Optimal encoding and decoding of a spin direction

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(Received 12 December 2000; published 17 April 2001)

For a system of N spins there are quantum states that can encode a direction in an intrinsic way. Information on this direction can later be decoded by means of a quantum measurement. We present here the optimal encoding and decoding procedure using the fidelity as a figure of merit. We compute the maximal fidelity and prove that it is directly related to the largest zeros of the Legendre and Jacobi polynomials. We show that this maximal fidelity approaches unity quadratically in $1/N$. We also discuss this result in terms of the dimension of the encoding Hilbert space.

DOI: 10.1103/PhysRevA.63.052309 PACS number(s): 03.67.−a, 03.65.Ta

I. INTRODUCTION

Entanglement and superposition are the most characteristic features of quantum states. They play a central role in the storage and transmission of information in the quantum world and are responsible for the many remarkable, and often intriguing, quantum effects that are constantly being discovered. These effects, in turn provide new insights in the difficult task of understanding quantum information.

Some time ago Peres and Wootters [1] posed an interesting question. Imagine a quantum system composed of several subsystems, which are not necessarily entangled. How can we learn more about this system? By performing measurements on the individual subsystems or on the system as a whole? They showed evidence that the latter, the so-called collective measurements, are more informative. Obviously entanglement is the property responsible for this. In this case, however, it is not explicit, since the system can be chosen to be in a product state, but hidden in the collective measurement.

Later Massar and Popescu [2] addressed a more concrete problem. Imagine Alice has a system of N parallel spins. She can use this system to tell Bob the direction along which some given unit vector $\vec{n}$ is pointing. She just has to rotate, or prepare in some other way, the state of her system so that it becomes an eigenstate of $\vec{n}\cdot\vec{S}$, the projection of the total spin in the $\vec{n}$ direction. The state is then sent to Bob, whose task is to determine the direction encoded in the state. He will need to perform a collective measurement and from each one of its outcomes, labeled with an index r, he will have a guess for Alice's direction given by a unit vector $\vec{n}_r$. To quantify the quality of this communication procedure Massar and Popescu used the average fidelity, which is defined by $F=\Sigma_r\int dn(1+\vec{n}\cdot\vec{n}_r)/2P_r(\vec{n})$, where $\vec{n}$ is assumed to come from an isotropic source. Here $P_r(\vec{n})$ is the probability of getting the outcome r if Alice's direction is $\vec{n}$, and dn is the rotationally invariant measure on the unit two-sphere. The authors proved that the maximal average fidelity Bob can achieve is $F=(N+1)/(N+2)$, which is readily seen to approach unity linearly: $F\sim 1-1/N$. Explicit realizations of the optimal measurements with a finite number of outcomes were obtained in [3] for arbitrary N and minimal versions of these measurements for N up to seven are in [4].

A surprise was recently presented in [5]. In this paper the authors consider $N=2$ and show that states with two antiparallel spins $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ provide a better encoding of Alice's directions than the two parallel-spin states used in [2–4]. The average fidelity is now $(3+\sqrt{3})/6$ which is larger than 3/4 for two parallel spins, i.e., Bob can have a better determination of Alice's direction if she uses antiparallel spins. This is a startling result, since classically one would expect that a direction is encoded equally as well in a state pointing one way as in one pointing the opposite way. The main reason why this is not so in the quantum world, as will become clear from our work, is the different dimensionality of the Hilbert spaces to which two parallel or two antiparallel spin states belong.

At this point, the obvious reaction is to ask ourselves what are the best states Alice can use to encode directions. Since the very natural state with only parallel spins is not optimal for $N=2$, we expect that neither will it be for arbitrary N. Hence, one has to search for the optimal encoding state among all the eigenstates of $\vec{n}\cdot\vec{S}$ that Alice can produce. These eigenstates have the obvious, and very useful, property of pointing along the direction given by $\vec{n}$ in an intrinsic way, namely, independently of any reference frame Alice and Bob may share. In short, they are the quantum analog of the gyroscope. One could use a much more general class of states to encode the information contained in $\vec{n}$ (see [6] and the last section of the present paper). However, all other possible encodings of $\vec{n}$ will necessarily require that Alice and Bob share a common reference frame. Hence, the whole procedure becomes less interesting, since one can argue that in this situation classical communication is more efficient.

In this paper we will present a very general analysis of these "quantum gyroscopes." We compute the maximal average fidelity (hereafter we will usually drop "average" when there is no ambiguity) for arbitrary N and show that it approaches unity quadratically in $1/N$, as compared to the linear behavior found in [2] for parallel spins. As a by-product, we also compute the maximal fidelity for encoding states of two arbitrary spins s such as two nuclei. A short description of the main results of this analysis was presented in [6]. These results have recently been corroborated by numerical analysis [7].

The paper is organized as follows. In Sec. II we introduce our notation and conventions and present a detailed calculation of the maximal fidelity for $N=2$. We show that the fidelity obtained by Gisin and Popescu in [5] is optimal (a result also obtained in [8] using different methods). In Sec. III we analyze the more general case of two states with equal but arbitrary spin s. The analysis for any number of spins is in Sec. IV and our results and discussion are in Sec. V. We conclude with an Appendix containing technical details.

II. TWO SPINS

We start by assuming that Alice has two spins in a general eigenstate of $\vec{n}\cdot\vec{S}$. (We skip the analysis of the simplest situation in which Alice has only one spin. The reader can find it in [2,6], and our general formulas of Sec. IV can also be specialized to this case.) We can think of it as a fixed eigenstate of $S_z=\vec{z}\cdot\vec{S}$ ($\vec{z}$ is the unit vector pointing along the z direction) that Alice has rotated into the direction $\vec{n}=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)$. It is convenient to work in the irreducible representations of SU(2). In the present case, $1/2\otimes 1/2=1\oplus 0$, the general form of this fixed eigenstate is

$$|A\rangle=A_+|1,1\rangle+A_0|1,0\rangle+A_-|1,-1\rangle+A_s|0,0\rangle, \qquad (1)$$

where, as usual, the normalized states of the basis, $|j,m\rangle$, are labeled by the total spin S and the third component S_z: $S^2|j,m\rangle=j(j+1)|j,m\rangle$ and $S_z|j,m\rangle=m|j,m\rangle$. In the following we stick to the general form (1) to treat all the cases jointly, but one should keep in mind that only combinations with definite S_z will be relevant for our analysis. The rotated state $U(\vec{n})|A\rangle$, where $U(\vec{n})$ is the element of the SU(2) group associated with the rotation $\vec{z}\to\vec{n}=R\vec{z}$, is precisely Alice's general eigenstate of $\vec{n}\cdot\vec{S}$. Obviously, $U(\vec{n})$ is reducible since it has the form $U(\vec{n})=U^{(1)}(\vec{n})\oplus U^{(0)}(\vec{n})$, where $U^{(j)}$ denotes the SU(2) irreducible representation of spin j.

Next, Alice sends the rotated state to Bob, who tries to determine $\vec{n}$ from his measurements. The most general one he can perform is a positive-operator-valued measurement (POVM). We specify this POVM by giving a set of positive Hermitian operators $\{O_r\}$, that are a resolution of the identity

$$\mathbb{I}=\sum_r O_r. \qquad (2)$$

For each outcome r, Bob makes a guess $\vec{n}_r$ for the direction. As we mentioned in the Introduction, the quality of the guess is quantified in terms of the fidelity, which we can view as a "score." To Bob's guess $\vec{n}_r$, we give the score $f=(1+\vec{n}\cdot\vec{n}_r)/2$. We see that the fidelity f is unity if Bob's guess coincides with Alice's direction and it is zero when they are opposite. Thus, if $\vec{n}$ is isotropically distributed the average fidelity can be written as

$$F=\sum_r \int dn\,\frac{1+\vec{n}\cdot\vec{n}_r}{2}\,\mathrm{tr}[\rho(\vec{n})O_r], \qquad (3)$$

where $\rho(\vec{n})=U(\vec{n})|A\rangle\langle A|U^\dagger(\vec{n})$ and dn was defined in the Introduction. The evaluation of F can be greatly simplified by exploiting the rotational invariance of the integral (3). If we define R_r through the relation

$$\vec{n}_r=R_r\vec{z} \qquad (4)$$

and make the change of variables

$$R_r^{-1}\vec{n}\to\vec{n}, \qquad (5)$$

we have

$$F=\sum_r \int dn\,\frac{1+\vec{n}\cdot\vec{z}}{2}\,\mathrm{tr}[\rho(\vec{n})\Omega_r], \qquad (6)$$

where

$$\Omega_r=U^\dagger(\vec{n}_r)O_rU(\vec{n}_r). \qquad (7)$$

Notice that in general $\Sigma_r\Omega_r\neq\mathbb{I}$. We can regard Ω_r as fixed or reference projectors associated with the single direction $\vec{z}$. In this sense, they are the counterpart of Alice's fixed state $|A\rangle$. Inserting four times the closure relation $\Sigma_k|k\rangle\langle k|=\mathbb{I}$, where $k=+,0,-,s$, and $\{|k\rangle\}$ is the basis of the representations $1\oplus 0$,

$$\begin{aligned}|\pm\rangle&=|1,\pm 1\rangle,\\ |0\rangle&=|1,0\rangle,\\ |s\rangle&=|0,0\rangle,\end{aligned} \qquad (8)$$

we obtain

$$F=\sum_{kijl} A_i^*A_l\omega_{kj}\int dn\,\frac{1+\cos\theta}{2}\,\mathfrak{D}_{ki}^*(\vec{n})\mathfrak{D}_{jl}(\vec{n}). \qquad (9)$$

Here the indices k, i, j, and l also run over $+,0,-,s$; $\mathfrak{D}_{kj}(\vec{n})=[\mathfrak{D}^{(1)}\oplus\mathfrak{D}^{(0)}]_{kj}(\vec{n})=\langle k|U(\vec{n})|j\rangle$ are the SU(2) rotation matrices in the $1\oplus 0$ representations, and

$$\omega_{kj}=\sum_r \langle k|\Omega_r|j\rangle. \qquad (10)$$

Now, one can easily evaluate the integrals and obtain the fidelity

$$F=\mathsf{A}^\dagger\mathsf{W}\mathsf{A}, \qquad (11)$$

where $\mathsf{A}=(A_+,A_0,A_-,A_s)^t$ and $\mathsf{A}^\dagger$ is its transposed complex conjugate. The matrix W is

$$\mathsf{W}=\begin{pmatrix} \frac{3\omega_{++}+2\omega_{00}+\omega_{--}}{12} & * & * & * \\ * & \frac{\omega_{++}+\omega_{00}+\omega_{--}}{6} & * & \frac{\omega_{0s}}{6} \\ * & * & \frac{\omega_{++}+2\omega_{00}+3\omega_{--}}{12} & * \\ * & \frac{\omega_{s0}}{6} & * & \frac{\omega_{ss}}{2} \end{pmatrix}, \tag{12}$$

where the entries marked with $*$ are not relevant for our analysis since we consider eigenstates of S_z only for the fixed states $|A\rangle$. These, and the corresponding rotated states $U(\vec{n})|A\rangle$, are the only ones that point along a definite direction in an absolute sense, i.e., even if Alice and Bob do not share a common reference frame. From its definition (10), it follows that ω_{jj} are real non-negative numbers but ω_{ij} are in general complex numbers for $i\neq j$. There are other constraints on ω_{ij} stemming from the condition $\Sigma_r O_r=\mathbb{1}$:

$$\omega_{ss}=1, \quad \sum_{l=+,0,-}\omega_{ll}=3. \tag{13}$$

Because of the Schwarz inequality, we also have

$$|\omega_{0s}|^2\leqslant\omega_{00}\omega_{ss}=\omega_{00}. \tag{14}$$

Let us discuss the implications of these equations for different values of m.

The $m=\pm1$ case

The fixed state $|A\rangle$ for $m=1$ is simply $|A\rangle=|1,1\rangle$, i.e., $A_+=1$ and $A_0=A_-=A_s=0$. In this case the fidelity is given by the element W_{++} of Eq. (12),

$$F=\mathsf{W}_{++}=\frac{3\omega_{++}+2\omega_{00}+\omega_{--}}{12}=\frac{3}{4}-\frac{\omega_{00}+2\omega_{--}}{12}\leqslant\frac{3}{4}, \tag{15}$$

where the second condition in Eq. (13) has been used. The maximal value, which we denote by F_+, is then

$$F_+=\tfrac{3}{4}. \tag{16}$$

This value occurs for

$$\omega_{--}=\omega_{00}=0\Rightarrow\omega_{++}=3. \tag{17}$$

The case $m=-1$, for which $|A\rangle=|1,-1\rangle$, is completely analogous with the index substitution $+\leftrightarrow-$. The maximal value of the fidelity is also $F_-=\frac{3}{4}$.

The $m=0$ case

For $m=0$ one has $|A\rangle=A_0|1,0\rangle+A_s|0,0\rangle$, with $|A_0|^2+|A_s|^2=1$. The maximal fidelity is the largest eigenvalue of the 2×2 submatrix of Eq. (12) corresponding to the $m=0$ subspace:

$$F=\frac{3+|\omega_{0s}|}{6}\leqslant\frac{3+\sqrt{\omega_{00}}}{6}. \tag{18}$$

It reaches its maximal value F_0 for

$$\omega_{00}=3\Rightarrow\omega_{++}=\omega_{--}=0. \tag{19}$$

Substituting back into Eq. (18) we obtain [5]

$$F_0=\frac{3+\sqrt{3}}{6}. \tag{20}$$

The corresponding eigenvector is

$$|A\rangle=\frac{1}{\sqrt{2}}|1,0\rangle+\frac{e^{i\delta}}{\sqrt{2}}|0,0\rangle, \tag{21}$$

where the phase is the unconstrained parameter $\delta=\arg\omega_{s0}$. Notice that the family of states (21) contains entangled as well as unentangled states. With the choice $e^{i\delta}=\pm1$ one obtains the product states $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$; precisely those considered by Gisin and Popescu [5], which led them to the conclusion that antiparallel spins are better than parallel spins for encoding a direction.

From this analysis one can also obtain important information about the optimal POVM. Taking into account that one can always take the projectors O_r to be one dimensional [9], we can write Bob's reference projectors Ω_r as

$$\Omega_r=c_r|\Psi_r\rangle\langle\Psi_r|, \tag{22}$$

where $|\Psi_r\rangle$ are normalized states and c_r are positive numbers. The values of ω_{ij} [see Eq. (10)] endow the information about the components of $|\Psi_r\rangle$ in the spherical basis (8). To be specific, consider states with $m=0$. The maximal-fidelity condition (19) implies that the states $|\Psi_r\rangle$ must also have $m=0$; hence $|\Psi_r\rangle=\alpha_r|1,0\rangle+\beta_r|0,0\rangle$. This result is, to some extent, what one expects: in order for a POVM to be optimal, the measurement must project on states as similar as possible to the signal state. Further, the Schwarz inequality

(14) becomes an equality if and only if $\alpha_r/\beta_r=\lambda$ for all r. If this is the case, the fidelity can reach the maximal value F_0. Then, imposing the POVM conditions (13), it is straightforward to verify that all $|\Psi_r\rangle$ must coincide with a single state, which we denote by $|B\rangle$,

$$|\Psi_r\rangle=|B\rangle=\frac{\sqrt{3}}{2}|1,0\rangle+\frac{e^{i\delta}}{2}|0,0\rangle. \tag{23}$$

The relative weight of the $|1,0\rangle$ and $|0,0\rangle$ components, $\sqrt{3}:1$, is easily understood as being the square root of the dimension of the Hilbert spaces corresponding to $j=1$ and 0. We therefore see that optimal POVMs can be obtained by rotating the single reference state $|B\rangle$. The weights c_r are free parameters except for the constraint

$$\sum_r c_r=4. \tag{24}$$

Because the Hilbert space has dimension 4, a POVM (optimal or not) must consist of at least four projectors. Let us show that indeed an optimal POVM with this minimal number of projectors exists. Since the number of projectors in the POVM equals the dimension of the Hilbert space, we are actually dealing with a von Neumann measurement, i.e.,

$$O_rO_s=O_r\delta_{rs}. \tag{25}$$

Hence, $\langle\Psi_r|\Psi_r\rangle=1\Rightarrow c_r=1$ for the four values of r, which is, of course, consistent with Eq. (24). Inverting Eq. (7) and taking into account Eq. (22), we see that the four unit vectors $\vec{n}_r$ have to be chosen so that

$$\sum_{r=1}^{4} O_r=\sum_{r=1}^{4} U(\vec{n}_r)|B\rangle\langle B|U^\dagger(\vec{n}_r)=\mathbb{I}. \tag{26}$$

By symmetry, they should correspond to the vertices of a tetrahedron inscribed in a unit sphere, i.e., $\vec{n}_r=(\cos\phi_r\cdot\sin\theta_r,\sin\phi_r\sin\theta_r,\cos\theta_r)$ with

$$\begin{gathered}\cos\theta_1=1,\quad \phi_1=0,\\ \cos\theta_r=-\tfrac{1}{3},\quad \phi_r=(r-2)\frac{2\pi}{3},\quad r=2,3,4.\end{gathered} \tag{27}$$

It is easy to verify that with this choice condition (26) is fulfilled and the maximal fidelity (20) is attained. One can check that the four projectors (26) are equal to those already considered by Gisin and Popescu in [5]. Our aim here was just to explain their choice of POVM. Finite optimal POVMs for $N>2$ are less straightforward to obtain. However, the results of [3,4], which enable us to construct finite POVMs for code states with maximal m, $|N/2,N/2\rangle=|\overset{N}{\uparrow\uparrow\cdots\uparrow}\rangle$, can also be used here for other values of m. We will comment on this issue in our last section.

After dwelling on minimal POVMs, it is convenient to consider also the other end of the spectrum: POVMs with infinitely many outcomes or continuous POVMs [10]. They will be used in the general analysis in the sections below, where they will prove very efficient. Recall that for any finite measurement on isotropic distributions it is always possible to find a continuous POVM that gives the same fidelity [3]. Therefore, restricting ourselves to this type of measurement does not imply any loss of generality. We illustrate this point for $N=2$ and $m=0$ to introduce the notation that will be used in the following sections.

We have seen that the matrix elements ω_{ij} contain all the information required for computing the fidelity, independently of any particular choice of POVM. Any measurement for which ω_{ij} satisfy the condition (17) for $m=1$ or (19) for $m=0$ is surely optimal. A continuous POVM is just a particularly simple and useful realization. It amounts to taking the index r to be continuous, i.e.,

$$\sum_r \to\int dn_B, \tag{28}$$

where the subindex B in the invariant measure refers to Bob (measuring device). Substituting Eq. (22) into Eq. (10) one obtains in the continuous version

$$\omega_{kj}=\int dn_B c(\vec{n}_B)\langle k|B\rangle\langle B|j\rangle, \tag{29}$$

where $|B\rangle$ is the normalized state (23) and $c(\vec{n}_B)$ is a continuous positive weight, which plays the role of c_r and according to Eq. (24) must satisfy

$$\int dn_B c(\vec{n}_B)=4. \tag{30}$$

We now show that in fact $c(\vec{n}_B)$ is a constant and, hence, equal to 4. Condition (26) reads

$$\int dn_B c(\vec{n}_B)U(\vec{n}_B)|B\rangle\langle B|U^\dagger(\vec{n}_B)=\mathbb{I}, \tag{31}$$

which is equivalent to

$$\frac{2j+1}{4}\int dn_B c(\vec{n}_B)\mathfrak{D}^{(j)}_{m0}(\vec{n}_B)\mathfrak{D}^{(j')*}_{m'0}(\vec{n}_B)=\delta_{jj'}\delta_{mm'},$$
$$j,j'=0,1. \tag{32}$$

Using the well-known orthogonality relation of the matrix representations of SU(2) [11],

$$\int dn\,\mathfrak{D}^{(j)}_{m_1m_2}(\vec{n})\mathfrak{D}^{(j')*}_{m_1'm_2}(\vec{n})=\frac{1}{2j+1}\delta_{jj'}\delta_{m_1m_1'}, \tag{33}$$

one obtains

$$c(\vec{n}_B)\equiv c=4, \tag{34}$$

which is just the total dimension $(3+1)$ of the Hilbert space to which the state (23) belongs. Therefore, the projectors $O(\vec{n}_B)=cU(\vec{n}_B)|B\rangle\langle B|U^\dagger(\vec{n}_B)$ in Eq. (31) describe an optimal continuous POVM. They are obtained from the fixed state (23) in a manner analogous to the construction of the minimal POVM in Eqs. (26) and (27), excepting the constant factor c required by the normalization of the matrix representations of SU(2).

To complete the analysis of $N=2$, we calculate the maximal fidelity for a given (nonoptimal) fixed state $|A\rangle$ with $m=0$. Without any loss of generality it can be written as

$$|A\rangle=|A_0||1,0\rangle+|A_s|e^{i\delta}|0,0\rangle, \quad |A_0|^2+|A_s|^2=1, \tag{35}$$

where we have used the same phase convention as in Eq. (21). From Eq. (12), and the constraints (13) and (14), it is straightforward to see that the maximal value of the fidelity is

$$F_A=\frac{1}{2}+\frac{|A_0||A_s|}{\sqrt{3}}. \tag{36}$$

To attain this value, Bob must perform an optimal POVM, characterized by Eq. (23). He may use, for instance, the minimal one [Eqs. (26)–(27)], or the continuous one $O(\vec{n}_B)$. From Eq. (36) it follows that for any fixed state (35) with $\frac{1}{2}<|A_0|<\sqrt{3}/2$ the fidelity is higher than that of the parallel case (i.e., $m=\pm 1$) for which $F=F_{\pm}=\frac{3}{4}$.

III. TWO ARBITRARY SPINS

Imagine now that Alice can use two equal but arbitrary spins $s_1=s_2=s$ to encode the directions. This can be seen as a generalization of the simple case studied in the preceding section. However, the most important feature of this analysis, as will be shown in Sec. IV, is that it provides the solution of our original problem, namely, that of obtaining the maximal fidelity when Alice has N spins at her disposal.

According to the Clebsch-Gordan decomposition, a normalized eigenvector of the total spin in the z direction with eigenvalue m_A can be written as

$$|A\rangle=\sum_{j=m_A}^{J} A_j|j,m_A\rangle, \quad \sum_{j=m_A}^{J}|A_j|^2=1, \tag{37}$$

where $J=2s$. The state $|A\rangle$ and its components A_j should carry the label m_A to denote the different eigenvalues of S_z; however, we will drop it to simplify the notation. A general eigenstate of $\vec{n}\cdot\vec{S}$ has the form $U(\vec{n})|A\rangle$, where $U(\vec{n})$ is now

$$U(\vec{n})=\bigoplus_{j=m_A}^{J} U^{(j)}(\vec{n}). \tag{38}$$

The POVM projectors can be constructed from a fixed state $|B\rangle$ of the form

$$|B\rangle=\sum_{j=m_B}^{J} B_j|j,m_B\rangle, \tag{39}$$

namely, $O(\vec{n}_B)=cU(\vec{n}_B)|B\rangle\langle B|U^{\dagger}(\vec{n}_B)$. Note that $|B\rangle$ is an eigenvector of S_z with eigenvalue m_B, although we also drop the label m_B here. The absolute value of the coefficients B_j and the positive weight c are determined by the completeness relation $\int dn_B\, O(\vec{n}_B)=\mathbb{I}$, which using Eq. (33) leads to the normalization condition

$$|B_j|=\sqrt{(2j+1)/c}, \tag{40}$$

and a value for c given by

$$c=(J+1)^2-m_B^2. \tag{41}$$

Notice that the factor $2j+1$ in Eq. (40) is just the dimension of the Hilbert space of the irreducible representation j of SU(2), and c is the dimension of the total Hilbert space. Thus, Eq. (39) is the straight generalization of the states (23). The fidelity can be written as

$$F=c\sum_{j,j'=m}^{J} A_j A_{j'}^{*} B_j^{*} B_{j'}\int dn\,\frac{1+\cos\theta}{2}\mathfrak{D}^{(j)}_{m_B m_A}(\vec{n}) \times\mathfrak{D}^{(j')*}_{m_B m_A}(\vec{n}), \tag{42}$$

where

$$m=\max(m_A,m_B). \tag{43}$$

The integral in Eq. (42) can be easily computed by noticing that $\cos\theta=\mathfrak{D}^{(1)}_{00}(\vec{n})$. Using again the orthogonality relations (33) we have

$$\int dn\cos\theta\,\mathfrak{D}^{(j)}_{m_1 m_2}(\vec{n})\mathfrak{D}^{(j')*}_{m_1' m_2}(\vec{n}) =\frac{1}{2j'+1}\langle 10;jm_1|j'm_1'\rangle\langle 10;jm_2|j'm_2\rangle, \tag{44}$$

where $\langle j_1m_1;j_2m_2|j_3m_3\rangle$ are the Clebsch-Gordan coefficients of $j_1\otimes j_2\to j_3$. The fidelity can be recast as

$$F=\frac{1}{2}+\frac{1}{2}\sum_{j=m}^{J}\mu_j|A_j|^2+\frac{1}{2}\sum_{j=m+1}^{J}(A_{j-1}^{*}A_j\nu_j^{*}+A_{j-1}A_j^{*}\nu_j) -\frac{1}{2}\sum_{j=m_A}^{m-1}|A_j|^2, \tag{45}$$

where the last term is zero for $m_A<m_B$ and the coefficients μ_j and ν_j are

$$\mu_j=\frac{m_A m_B}{j(j+1)}, \tag{46}$$

$$\nu_j=\frac{e^{i\delta_j}}{j}\left(\frac{(j^2-m_A^2)(j^2-m_B^2)}{4j^2-1}\right)^{1/2}. \tag{47}$$

The phases δ_j in Eq. (47) are arbitrary. Th ey are just the generalization of the single free phase of Eq. (23). Here we have $\delta_j=\arg(B_j^{*}B_{j-1})$. The maximal fidelity is achieved by choosing δ_j equal to the phases of the signal state $|A\rangle$:

$$\delta_j=\arg(B_j^{*}B_{j-1})=\arg(A_j^{*}A_{j-1}). \tag{48}$$

We see now that all terms in Eq. (45) are explicitly positive with the exception of the last one, which necessarily vanishes for optimal states $|A\rangle$, i.e., $A_j=0$ for $j<m$. Gathering all these results, we obtain for the fidelity

$$F=\tfrac{1}{2}+\tfrac{1}{2}\mathsf{A}^t\mathsf{M}\mathsf{A}. \tag{49}$$

Here $\mathsf{A}^t=(|A_J|,|A_{J-1}|,|A_{J-2}|,\dots)$ is the transpose of A, and M is a real matrix of tridiagonal form,

$$\mathsf{M}=\begin{pmatrix} d_l & c_{l-1} & & & & \\ c_{l-1} & \ddots & \ddots & & \underline{0} & \\ & \ddots & d_3 & c_2 & & \\ \underline{0} & & c_2 & d_2 & c_1 \\ & & & c_1 & d_1 \end{pmatrix}, \tag{50}$$

with

$$l=J+1-m, \tag{51}$$

and

$$\begin{aligned} d_k&=\mu_{k+m-1},\\ c_k&=|\nu_{k+m}|. \end{aligned} \tag{52}$$

The largest eigenvalue x_l of M determines the maximal fidelity through the relation

$$F=\frac{1+x_l}{2}. \tag{53}$$

To find x_l, we set up a recursion relation for the characteristic polynomial of M:

$$Q_{l+1}(x)=(d_{l+1}-x)Q_l(x)-c_l^2Q_{l-1}(x), \tag{54}$$

with the initial values $Q_{-1}(x)=0$ and $Q_0(x)=1$. Equation (54) resembles the recursion relation of orthogonal polynomials, but at first sight the solution does not seem straightforward at all. We thus work out in detail the simplest case for which $m_A=m_B=0$. For this particular instance Eq. (54) reads

$$Q_{l+1}(x)=-xQ_l(x)-\frac{l^2}{4l^2-1}Q_{l-1}(x), \tag{55}$$

where we have used the definitions (46), (47), and (52). We can rewrite Eq. (55) as

$$\begin{aligned}(l+1)&\left[-\frac{(2l+1)(2l-1)}{(l+1)}Q_{l+1}(x)\right]\\ &=(2l+1)x\left[\frac{2l-1}{l}Q_l(x)\right]-l[-Q_{l-1}(x)].\end{aligned} \tag{56}$$

It is now apparent that the terms inside the square brackets can be absorbed into a redefinition of the characteristic polynomial through an x-independent change of normalization, namely,

$$Q_l(x)\equiv(-1)^l\frac{l!}{(2l-1)!!}P_l(x)=(-1)^l\frac{2^l(l!)^2}{(2l)!}P_l(x). \tag{57}$$

This leads us to the recursion relation of the Legendre polynomials:

$$(l+1)P_{l+1}(x)=(2l+1)xP_l(x)-lP_{l-1}(x). \tag{58}$$

Working along the same lines, it is easy to convince oneself that the general solution of Eq. (54) is, up to a normalization factor, the Jacobi polynomial $P_l^{a,b}(x)$ [12]:

$$Q_l(x)=(-1)^l\frac{2^l l!(l+2m)!}{(2l+2m)!}P_l^{a,b}(x), \tag{59}$$

where

$$a=|m_B-m_A|,\quad b=m_B+m_A, \tag{60}$$

and m is defined in Eq. (43). Note that m can be written simply as $m=(a+b)/2$. Note also that $P_l^{0,0}$ is the Legendre polynomial P_l.

From the result (A12) in the Appendix it turns out that the maximal value of the fidelity (53) is attained for $m_A=m_B=0$, i.e., precisely the particular case of Legendre polynomials discussed above. Thus, from Eq. (53) we have

$$F_{\max}=\frac{1+x_{J+1}^{0,0}}{2}, \tag{61}$$

where $x_n^{a,b}$ stands for the largest zero of $P_n^{a,b}(x)$. The fact that $m_A=m_B=0$ implies that maximal fidelity can be translated into physical terms by saying that Alice's states and Bob's projectors must *effectively* span the largest possible Hilbert space. For a fixed choice of m_A, not necessarily optimal, the best m_B is that for which the Hilbert spaces spanned by $U(\vec{n})|A\rangle$ and $U(\vec{n}_B)|B\rangle$ coincide, i.e., $m_A=m_B=m$. In this case, the maximal value of the fidelity is given by Eq. (53), with $x_l=x_{J+1-m}^{0,2m}$, i.e., $F=(1+x_{J+1-m}^{0,2m})/2<F_{\max}$. One reaches the same conclusion if m_B is fixed and m_A can be adjusted for optimal results [see the discussion in the Appendix after Eq. (A12)].

IV. GENERAL CASE: *N* SPINS

We now show that the solution we obtained in the preceding section is in fact of general validity. Recall that in our original problem Alice has N spins. Let us suppose that N is even (odd N will be considered below). As usual, Alice constructs her states by rotating a fixed eigenstate of S_z. In terms of the irreducible representations of SU(2), such states can be written as

$$|A\rangle=\sum_{j=m_A}^{N/2}\left(\sum_{\alpha} A_j^{\alpha}|j,m_A;\alpha\rangle\right),\quad \sum_{j=m_A}^{N/2}\sum_{\alpha}|A_j^{\alpha}(m)|^2=1. \tag{62}$$

The main difference from the previous example of two equal spins s is that for $j<N/2$ the irreducible representations $U^{(j)}$ appear more than once in the Clebsch-Gordan decomposition of $(1/2)^{\otimes N}$. Hence, we label the different occurrences with the index α, which we can view as a new quantum number required to break the degeneracy of Alice's system of spins under global rotations. Similarly, the expression for Bob's fixed state $|B\rangle$ is

$$|B\rangle=\sum_{j=m_B}^{N/2}\left(\sum_{\beta} B_j^{\beta}|j,m_B,\beta\rangle\right). \tag{63}$$

However, it is known that equivalent matrix representations

$$\mathfrak{D}_{mm'}^{(j,\alpha)}(\vec{n})=\langle j,m;\alpha|U(\vec{n})|j,m';\alpha\rangle \tag{64}$$

are not orthogonal under the group integration, i.e., for $\alpha\neq\beta$ one has in general

$$\int dn\,\mathfrak{D}_{mm'}^{(j,\alpha)}(\vec{n})\mathfrak{D}_{mm'}^{(j,\beta)*}(\vec{n})\neq 0, \tag{65}$$

and the completeness relation $\int dn_B\,O(\vec{n}_B)=\mathbb{I}$ does *not* hold for the simple choice of projectors $O(\vec{n}_B)=c\,U(\vec{n}_B)|B\rangle\langle B|U^{\dagger}(\vec{n}_B)$. We can circumvent this difficulty by introducing several copies of $|B\rangle\langle B|$. A single direction (unit vector) $\vec{n}_B$ is thus associated with

$$O(\vec{n}_B)=U(\vec{n}_B)[|B\rangle\langle B|+|B'\rangle\langle B'|+|B''\rangle\langle B''|+\cdots]U^{\dagger}(\vec{n}_B). \tag{66}$$

The fixed projectors in the square brackets will be judiciously chosen to eliminate the off-diagonal terms coming from the mixing of equivalent representations in the closure relation. The projectors $O(\vec{n}_B)$ are explicitly of rank higher than 1. However, recalling [9], we can view the right-hand side of Eq. (66) as defining a sum of rank-1 projectors $O(\vec{n}_B)+O'(\vec{n}_B)+O''(\vec{n}_B)+\cdots$. The two points of view are equivalent if the averaged fidelity is used as a figure of merit. In a suggestive compact notation we can write

$$|B\rangle\langle B|+|B'\rangle\langle B'|+|B''\rangle\langle B''|+\cdots\equiv|\mathbf{B}\rangle\cdot\langle\mathbf{B}|, \tag{67}$$

where

$$|\mathbf{B}\rangle\equiv\sum_{j=m_B}^{N/2}\left(\sum_{\beta}\mathbf{B}_j^{\beta}|j,m_B,\beta\rangle\right), \tag{68}$$

and

$$\mathbf{B}_j^{\beta}\equiv(B_j^{\beta},B_j'^{\beta},B_j''^{\beta},\ldots). \tag{69}$$

Next, we introduce a set of orthonormal vectors $\{\mathbf{b}_j^{\alpha}\}$,

$$\mathbf{b}_j^{\alpha}\cdot\mathbf{b}_j^{\beta}=\delta^{\alpha\beta}, \tag{70}$$

and define the vectors $\mathbf{B}_j^{\alpha}$ as

$$\mathbf{B}_j^{\alpha}=\sqrt{2j+1}\,\mathbf{b}_j^{\alpha}. \tag{71}$$

Note that for convenience we henceforth use a different normalization of the states $|B\rangle,|B'\rangle,\ldots$ [see Eq. (40)]. With the above definitions one can easily see that $\int dn_B\,O(\vec{n}_B)=\mathbb{I}$ and, hence, the set of projectors (66) defines a POVM.

The fidelity can be read off from Eq. (45) and is given by

$$F=\frac{1}{2}+\frac{1}{2}\sum_{j=m}^{N/2}\sum_{\alpha}\mu_j(A_j^{\alpha})^2+\sum_{j=m+1}^{N/2}\sum_{\alpha\beta}A_{j-1}^{\alpha}(\mathbf{b}_{j-1}^{\alpha}\cdot\mathbf{b}_j^{\beta})A_j^{\beta}\nu_j-\frac{1}{2}\sum_{j=m_A}^{m-1}\sum_{\alpha}(A_j^{\alpha})^2, \tag{72}$$

where the phases have been chosen so that ν_j, A_j^{α}, and B_j^{α} are real. In general $\mathbf{b}_j^{\alpha}\in\mathbb{R}^k$, where k must be greater than or equal to the highest degeneracy of the irreducible representations in the Clebsch-Gordan series of $(1/2)^{\otimes N}$, since otherwise Eq. (70) could not be satisfied. Equation (72) suggests the definition

$$\mathbf{A}_j=\sum_{\alpha}A_j^{\alpha}\mathbf{b}_j^{\alpha}, \tag{73}$$

which enables us to write

$$F=\frac{1}{2}+\frac{1}{2}\sum_{j=m}^{N/2}\mu_j|\mathbf{A}_j|^2+\sum_{j=m+1}^{N/2}\mathbf{A}_{j-1}\cdot\mathbf{A}_j\nu_j-\frac{1}{2}\sum_{j=m_A}^{m-1}|\mathbf{A}_j|^2. \tag{74}$$

Using the Schwarz inequality we have

$$F\leqslant\frac{1}{2}+\frac{1}{2}\sum_{j=m}^{N/2}\mu_j|\mathbf{A}_j|^2+\sum_{j=m+1}^{N/2}|\mathbf{A}_{j-1}||\mathbf{A}_j|\nu_j-\frac{1}{2}\sum_{j=m_A}^{m-1}|\mathbf{A}_j|^2. \tag{75}$$

The right-hand side is exactly the fidelity (45) of the preceding section with the substitution

$$A_j\to\tilde{A}_j\equiv|\mathbf{A}_j|=\left(\sum_{\alpha}(A_j^{\alpha})^2\right)^{1/2}. \tag{76}$$

This equation shows that the existence of several equivalent representations in the Clebsch-Gordan decomposition of Alice's Hilbert space cannot be used to increase the value of the fidelity already obtained in Sec. III. The equality holds when all vectors $\mathbf{A}_j$ are parallel, in which case we recover Eq. (45). The square root on the right-hand side of Eq. (76) plays the role of an effective component of $|A\rangle$ on the Hilbert space of a *single* irreducible representation j. The specific ways $|A\rangle$ projects on each one of the equivalent representations are of no relevance, provided $\tilde{A}_j$ do not change. As far as the fidelity is concerned, all of them are equivalent to taking a state $|\tilde{A}\rangle$ that belongs to $N/2\oplus(N/2-1)\oplus(N/2-2)\oplus\cdots$ (no duplications), with the corresponding components given by $\tilde{A}_j$.

TABLE I. Maximal fidelities as a function of the number of spins.

N	1	2	3	4	5	6	7
F_N	$\frac{2}{3}$	$\frac{3+\sqrt{3}}{6}$	$\frac{6+\sqrt{6}}{10}$	$\frac{5+\sqrt{15}}{10}$	0.9114	0.9306	0.9429

As we have just seen, the maximal fidelity can be achieved from a code state containing only one of each irreducible representation. These types of state are formally the same as those considered in the simplified example of two equal spins $s_1=s_2=s$ studied in Sec. III, for which $s\otimes s=J\oplus(J-1)\oplus\cdots\oplus 0$, with $J=2s=N/2$. The problem of an even number of spins is thus completely solved: according to Eq. (53) the maximal fidelity is given by

$$F_N=\frac{1+x_{N/2+1}^{0,0}}{2} \quad \text{for } N \text{ even}, \tag{77}$$

where $x_{N/2+1}^{0,0}$ is the largest zero of the (Legendre) polynomial $P_{N/2+1}(x)=P_{N/2+1}^{0,0}(x)$.

For an odd number of spins we can proceed as in Sec. III but considering now states with two different spins: $s_1=s$, $s_2=s+\frac{1}{2}$. The corresponding Clebsch-Gordan decomposition is also nondegenerate: $s\otimes(s+1/2)=J\oplus(J-1)\oplus\cdots\oplus 1/2$, with $J=2s+\frac{1}{2}=N/2$. The results from Eqs. (37)–(54) are still valid (for the value of J we have just specified). The optimal values of m_A and m_B are again the minimal ones: $m_A=m_B=\frac{1}{2}$. The maximal fidelity is

$$F_N=\frac{1+x_{N/2+1/2}^{0,1}}{2} \quad \text{for } N \text{ odd}, \tag{78}$$

where $x_{N/2+1/2}^{0,1}$ stands for the largest zero of the Jacobi polynomial $P_{N/2+1/2}^{0,1}(x)$. This completes the solution of the general problem.

It is physically obvious that the larger the number of spins Alice can use the better she should be able to encode $\vec{n}$. One thus expects that the maximal fidelity should increase monotonically with N. It is interesting to obtain this result from the properties of the zeros of the Jacobi polynomials. For an even number of spins, $N=2n-2$, the corresponding zero is $x_n^{0,0}$, whereas for $N+1$ it is $x_n^{0,1}$, and $x_{n-1}^{0,1}$ for $N-1$. Proving that $F_{N-1}<F_N<F_{N+1}$ amounts to showing that

$$x_{n-1}^{0,1}<x_n^{0,0}<x_n^{0,1}, \tag{79}$$

but this is just a particular case of Eq. (A9) for $a=0$ and $b=1$.

Not only the optimal strategy Alice can devise with N spins leads to a fidelity larger than F_{N-1}. She can also use nonoptimal ones and still improve on F_{N-1}. For example, for $N=4$, the choice $m_A=m_B=1$, which is nonoptimal, gives a fidelity $F=(10+\sqrt{10})/15>(6+\sqrt{6})/10=F_3$. This is also a trivial consequence of Eq. (A9) as in this case one has $x_2^{0,2}>x_2^{0,1}$. In physical terms, this tells us that the dimension of the Hilbert space spanned by $U(\vec{n})|A\rangle$ and $U(\vec{n}_B)|B\rangle$ when $N=4$ and $m_A=m_B=1$ (including equivalent spin representations only once) is still larger than the maximal available dimension for $N=3$.

V. DISCUSSION AND OUTLOOK

In this paper we have addressed the problem of optimizing strategies for encoding and decoding directions on the quantum states of a system of N spins. We have restricted ourselves to states that point along a definite direction in an intrinsic way, namely, to eigenstates of $\vec{n}\cdot\vec{S}$. This case is of great interest since no prior knowledge of any sender's (Alice's) reference state or frame by the recipient (Bob) is needed at all for a viable transfer of the information. We have optimized both Alice's states and Bob's measurements. Our results are summarized in Eqs. (77) and (78), where we give the maximal averaged fidelities F_N. Interestingly enough, these results can be written in terms of the largest zeros of the Jacobi polynomial, which are known to play an important role in angular momentum theory and are intimately related to the matrix representations of SU(2). The states that lead to the maximal fidelities are among those that have the smallest (non-negative) values of $\vec{n}\cdot\vec{S}$, namely, $m=0$ for N even and $m=\frac{1}{2}$ for N odd, but still span the largest Hilbert space under rotations.

We display the values of the maximal fidelity for N up to 7 in Table I for illustrational purposes. It shows, e.g., that the optimal encoding with three spins ($m=\frac{1}{2}$) gives $F_3=(6+\sqrt{6})/10\sim 0.845$, which is already larger than the corresponding maximal value for four *parallel* spins ($m=2$): $F=\frac{5}{6}\sim 0.833$ [2]. This illustrates a general feature: the optimal strategies discussed here lead to fidelities that increase with N much faster than that of sending parallel spins. In fact, Eq. (A13) shows that F_N approaches unity quadratically in the number of spins, namely,

$$F_N\sim 1-\frac{\xi^2}{N^2}, \tag{80}$$

where $\xi\sim 2.4$ is the first zero of the Bessel function $J_0(x)$. In contrast, if parallel spins are used the maximal fidelity approaches unity only linearly, $F\sim 1-1/N$.

This can be understood in terms of the dimension d of the Hilbert space used effectively in each case, which is a direct sum of the Hilbert spaces of the irreducible representations of SU(2) involved. Here ''effectively'' means ''nonredundantly;'' thus equivalent representations count only once. Encoding with N parallel spins uses only the Hilbert space of the representation $J=N/2$, whose dimension is $d=N+1$, whereas our optimal strategy uses a much larger Hilbert space, with $d=(N/2+1)^2$ for N even and $d=(N/2+1)^2$

$-\frac{1}{4}$ for N odd; in both cases $d \sim N^2$. We are led to the conclusion that the fidelity as a function of d tends to unity as

$$F \approx 1 - \frac{a}{d}, \tag{81}$$

where a is of order 1 and depends on the particular strategy.

Improvements on the approach discussed in this paper can only come from encoding and decoding procedures that make extensive use of the available Hilbert space, namely, strategies that use the redundant equivalent representations. In [6] we presented a strategy for which the maximal fidelity approaches unity exponentially in the number of spins, i.e., $F \sim 1 - 2^{-N}$. We argued there that this encoding is likely to lead to the maximal fidelity one can possibly achieve with N spins, since it makes effective use of the whole Hilbert space of the system, for which $d = 2^N$ [thus, Eq. (81) also holds in this case]. The corresponding encoding process, however, involves complicated unitary operations and, moreover, it seems to require that Alice and Bob share a common reference frame [13].

We have obtained our general results using continuous POVMs, but finite ones can also be designed. For N parallel spins ($m_A = m_B = N/2$), a general recipe for finite optimal POVMs exists [3], and minimal versions for up to $N = 7$ can be found in [4]. The unit vectors $\vec{n}_r$ associated with the outcomes of these POVMs are the vertices of certain polyhedra inscribed in the unit sphere. For $N \leqslant 7$ we have explicitly verified that these very same polyhedra can be used to design finite optimal POVMs for any value of $m_A = m_B \leqslant N/2$. Moreover, the minimal POVMs of [4] remain minimal for the states considered here. We have discussed this issue in detail for $N = 2$ in Sec. II. For $N = 3$ the polyhedron corresponding to the minimal POVM is the octahedron [4]. One can easily verify that $O_r = U(\vec{n}_r)|B\rangle\langle B|U^\dagger(\vec{n}_r)$ satisfy the completeness condition [(2) for both $m_B = \frac{1}{2}$ and $m_B = \frac{3}{2}$, where $|B\rangle$ is given in Eq. (39)]. We hence believe that the discretization of a continuous POVM is a geometrical problem, i.e., it seems to be independent of the states $|B\rangle$.

The optimal states $|A\rangle$ can easily be computed from the matrix M in Eq. (50), as they are the eigenvectors corresponding to the maximal eigenvalue. Recall that for $N = 2$ one obtains the one-parameter family of states (21) which includes the product states $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$. For $N > 2$, product states of the type $|\uparrow\downarrow\uparrow\uparrow\downarrow\cdots\rangle$ do not seem to be optimal. Consider, e.g., $N = 4$. The optimal eigenvector of M is

$$|A\rangle = \frac{\sqrt{2}}{3}|2,0\rangle + e^{i\gamma_1}\frac{1}{\sqrt{2}}|1,0\rangle + e^{i\gamma_0}\sqrt{\tfrac{5}{18}}|0,0\rangle, \tag{82}$$

which is clearly not a product state of the individual spins for any choice of the phases (it is also entangled if considered as a bipartite system of two spin-1 subsystems). One could argue that this solution is not entirely general because the Clebsch-Gordan series of $(1/2)^{\otimes 4}$ contains the representation 1 three times and 0 twice, whereas in Eq. (82) they appear only once. However, any optimal state has the same "effective" components $\tilde{A}_j$ [see Eqs. (75) and (76)], which can be read off from Eq. (82):

$$\tilde{A}_2 = \frac{\sqrt{2}}{3}, \quad \tilde{A}_1 = \frac{1}{\sqrt{2}}, \quad \tilde{A}_0 = \sqrt{\tfrac{5}{18}}. \tag{83}$$

Note now that *any* product state with $m = 0$ (two spins up and two spins down), e.g., $|\uparrow\uparrow\downarrow\downarrow\rangle$, $|\uparrow\downarrow\downarrow\uparrow\rangle$, has an "effective" Clebsch-Gordan decomposition given by $\tilde{A}_2 = \tilde{A}_1 = \tilde{A}_0 = 1/\sqrt{3}$, which are not the values in Eq. (83). Therefore, these product states cannot be optimal. Nevertheless, they lead to a maximal fidelity $F = (15 + 5\sqrt{2} + 2\sqrt{5})/30 \approx 0.885$, which is remarkably close to $F_4 \approx 0.887$. This is likely to be the case for arbitrary N. These issues are currently under investigation.

ACKNOWLEDGMENTS

We thank S. Popescu, A. Bramon, G. Vidal, and W. Dür for stimulating discussions, and M. Lavelle for a critical reading of the manuscript. Financial support from CICYT Contracts No. AEN98-0431 and No. AEN99-0766, CIRIT Contracts No. 1998SGR-00026, No. 1998SGR-00051, and No. 1999SGR-00097, and EC Contract No. IST-1999-11053 is acknowledged.

APPENDIX

In this Appendix we collect the mathematical properties of the Jacobi polynomials $P_n^{a,b}(x)$ that we use in the text. We are concerned only with integer values of a and b such that $b \geqslant a \geqslant 0$. Further properties can be found in [12] and [14].

For fixed a and b, $\{P_n^{a,b}(x)\}$ is a set of orthogonal polynomials, where n labels the degree of each polynomial in the set. A convenient definition can be stated in terms of their Rodrigues formula:

$$P_n^{a,b}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-a}(1+x)^{-b}\frac{d^n}{dx^n} \times [(1-x)^{n+a}(1+x)^{n+b}]. \tag{A1}$$

From Eq. (A1) follows the recursion relation

$$xP_n^{a,b}(x) = \alpha_n P_{n+1}^{a,b}(x) + \beta_n P_n^{a,b}(x) + \gamma_n P_{n-1}^{a,b}(x), \tag{A2}$$

with

$$\alpha_n = \frac{2(n+1)(n+a+b+1)}{(2n+a+b+1)(2n+a+b+2)},$$

$$\beta_n = \frac{b^2 - a^2}{(2n+a+b)(2n+a+b+2)}, \tag{A3}$$

$$\gamma_n = \frac{2(n+a)(n+b)}{(2n+a+b)(2n+a+b+1)}.$$

Equation (A1) also implies that

$$\frac{dP_n^{a,b}(x)}{dx}=\frac{n+a+b+1}{2}P_{n-1}^{a+1,b+1}(x). \tag{A4}$$

The normalization is chosen so that the coefficient A_n of the highest power of $P_n^{a,b}(x)=A_nx^n+B_nx^{n-1}+\cdots$ is

$$A_n=\frac{\Gamma(2n+a+b+1)}{2^n n!\Gamma(n+a+b+1)}. \tag{A5}$$

The following two relations can also be obtained from the definition (A1):

$$(2n+a+b)P_n^{a,b-1}(x)=(n+a+b)P_n^{a,b}(x)+(n+a)P_{n-1}^{a,b}(x), \tag{A6}$$

$$(n+b+a+1)\frac{1+x}{2}P_n^{a,b+1}(x)=(n+1)P_{n+1}^{a,b-1}(x)+bP_n^{a,b}(x). \tag{A7}$$

Let us recall some basic facts about the zeros of orthogonal polynomials. (i) Any nth-order orthogonal polynomial P_n has n real simple zeros. For Jacobi polynomials these zeros lie in the interval $(-1,1)$. (ii) The zeros of P_n and P_{n+1} are interlaced. (iii) For x greater than the largest zero, the polynomial is a monotonically increasing function [if the polynomial is normalized as in Eq. (A5), where $A_n>0$]. In particular, $P_n(x)$ must be positive in this region.

Now we can prove the results needed in the text. As there, we denote by $x_n^{a,b}$ the largest zero of the polynomial $P_n^{a,b}(x)$. Let us start by showing that

$$x_{n-1}^{a+1,b+1}<x_n^{a,b}. \tag{A8}$$

From property (iii) above it follows that the left-hand side of Eq. (A4) is manifestly positive for $x>x_n^{a,b}$. Hence, so is the right-hand side. We conclude that $x_{n-1}^{a+1,b+1}$ cannot belong to this region and Eq. (A8) follows. ■

Next, we prove the inequality

$$x_{n-1}^{a,b}<x_n^{a,b-1}<x_n^{a,b}. \tag{A9}$$

We evaluate Eq. (A6) at $x=x_n^{a,b}$ and use properties (ii) ($\Rightarrow x_{n-1}^{a,b}<x_n^{a,b}$) and (iii), which imply that $P_{n-1}^{a,b}(x_n^{a,b})>0$, to show that $P_n^{a,b-1}(x_n^{a,b})>0$. We repeat the process for $x=x_{n-1}^{a,b}$ and conclude that $P_n^{a,b-1}(x_{n-1}^{a,b})<0$. Hence $P_n^{a,b-1}$ has a zero in the interval $(x_{n-1}^{a,b},x_n^{a,b})$. This is necessarily the largest zero $x_n^{a,b-1}$ since, according to Eq. (A6) and properties (ii) and (iii) $P_n^{a,b-1}(x)>0$ for $x>x_n^{a,b}$. Thus Eq. (A9) follows. ■

The inequality

$$x_n^{a,b+1}<x_{n+1}^{a,b-1} \tag{A10}$$

can be proven as follows. Evaluate Eq. (A7) at $x=x_n^{a,b+1}$ so that the left-hand side of this equation is zero. The second inequality in Eq. (A9) and property (iii) imply that $P_n^{a,b}(x_n^{a,b+1})>0$. Hence the first term on the right-hand side of Eq. (A7) must be negative, i.e., $P_{n+1}^{a,b-1}(x_n^{a,b+1})<0$, and Eq. (A10) follows immediately, since otherwise property (iii) would not hold for $P_{n+1}^{a,b-1}$. ■

For two given integers l,m consider now the following set of zeros:

$$C_m^l=\{x_{l-m''}^{m''-m',m''+m'}:m\le m'\le m''\le l\}. \tag{A11}$$

We want to prove that

$$\max C_m^l=x_{l-m}^{0,2m} \tag{A12}$$

According to Eq. (A8), lowering m'' by 1 leads us to a larger zero. The maximum is then in the subset $\{x_{l-m'}^{0,2m'}:m\le m'\le l\}$. The inequality (A10) now implies (A12). ■

Finally, we give the large-n (asymptotic) behavior of $x_n^{a,b}$ [12]:

$$x_n^{a,b}=1-\frac{\xi_a^2}{2n^2}+O\left(\frac{1}{n^3}\right), \tag{A13}$$

where ξ_a is the first zero of the Bessel function $J_a(x)$. For $a=0$, which is relevant for our discussion in Sec. V, we also give the subleading term:

$$x_n^{0,b}=1-\frac{\xi_0^2}{2n^2}\left(1-\frac{b+1}{n}\right)+O\left(\frac{1}{n^4}\right), \tag{A14}$$

where

$$\xi_0=\xi=2.405. \tag{A15}$$

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