work on the radiative corrections of order $\alpha^2(Z\alpha)^5 m$ [8],

$$\Lambda_\mu(p,p-k) = \frac{\alpha}{4\pi}\sum_{n=1}^{2}\frac{F_\mu^{(n)}}{\Delta^n}, \tag{15}$$

where

$$F_\mu^{(1)} = 3\gamma_\mu[k^2 - 2pk + (2-x)\Delta]$$
$$- x[3\gamma_\alpha(\hat{p}+m)\gamma_\mu(\hat{p}-\hat{k}+m)\gamma^\alpha - 6(p-k)Q\gamma_\mu$$
$$+ \gamma_\alpha\hat{Q}\gamma_\mu(\hat{p}-\hat{k}+m)\gamma^\alpha + \gamma_\alpha(\hat{p}+m)\gamma_\mu\hat{Q}\gamma^\alpha$$
$$+ 2\gamma_\mu(\hat{p}-\hat{k}+m)\hat{Q} + 2\hat{Q}(\hat{p}+m)\gamma_\mu]$$
$$+ x^2(2\hat{Q}\gamma_\mu\hat{Q} + Q^2\gamma_\mu), \tag{16}$$

$$F_\mu^{(2)} = 2x(1-x)\gamma_\mu(\hat{p}-\hat{k}+m)[\hat{Q}\hat{p}\hat{Q} - \hat{p}Q^2], \tag{17}$$

$$\Delta = m^2x + 2pk(1-x)z - k^2z(1-xz)$$
$$\equiv z(1-xz)(-k^2 + 2bk_0 + a^2), \tag{18}$$

and $Q = -p + kz, \quad p^2 = m^2$.

According to Eq. (7) the radiative-recoil contribution of order $\alpha(Z\alpha)^5(m/M)m$ generated by the vertex insertion has the form (we use dimensionless integration momenta below)

$$\Delta E_\Lambda = 2\frac{\alpha(Z\alpha)^5}{\pi^2 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\int_0^1 dx\int_0^1 dz\int\frac{d^4k}{i\pi^2}\frac{1}{k^4}\frac{1}{k^2-2k_0}$$
$$\times\sum_{n=1}^{2}\frac{1}{\Delta^n}\left[V_0^{(n)} + V_1^{(n)}\frac{1}{k_0} + V_2^{(n)}\mathrm{P}\left(\frac{1}{k_0^2}\right)\right], \tag{19}$$

where

$$V_0^{(n)} \equiv \frac{1}{4}\mathrm{Tr}[(1+\gamma_0)F_\mu^{(n)}(\hat{p}-\hat{k}+1)\gamma_\nu]g_{\mu\nu},$$

$$V_1^{(n)} \equiv \frac{1}{4}\mathrm{Tr}[(1+\gamma_0)F_\mu^{(n)}(\hat{p}-\hat{k}+1)\gamma_\nu](-g_{\mu0}k_\nu - g_{\nu0}k_\mu),$$

$$V_2^{(n)} \equiv \frac{1}{4}\mathrm{Tr}[(1+\gamma_0)F_\mu^{(n)}(\hat{p}-\hat{k}+1)\gamma_\nu]k^2 g_{\mu0}g_{\nu0}.$$

Calculating traces and contracting the Lorentz indices we obtain the numerator factors in the square brackets in Eq. (19),

$$V_0^{(1)} + V_1^{(1)}\frac{1}{k_0} + V_2^{(1)}\mathrm{P}\left(\frac{1}{k_0^2}\right)$$
$$= 4x(1-x)k^2\left[1 + \frac{1}{k_0}\right]$$
$$+ 2x(5 - 9z + 6z^2 + 4xz - 4xz^2)k^2\left[k_0 + \frac{2}{k_0}\right] + (k^2$$
$$- 2k_0)\Big\{[-6(1-2z) + 2x(-1 - 10z + 2x + 2xz)]$$
$$+ [6(1-2z) + 2x(1 + z + 6z^2 + 2xz - 4xz^2)]k^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right)$$
$$+ [6(1-2z) + 2x(-5 + 10z - 4xz)]k_0$$
$$+ [3(1-2z) + x(z + 6z^2 - 4xz^2)]\frac{k^2}{k_0}\Big\}, \tag{20}$$

$$V_0^{(2)} + V_1^{(2)}\frac{1}{k_0} + V_2^{(2)}\mathrm{P}\left(\frac{1}{k_0^2}\right)$$
$$= -4x(1-x)z\mathbf{k}^2\left\{4(1-z)\left[1 + k^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right)\right] + 2(1-z)\right.$$
$$\times(2k_0^2 - k^2)\frac{1}{k_0} + (k^2 - 2k_0)\left[-2z + 2(1-2z)\frac{1}{k_0}\right.$$
$$\left.\left. + zk^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right)\right]\right\}. \tag{21}$$

As a simple example let us consider calculation of the contribution to the energy shift generated by the terms in the last line in Eq. (20),

$$\Delta E'_\Lambda = 2C\int_0^1 dx\int_0^1 dz\int \frac{d^4k}{i\pi^2}\frac{1}{k^4\Delta}\times\Big\{[6(1-2z)+2x(-5+10z-4xz)]k_0 + [3(1-2z)+x(z+6z^2-4xz^2)]\frac{k^2}{k_0}\Big\}$$
$$= 2C\int_0^1 dx\int_0^1 dz\frac{b}{z(1-xz)}\int_0^\infty dk^2\frac{2}{\pi}\times\int_0^\pi d\theta\frac{\sin^2\theta}{(k^2+a^2)^2+4b^2k^2\cos^2\theta}$$
$$\{[6(1-2z)+2x(-5+10z-4xz)]\cos^2\theta+[3(1-2z)+x(z+6z^2-4xz^2)]\}. \qquad (22)$$

The last integral, as all other integrals in this paper, may be written as a linear combination of integrals of the form

$$\int_0^\infty dk^2\frac{2}{\pi}\int_0^\pi d\theta\frac{\sin^2\theta\cos^{2l}\theta(k^2)^m(k^2+a^2)^n}{[(k^2+a^2)^2+4b^2k^2\cos^2\theta]^p}, \qquad (23)$$

where $l,m,n=-1,0,1$ and $p=1,2,3$. For $l=-1$ the integrals with $\cos^2\theta$ in the denominator should be interpreted as principal-value integrals. All these integrals may be calculated in terms of four standard functions of the parameters a,b,

$$L_0=\ln\frac{a^2+b^2}{a^2},\quad L_1=1-\frac{a^2}{b^2}L_0,$$
$$L_2=1-\frac{2a^2}{b^2}L_1,\quad \frac{b}{a}\arctan\frac{b}{a}. \qquad (24)$$

The result of integration for our example in Eq. (22) is

$$\Delta E'_\Lambda = 2C\int_0^1 dx\int_0^1 dz\frac{1}{z(1-x)}\Big\{[6(1-2z)+2x(-5+10z-4xz)]\left(\frac{1}{2}L_0-\frac{1}{2}L_1\right)+[3(1-2z)+x(z+6z^2-4xz^2)]\times\left(-L_0+\frac{2b}{a}\arctan\frac{b}{a}\right)\Big\}$$
$$=2C\left[-\frac{11}{2}\zeta(3)+7\pi^2\ln 2-\frac{53\pi^2}{12}+7\right]. \qquad (25)$$

In the contribution to the energy shift in Eq. (22) the electron denominator canceled with a similar term in the numerator, and as a result all denominators were combined with the help of only two Feynman parameters x,z. Let us turn now to a simple example where such a cancellation does not take place, and we need to introduce a third Feynman parameter t. The contribution generated by the first term in Eq. (20) may be written in the form

$$\delta E''_\Lambda = 8C\int_0^1 dx\int_0^1 dz\int\frac{d^4k}{i\pi^2}\frac{1}{k^2(k^2-2k_0)}\frac{x(1-x)}{\Delta}$$
$$=-8C\int_0^1 dx\int_0^1 dz(1-x)a^2\int_0^1 dt\frac{\partial}{\partial a_t^2}\int_0^\infty dk^2\frac{2}{\pi}\int_0^\pi d\theta\times\sin^2\theta\frac{k^2+a_t^2}{(k^2+a_t^2)^2+4b_t^2k^2\cos^2\theta}, \qquad (26)$$

where $a_t^2=a^2t$ and $b_t=1-z(1-z)a^2t$.

The integral over angles and momenta in Eq. (26) is just of the standard form Eq. (23), and may easily be calculated. Integration over t is facilitated by the simple observation that the t integral may be written as an integral over a_t^2 with the upper limit a^2. After this transformation, all dependence of the new integral on x is hidden in this upper integration limit, and we can get rid of the third Feynman parameter integrating by parts over x (for more details see [7]),

$$\Delta E''_\Lambda = 8C\int_0^1 dx\int_0^1 dz(1-x)a^2\int_0^1\frac{dt}{b_t^2}L_{0t}$$
$$=8C\int_0^1 dx\int_0^1 dz(1-x)\int_0^{a^2}\frac{da_t^2}{b_t^2}L_{0t}$$
$$=4C\int_0^1 dx\int_0^1\frac{dz}{z}L_0=2C\left[\frac{\pi^2}{2}-2\right]. \qquad (27)$$

Contributions of the other terms in Eq. (20) are calculated in the same fashion, and the total contribution of all terms with $n=1$ in Eq. (19) is equal to

$$\Delta E_\Lambda^{(1)}=2\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\left[-9\ln\frac{1}{\gamma}+\frac{3}{8}\zeta(3)+\frac{9\pi^2}{4}\times\ln 2-\frac{3\pi^2}{4}+4\right]. \qquad (28)$$

For the total contribution of all terms with $n=2$ in Eq. (19) we obtain

$$\Delta E_\Lambda^{(2)}=2\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\left[\frac{21}{8}\zeta(3)-\frac{13\pi^2}{4}\ln 2+\frac{9\pi^2}{8}-\frac{35}{4}\right]. \qquad (29)$$

The total vertex-insertion contribution to the radiative-recoil correction of order $\alpha(Z\alpha)^5(m/M)m$ is given by the sum of the contributions in Eq. (28) and Eq. (29),

$$\Delta E_\Lambda = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\left[-18\ln\frac{1}{\gamma}+6\zeta(3)-2\pi^2\ln 2 + \frac{3\pi^2}{4}-\frac{19}{2}\right]. \tag{30}$$

IV. SPANNING PHOTON CONTRIBUTION

The contribution to the Lamb shift generated by the spanning photon insertion in the electron line in Fig. 1 is calculated with the help of an explicit expression for the jellyfish-shaped diagram. The small k behavior of the jellyfish diagram is one of our primary concerns in further calculations, since the contribution of the previous order is connected just with this infrared region. The jellyfish diagram is finite at small k in the Yennie gauge, and this is one of the reasons why we are working in this gauge. We need a representation for the jellyfish diagrams where not only the diagram as a whole, but all entries are finite at $k=0$. The compact expression for the jellyfish diagram with such properties was used in our earlier work on the radiative corrections of order $\alpha^2(Z\alpha)^5 m$ [8],

$$L^{\Xi}_{\mu\nu} = \frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dz x(1-z)\sum_{n=1}^{3}\frac{M^{(n)}_{\mu\nu}}{\Delta^n}, \tag{31}$$

where

$$M^{(1)}_{\mu\nu} = -2N^{(a)}_{\mu\nu}, \tag{32}$$

$$M^{(2)}_{\mu\nu} = N^{(b)}_{\mu\nu} + 2(1-x)N^{(c)}_{\mu\nu} + 3N^{(sing)}_{\mu\nu},$$

$$M^{(3)}_{\mu\nu} = -2(1-x)N^{(d)}_{\mu\nu},$$

$$\begin{aligned}
N^{(a)}_{\mu\nu} &= \gamma_\mu(5\hat{p}-3\hat{k})\gamma_\nu + 4mg_{\mu\nu} \\
&\quad + x[\hat{Q}\gamma_\nu\gamma_\mu + \gamma_\nu\gamma_\mu\hat{Q} + 4\gamma_\mu\hat{Q}\gamma_\nu],\\
N^{(b)}_{\mu\nu} &= 2m[\hat{Q}\gamma_\mu(\hat{p}+x\hat{Q}-\hat{k}+m)\gamma_\nu \\
&\quad + \gamma_\mu(\hat{p}+x\hat{Q}-\hat{k}+m)\gamma_\nu\hat{Q}] \\
&\quad - 12(1-2x)m^2\gamma_\mu\hat{Q}\gamma_\nu - 2x\hat{Q}\gamma_\nu \\
&\quad \times(\hat{p}+x\hat{Q}-\hat{k})\gamma_\mu\hat{Q} + 4xmQ^2 g_{\mu\nu} \\
&\quad + [2xQ^2 + 8(pQ)]\gamma_\mu(\hat{p}+x\hat{Q}-\hat{k} \\
&\quad + m)\gamma_\nu,
\end{aligned} \tag{33}$$

$$N^{(c)}_{\mu\nu} = 4(pQ)\gamma_\mu\hat{p}\gamma_\nu + 2m^2\gamma_\mu\hat{Q}\gamma_\nu,$$

$$N^{(d)}_{\mu\nu} = 8[(pQ)^2 - m^2Q^2]\gamma_\mu(\hat{p}+x\hat{Q}-\hat{k}+m)\gamma_\nu,$$

$$N^{(sing)}_{\mu\nu} = 4m^2\gamma_\mu(\hat{p}-\hat{k}+m)\gamma_\nu,$$

and a^2, b, and Q were defined in Eq. (18). It should be noted that all $M^{(i)}_{\mu\nu}$ are infrared finite even at $k=0$.

The general expression for the energy shift induced by the spanning photon insertion has the form [see Eq. (7)]

$$\Delta E_\Xi = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\int_0^1 dx\int_0^1 dz x(1-z)\int\frac{d^4k}{i\pi^2}\frac{1}{k^4} \sum_{n=1}^{3}\frac{1}{\Delta^n}\left[T_0^{(n)} + T_1^{(n)}\frac{1}{k_0} + T_2^{(n)}\mathrm{P}\left(\frac{1}{k_0^2}\right)\right], \tag{34}$$

where

$$T_0^{(n)} \equiv \frac{1}{4}\mathrm{Tr}[(1+\gamma_0)M^{(n)}_{\mu\nu}]g_{\mu\nu},$$

$$T_1^{(n)} \equiv \frac{1}{4}\mathrm{Tr}[(1+\gamma_0)M^{(n)}_{\mu\nu}](-g_{\mu0}k_\nu - g_{\nu0}k_\mu),$$

$$T_2^{(n)} \equiv \frac{1}{4}\mathrm{Tr}[(1+\gamma_0)M^{(n)}_{\mu\nu}]k^2 g_{\mu0}g_{\nu0}. \tag{35}$$

Calculating traces and contracting the Lorentz indices we obtain the numerator factors in the square brackets in Eq. (34),

$$\begin{aligned}
T_0^{(1)} + T_1^{(1)}\frac{1}{k_0} + T_2^{(1)}\mathrm{P}\left(\frac{1}{k_0^2}\right) &= 24(1-x) + 4(-3+2xz)k_0 + 2(-3+2xz)\frac{k^2}{k_0} \\
&\quad + 6(-3+2x)k^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right),
\end{aligned} \tag{36}$$

$$\begin{aligned}
T_0^{(2)} + T_1^{(2)}\frac{1}{k_0} + T_2^{(2)}\mathrm{P}\left(\frac{1}{k_0^2}\right) &= 12x - 8z(1-x)(4-x)k_0 + 8z(2-x-xz)k_0^2 + 4z \\
&\quad \times(4+x-4xz+2x^2z)k^2 + 8z(1-x)(4-x)\frac{k^2}{k_0} \\
&\quad + 6xk^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right) + 2z(2+2x+3xz-4x^2z)k^4 \\
&\quad \times\mathrm{P}\left(\frac{1}{k_0^2}\right),
\end{aligned} \tag{37}$$

$$T_0^{(3)}+T_1^{(3)}\frac{1}{k_0}+T_2^{(3)}\mathrm{P}\left(\frac{1}{k_0^2}\right)$$

$$=-16(1-x)z^2\mathbf{k}^2\Bigg[2(-1+2x)+2(1-xz)k_0+(1-xz)\frac{k^2}{k_0}+(2-x)k^2\,\mathrm{P}\left(\frac{1}{k_0^2}\right)\Bigg]. \tag{38}$$

We will use the integral

$$\Delta E'_{\Xi}=16C\int_0^1 dx\int_0^1 dz z(1-z)\int\frac{d^4k}{i\pi^2}\frac{\mathbf{k}^2}{k^2}\frac{-2xz}{\Delta^3}\mathrm{P}\left(\frac{1}{k_0^2}\right), \tag{39}$$

describing one of the contributions with $n=3$ in order to illustrate one more subtlety encountered in our calculations. This infrared-divergent integral contains in the integrand the term x/Δ^3. At small $k\to 0$ the denominator $\Delta\to x$, and integration over x becomes too singular. The singular factor $\mathrm{P}(1/k_0^2)$ in the integrand makes things even worse, and we risk ending up with a divergent integral over the Feynman parameter x instead of an infrared-divergent integral over the momentum. While linearly infrared divergent integrals over k have a transparent physical interpretation as contributions of the previous order in $Z\alpha$, divergent integrals over x could lead to uncontrollable contributions. We separate the infrared-divergent part of the momentum integral with the help of the identity

$$-2z\int_0^1 dx\frac{x}{\Delta^3}$$

$$=\frac{1}{k^2-2k_0}+\frac{2z}{1-k^2z(1-z)}+\frac{z^2(1-z)k^2}{[1-k^2z(1-z)]^2} \tag{40}$$

$$-z(1-Q^2)\int_0^1 dx\left(\frac{1}{\Delta^2}+\frac{2x}{\Delta^3}\right).$$

Then the singular integration over momentum decouples, and we easily obtain

$$\Delta E'_{\Xi}=16C\left[\frac{4}{3\gamma}-\frac{3}{4}\right]. \tag{41}$$

After tedious calculations we obtain

$$\Delta E_{\Xi}^{(1)}=\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\Bigg[9\ln\frac{1}{\gamma}-\frac{123}{4}\zeta(3)+\frac{3\pi^2}{2}\ln 2-\frac{15\pi^2}{16}+\frac{159}{4}\Bigg], \tag{42}$$

$$\Delta E_{\Xi}^{(2)}=\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\left[12\zeta(3)+\frac{3\pi^2}{8}-16\right], \tag{43}$$

$$\Delta E_{\Xi}^{(3)}=\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\Bigg[\frac{75}{4}\zeta(3)-\frac{3\pi^2}{2}\times\ln 2+\frac{9\pi^2}{16}-\frac{113}{4}\Bigg], \tag{44}$$

for the contributions with $n=1,2,3$ in Eq. (34), respectively. Let us emphasize once again that we have thrown away the linearly infrared divergent term $1/\gamma$ in $\Delta E_{\Xi}^{(3)}$, which corresponds to the contribution of the previous order, but we have preserved all logarithmically divergent terms that should cancel automatically in the final result for the energy shift.

The total contribution to the energy shift generated by the spanning photon insertion in Fig. 1 is equal to

$$\delta E_{\Xi}=\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m\left[9\ln\frac{1}{\gamma}-\frac{9}{2}\right]. \tag{45}$$

V. SUMMARY

Collecting all contributions to the energy shift in Eqs. (14), (30), and (45) we see that all logarithmically infrared divergent contributions cancel in the sum, and obtain the total radiative-recoil correction of order $\alpha(Z\alpha)^5(m/M)m$,

$$\Delta E=\left(6\zeta(3)-2\pi^2\ln 2+\frac{3\pi^2}{4}-14\right)\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m \tag{46}$$

$$\approx -13.067\,632\,2\ldots\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 m.$$

This result is in excellent agreement with the numerical result in [3], and this resolves the long-standing discrepancy on the magnitude of the radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)m$ to the Lamb shift. When this paper was in preparation we learned that the same result was just obtained in the framework of nonrelativistic quantum dynamics [9].

Numerically, the correction in Eq. (46) contributes

$$\Delta E(1S)=-13.43\ \ \mathrm{kHz} \tag{47}$$

to the $1S$ Lamb shift in the ground state of hydrogen. The discrepancy between the theoretical predictions for the $1S$ Lamb shift calculated according to [4–6] and [3] is about 6 kHz. This discrepancy is not too important for the $1S$ Lamb shift measurements, since the error bars of even the best current experimental results are still a few times larger (see, e.g., review in [1]). What is much more important from the phenomenological point of view, is that the radiative-recoil correction is linear in the electron-nucleus mass ratio, and it directly contributes to the hydrogen-deuterium isotope shift. The discrepancy between the theoretical values of the isotope shift calculated according to [4–6] and [3] is about 18 times larger than the experimental uncertainty 0.15 kHz of the isotope shift [10]. Thus the analytic result in Eq. (46) eliminates the largest source of the theoretical uncertainty in the magnitude of the deuterium-hydrogen isotope shift. One can use this radiative-recoil correction, and the latest experimental data in order to obtain a value for the difference of charge radii squared of the deuteron and proton, but we will not enter in the detailed discussion of the phenomenological implications here, since they were exhaustively discussed in our recent review [1].

ACKNOWLEDGMENTS

We are grateful to A. Czarnecki and K. Melnikov for communicating us their results prior to publication. This work was supported by NSF Grant No. PHY-0049059. The work of V.A.S. was also supported in part by RFBR Grant No. 00-02-16718.

[1] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rep. **342**, 63 (2001).
[2] M. I. Eides and H. Grotch, Phys. Rev. A **52**, 1757 (1995).
[3] K. Pachucki, Phys. Rev. A **52**, 1079 (1995).
[4] G. Bhatt and H. Grotch, Phys. Rev. A **31**, 2794 (1985).
[5] G. Bhatt and H. Grotch, Phys. Rev. Lett. **58**, 471 (1987).
[6] G. Bhatt and H. Grotch, Ann. Phys. (N.Y.) **178**, 1 (1987).
[7] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Ann. Phys. (N.Y.) **205**, 231 (1991).
[8] M. I. Eides and V. A. Shelyuto, Pis'ma Zh. Éksp. Teor. Fiz. **61**, 465 (1995) [JETP Lett. **61**, 478 (1995)]; Phys. Rev. A **52**, 954 (1995).
[9] A. Czarnecki and K. Melnikov, e-print hep-ph/0012053.
[10] A. Huber *et al.*, Phys. Rev. Lett. **80**, 468 (1998).