Generalized measurements of atomic qubits

Sonja Franke-Arnold,[1] Erika Andersson,[2] Stephen M. Barnett,[1] and Stig Stenholm[2]
[1]*Department of Physics and Applied Physics, University of Strathclyde, Glasgow G4 0NG, Scotland*
[2]*Department of Physics, Royal Institute of Technology, Lindstedtsvägen 24, S-10044 Stockholm, Sweden*
(Received 22 November 2000; published 12 April 2001)

We discuss how generalized measurements may be performed on various atomic systems including cold ions and Rydberg atoms. The suggested methods rely on the fact that any generalized measurement can be identified with a unitary transformation followed by a projective measurement in a higher-dimensional Hilbert space. If the information to be measured is stored in the form of atomic qubits this unitary transformation can be accomplished by redistributing the population of the relevant atomic levels.

DOI: 10.1103/PhysRevA.63.052301 PACS number(s): 03.65.Ta, 03.67.−a

I. INTRODUCTION

Sparked by the fast development in the areas of quantum computation and quantum cryptography [1], an interest in quantum observations beyond von Neumann measurements has been arising. The formalism of probability operator measures (POM) [2–4] was developed to describe general quantum measurements and allows us to interrogate qubits in more general ways.

The standard way of picturing a quantum measurement is the von Neumann description [5], where measuring the observable A corresponds to a projection onto the eigenstates of the associated Hermitian operator $\hat{A}$. In an ideal von Neumann measurement the wave function of the system is imagined to ''collapse'' into one of these eigenstates and the measurement result is given by the corresponding eigenvalue. The information that can be provided by a von Neumann measurement is restricted in principle. In particular, observables can be determined at the same time only if they can be described in the same basis of eigenstates. The simultaneous measurement of noncommuting observables, such as the position and momentum of a particle, requires a more general description [6,7] involving the introduction of observables associated with another quantum system.

Another example suggesting the necessity of a generalized measurement is the discrimination between nonorthogonal states, a task interesting in the light of quantum cryptographic protocols based on two nonorthogonal states [8]. Only if a system is prepared in one of two (or more) orthogonal states is it possible to unambiguously determine this state by means of a von Neumann measurement. In this case, the only source of uncertainty associated with the measurement is the technical imperfection of the detection process. If, however, the states of interest are nonorthogonal, their finite overlap causes an intrinsic uncertainty in the measurement. We may optimize the von Neumann measurement in order to minimize the overall probability to obtain an erroneous result [2,9]. Instead, we may choose to perform an error-free measurement at the expense of sometimes obtaining an inconclusive result [10–13]. This is not a von Neumann measurement, since the number of potential results is greater than the dimensionality of the system. A particular example of the discrimination between nonorthogonal states is the distinction between overcomplete states. If the number of states exceeds the dimensionality of the system, an unambiguous state discrimination becomes impossible even if we allow for inconclusiveness [14,15], and the optimal measurement is not a straightforward projection onto orthogonal states.

Given a measurement problem, it is usually a nontrivial task to find the optimal strategy. Only a few experiments implementing state discrimination protocols, and only on photons, have been performed so far [16–19]. In this paper we will concentrate on how to accomplish generalized measurements on atomic systems. We will first, in Sec. II, briefly describe the theory of probability operator measures and illustrate it with the examples of unambiguous state discrimination and distinction between trine states. In Sec. III we will introduce a measurement strategy involving a unitary transformation on an extension of the original system. In Secs. IV and V we will explicitly show how POMs can be realized on different atomic systems and describe the experimental requirements.

II. GENERALIZED MEASUREMENTS

A generalized measurement can be expressed in the language of probability operator measures (POM) [2–4]. In this formalism, each possible outcome of the measurement, labeled by j, is associated with a Hermitian operator $\hat{\Pi}_j$. The probability P_j to obtain the result labeled by j is given by

$$P_j = \mathrm{Tr}(\hat{\rho}\hat{\Pi}_j), \tag{1}$$

where $\hat{\rho}$ is the premeasurement density operator of the system. As a result, all the eigenvalues of the operators $\hat{\Pi}_j$ have to be positive or zero, and $\hat{\Pi}_j$ forms a decomposition of the identity operator,

$$\sum_j \hat{\Pi}_j = \hat{\mathbf{1}}. \tag{2}$$

The POM elements can always be represented by linear combinations of pure state projectors $|\Psi_j\rangle\langle\Psi_j|$. For the applications considered in this paper, it is sufficient [20] to consider POM elements of the form

$$\hat{\Pi}_j = |\Psi_j\rangle\langle\Psi_j|. \tag{3}$$

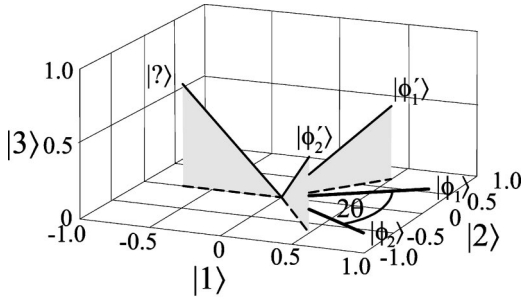

FIG. 1. Unambiguous discrimination between the nonorthogonal states $|\phi_1\rangle$ and $|\phi_2\rangle$ corresponds to a projection onto the states $|\phi_1'\rangle$, $|\phi_2'\rangle$, and $|?\rangle$.

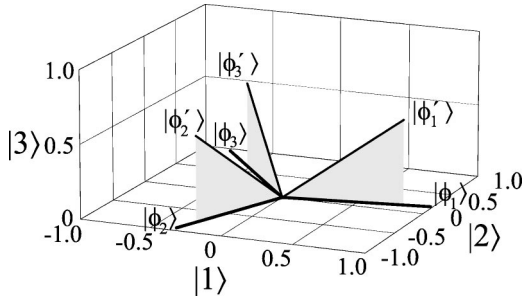

FIG. 2. Discrimination between the trine states $|\phi_j\rangle$ with $j=1,2,3$. The measurement corresponds to a projection onto the three mutually orthogonal states $|\Psi_j\rangle$.

For a von Neumann measurement, the POM elements are the projectors onto the orthonormal eigenstates of the observable. In general, however, the states $|\Psi_j\rangle$ are neither orthogonal nor normalized.

In the following we will give two explicit examples of measurement situations and the corresponding optimal measurement strategies. We will first consider the unambiguous distinction between two nonorthogonal states. We assume that a quantum system was prepared in one of the states

$$|\phi_1\rangle=\cos\theta|1\rangle+\sin\theta|2\rangle,$$
$$|\phi_2\rangle=\cos\theta|1\rangle-\sin\theta|2\rangle, \tag{4}$$

where $0<\theta<\pi/4$, and $|1\rangle$ and $|2\rangle$ are orthonormal. As the states $|\phi_1\rangle$ and $|\phi_2\rangle$ are nonorthogonal they cannot be distinguished from each other with certainty [10–13]. In order to obtain an error-free measurement, we have to allow for inconclusiveness. The minimum probability for an inconclusive result is given by the overlap $\langle\phi_1|\phi_2\rangle=\cos 2\theta$.

We will know for certain that the system was prepared in state $|\phi_1\rangle$ if we detect that it is not in state $|\phi_2\rangle$. This corresponds to a positive measurement outcome for a projection in a direction orthogonal to $|\phi_2\rangle$. Similarly we have to measure in a direction orthogonal to $|\phi_1\rangle$ in order to know that the system was initially prepared in state $|\phi_2\rangle$. Both measurements can be performed at the same time only if the two detection states are orthogonal to each other. In order to achieve this, we have to introduce a third dimension orthogonal to the plane spanned by $|1\rangle$ and $|2\rangle$ as depicted in Fig. 1, and the states we want to project on are

$$|\phi_1'\rangle=\frac{1}{\sqrt{2}}[\tan(\theta)|1\rangle+|2\rangle+\sqrt{1-\tan^2(\theta)}|3\rangle],$$

$$|\phi_2'\rangle=\frac{1}{\sqrt{2}}[\tan(\theta)|1\rangle-|2\rangle+\sqrt{1-\tan^2(\theta)}|3\rangle].$$

The state orthogonal to these two,

$$|\phi_?'\rangle=-\sqrt{1-\tan^2(\theta)}|1\rangle+\tan(\theta)|3\rangle,$$

corresponds to an inconclusive result. The respective POM elements are derived from the projection of these states back into the two-dimensional subspace, $\hat{\Pi}_1=|\Psi_1\rangle\langle\Psi_1|$, $\hat{\Pi}_2=|\Psi_2\rangle\langle\Psi_2|$, and $\hat{\Pi}_?=|\Psi_?\rangle\langle\Psi_?|$, where

$$|\Psi_1\rangle=\frac{1}{\sqrt{2}}[\tan\theta|1\rangle+|2\rangle],$$

$$|\Psi_2\rangle=\frac{1}{\sqrt{2}}[\tan\theta|1\rangle-|2\rangle], \tag{5}$$

$$|\Psi_?\rangle=-\sqrt{1-\tan^2\theta}|1\rangle.$$

A positive measurement outcome corresponding to $\hat{\Pi}_{1,2}$ tells us that the system was prepared in the state $|\phi_{1,2}\rangle$, respectively, while $\hat{\Pi}_?$ indicates an inconclusive result.

Another example of a measurement situation, which cannot be optimally solved by means of a von Neumann measurement, is the distinction between linearly dependent states. Assume that our quantum system is prepared in one of the three nonorthogonal, symmetric states

$$|\phi_j\rangle=\cos(\alpha_j)|1\rangle+\sin(\alpha_j)|2\rangle, \tag{6}$$

with the angles $\alpha_1=\alpha, \alpha_2=\alpha+2\pi/3$, and $\alpha_3=\alpha+4\pi/3$. These states are usually referred to as the trine states [21–23]. In order to measure in which trine state the system was prepared, we again need to introduce a third state, $|3\rangle$, orthogonal to $|1\rangle$ and $|2\rangle$. We now associate the nonorthogonal trine states with three orthogonal states in three dimensions, in such a way that each trine state has a large overlap with one of the orthogonal states and only a small overlap with the other two, as shown in Fig. 2. These three orthogonal states are given by $\sqrt{2/3}|\phi_j\rangle+\sqrt{1/3}|3\rangle$, with $j=1,2,3$. Their projection onto the two-dimensional initial subspace results in the states $|\Psi_j\rangle=\sqrt{2/3}|\phi_j\rangle$. It can be shown that, if the states $|\phi_j\rangle$ are equiprobable, the POM that minimizes the error probability indeed has the elements

$$\hat{\Pi}_j=\frac{2}{3}|\phi_j\rangle\langle\phi_j|, \tag{7}$$

where $j=1,2,3$ [2,24,25]. We note that such measurements are also important in quantum communication problems [20,27,28].

III. GENERAL MEASUREMENTS BY BASIS TRANSFORMATION

Note that the unambiguous discrimination between two nonorthogonal states as well as between trine states involved the introduction of a third dimension. Neumark's theorem [26] implies that any POM measurement can be realized as a von Neumann measurement in an extended Hilbert space. We need to introduce auxiliary states until the dimension of the Hilbert space coincides with the number of required measurements N. We denote these states that span the extended Hilbert space with $|j\rangle$, where $j=1,\ldots,N$. This allows us to associate each of the N required measurement outcomes with one of the orthonormal states $|j\rangle$. In the following we will outline the required measurement procedure. For simplicity we restrict ourselves to measurements on a two-dimensional system; however, the same considerations can easily be applied to systems of a higher dimension.

In two dimensions we denote the N POM elements $\hat{\Pi}_j = |\Psi_j\rangle\langle\Psi_j|$ with $j=1,\ldots,N$ where

$$|\Psi_j\rangle = \psi_{j1}|1\rangle + \psi_{j2}|2\rangle. \tag{8}$$

Here $|1\rangle$ and $|2\rangle$ are the orthonormal basis states of the initial system on which the measurement should be carried out. The fact that $\hat{\Pi}_j$ form a decomposition of the identity operator (2) implies that

$$\begin{aligned} \sum_{j=1}^{N} |\psi_{j1}|^2 &= \sum_{j=1}^{N} |\psi_{j2}|^2 = 1, \\ \sum_{j=1}^{N} \psi_{j1}\psi_{j2}^* &= \sum_{j=1}^{N} \psi_{j1}^*\psi_{j2} = 0. \end{aligned} \tag{9}$$

Our aim is to represent the states $|\Psi_j\rangle$ as projections onto the space spanned by $|1\rangle$ and $|2\rangle$ of orthonormal states $|\Psi_j'\rangle$ in the extended N-dimensional Hilbert space. In order to show that this is always possible we consider the following vectors:

$$|\Phi_i\rangle = \sum_{j=1}^{N} \psi_{ji}^*|j\rangle, \tag{10}$$

for $i=1,\ldots,N$. It follows from the conditions (9) that $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are orthonormal. It is always possible to choose $N-2$ additional orthonormal vectors $|\Phi_3\rangle,\ldots,|\Phi_N\rangle$, so that $\{|\Phi_i\rangle\}$, $i=1,\ldots,N$ form an orthonormal basis spanning the N-dimensional Hilbert space. The fact that $|\Phi_i\rangle$ are orthonormal means in particular that $\Sigma_{i=1}^{N}\psi_{ji}^*\psi_{ik} = \delta_{jk}$. With the help of this we can show that the states

$$|\Psi_j'\rangle = \sum_{i=1}^{N} \psi_{ji}|i\rangle = |\Psi_j\rangle + \sum_{i=3}^{N} \psi_{ji}|i\rangle \tag{11}$$

also form an orthonormal basis of the N-dimensional Hilbert space. We can associate the POM elements in the two-dimensional subspace with measurements in the N-dimensional Hilbert space by $P_j = \mathrm{Tr}(\hat{\rho}\hat{\Pi}_j) = \mathrm{Tr}(\hat{\rho}\hat{\Pi}_j')$, where $\hat{\Pi}_j' = |\Psi_j'\rangle\langle\Psi_j'|$. This means that the generalized measurement can be performed as a von Neumann projection onto the states $|\Psi_j'\rangle$.

In an experiment we perform measurements by monitoring the populations of the atomic states $|j\rangle$. We therefore have to map the states $|\Psi_j'\rangle$ onto the ''read-out'' states $|j\rangle$ by means of a unitary transformation

$$\hat{U} = \sum_{j=1}^{N} |j\rangle\langle\Psi_j'| = \sum_{i=1}^{N}\sum_{j=1}^{N} \psi_{ji}^*|j\rangle\langle i|. \tag{12}$$

Unitarity follows from orthonormality and completeness of $|j\rangle$ and $|\Psi_j'\rangle$.

Applied to an initial state $|\psi\rangle$ in the subspace of the original system, the required transformation is

$$\hat{U}|\psi\rangle = \sum_{j=1}^{N} |j\rangle\langle\Psi_j'|\psi\rangle = \sum_{j=1}^{N} |j\rangle\langle\Psi_j|\psi\rangle. \tag{13}$$

This means that the probability to find the system in state $|j\rangle$ is exactly $P_j = \mathrm{Tr}(\hat{\rho}\hat{\Pi}_j)$, with $\hat{\rho} = |\psi\rangle\langle\psi|$. Thus, in order to perform the desired POM measurement, we need to effect the unitary transform $\hat{U}$, and then measure the population in the basis states $|j\rangle$.

It has been shown for single photons or other propagating particles [29] that any discrete unitary operator can be implemented by a sequence of beam splitters, wave plates, and mirrors acting on two-dimensional subspaces of the whole Hilbert space. In the following we describe an analogous scheme employing the electronic states of atoms.

IV. GENERALIZED MEASUREMENTS ON TRAPPED IONS

Quantum information may be inscribed onto the population amplitudes of atomic levels. This is a key feature of the quantum computation scheme by Cirac and Zoller in which the electronic states of trapped cold ions are the carriers of qubits. Many variations of this original proposal have been devised since then, including schemes that allow ions that are not cooled to their motional ground state [30]. In this section we will describe how POM measurements can be performed on an atomic system of this kind.

For simplicity we will assume that quantum information is stored as a superposition of the two electronic states $|1\rangle$ and $|2\rangle$. In order to allow N independent measurement outcomes we will require N read-out states that will be the two initial and $N-2$ additional states. During the measurement process, population amplitudes have to be transferred from the two initial states to the N read-out states as described in Eq. (13). This can be achieved by a sequence of Raman transitions [31], coupling two of the atomic states at a time. Each Raman transition may be understood as a unitary transformation in Hilbert space in the plane spanned by the two involved basis states. The rotation angle is determined by the coupling strength Ω and the interaction time τ, and an additional phase factor arises from the phase difference φ between the two Raman pulses. The transformation via a Ra-

man transition is described by

$$\hat{R}(\Omega\tau,\varphi)=\begin{pmatrix}\cos[\Omega\tau/2] & -ie^{i\varphi}\sin[\Omega\tau/2]\\ -ie^{-i\varphi}\sin[\Omega\tau/2] & \cos[\Omega\tau/2]\end{pmatrix}, \tag{14}$$

where the matrix elements are defined according to $R_{ij}=\langle i|\hat{R}|j\rangle$ throughout the paper. The order of the transformations and their respective angle and phase is determined by the POM elements. The final level occupation may be obtained via electron shelving [32] and corresponds to the result of the generalized measurements.

The maximal number of measurement outcomes that can be obtained is limited to the number of atomic levels that can be addressed. In the following we describe the sequences of Raman pulses required for the unambiguous discrimination between two nonorthogonal states and the measurement of trine states.

Examples

Let an atom be prepared in one of the two nonorthogonal states $|\phi_1\rangle$ and $|\phi_2\rangle$ described in Eq. (4). These states can be realized as superpositions of the atomic levels $|1\rangle$ and $|2\rangle$. In order to detect unambiguously in which of these states the system was prepared, we need to employ a third atomic level, $|3\rangle$. Our aim is to associate the states $|\phi_1\rangle$ and $|\phi_2\rangle$ with the orthogonal states $|\phi_1'\rangle$ and $|\phi_2'\rangle$ as depicted in Fig. 1. These have to be transformed into the atomic states $|1\rangle$ and $|2\rangle$, whose population can then be detected by electron shelving. The required unitary transformation is achieved by a Raman transition of the effective pulse length $\Omega\tau/2=\arccos\sqrt{1-\tan^2\theta}$ and with the phase difference $\varphi=\pi/2$,

$$\hat{U}_{\mathrm{UD}}=\begin{pmatrix}\sqrt{1-\tan^2\theta} & 0 & \tan^2\theta\\ 0 & 1 & 0\\ -\tan^2\theta & 0 & \sqrt{1-\tan^2\theta}\end{pmatrix}.$$

The failure to discriminate between states $|\phi_1\rangle$ and $|\phi_2\rangle$ will correspond to the atom being left in state $|3\rangle$.

In our second example we want to discriminate between the three trine states

$$|\phi_j\rangle=\cos(\alpha_j)|1\rangle+\sin(\alpha_j)|2\rangle, \tag{15}$$

with the angles $\alpha_1=\alpha,\alpha_2=\alpha+2\pi/3$, and $\alpha_3=\alpha+4\pi/3$. As for the nonorthogonal states, these states will be realized as superpositions of the populations in the atomic levels $|1\rangle$ and $|2\rangle$ and we want to associate the three trine states with the three orthogonal atomic levels $|1\rangle$, $|2\rangle$, and $|3\rangle$. The population of these states will then provide a measurement of the trine states $|\phi_j\rangle$. As these are not orthogonal they cannot be distinguished with 100% accuracy. The best one can possibly do is to transfer each of the trine states in 2/3 of all cases to one of the detection states and in 1/6 of all cases to each one of the two remaining ''wrong'' detection states. This is achieved with the following sequence of Raman transitions, depicted in Fig. 3. First the initial trine states are rotated so that one of them, here $|\phi_1\rangle$, lies in the direction of the basis

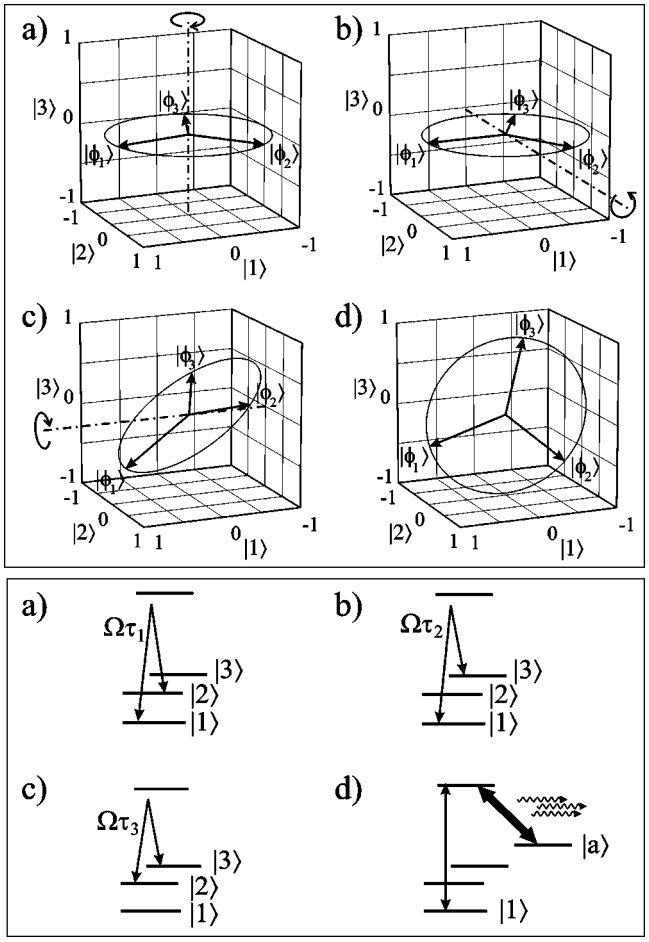

FIG. 3. Pulse sequence for measuring a POM with atoms for the example of discrimination between trine states. Above: Subsequent rotations of the atomic states. Below: Corresponding Raman transitions between the atomic levels (a)–(c) followed by a detection via electron shelving (d).

state $|1\rangle$. This can be done by using a Raman pulse between the atomic levels $|1\rangle$ and $|2\rangle$ of the effective pulse length $\Omega\tau/2=\alpha$ and the phase difference $\varphi=\pi/2$,

$$\hat{R}_1=\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The states are then rotated between the levels $|1\rangle$ and $|3\rangle$ using a Raman transition with $\Omega\tau/2=\arccos(\sqrt{2/3})$ and $\varphi=\pi/2$,

$$\hat{R}_2=\begin{pmatrix}\sqrt{2/3} & 0 & \sqrt{1/3}\\ 0 & 1 & 0\\ -\sqrt{1/3} & 0 & \sqrt{2/3}\end{pmatrix}.$$

The final rotation takes place between levels $|2\rangle$ and $|3\rangle|$ with a pulse duration of $\Omega\tau/2=\arccos(\sqrt{1/2})$,

$$\hat{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1/2} & \sqrt{1/2} \\ 0 & -\sqrt{1/2} & \sqrt{1/2} \end{pmatrix}.$$

The required unitary transformation is $\hat{U} = \hat{R}_3 \hat{R}_2 \hat{R}_1$.

In our examples we have employed only Raman pulses with a phase difference of $\varphi = \pi/2$ which reflects that we had to determine real population amplitudes. In general, phase differences between the Raman pulses will allow the discrimination between states with complex amplitudes.

V. GENERALIZED MEASUREMENTS ON RYDBERG ATOMS

In contrast to the scheme introduced in the preceding section, here we will devise a procedure where all the read-out states are different from the states in which the quantum information is stored. This is necessary, for example, if the read out is done via strongly decaying states. Alternatively, it might be useful to employ Rydberg levels as read-out states which will be tested by ionization.

Again, we restrict ourselves to a two-dimensional subspace spanned by $|1\rangle$ and $|2\rangle$, in which the quantum information is initially stored. For any state in this subspace represented by the density operator $\hat{\rho}$ we wish to perform a measurement with N possible outcomes. The probability for outcome j should be $P_j = \mathrm{Tr}(\hat{\rho}\hat{\Pi}_j) = \langle \Psi_j | \hat{\rho} | \Psi_j \rangle$ with $|\Psi_j\rangle = \psi_{j1}|1\rangle + \psi_{j2}|2\rangle$ as in Eq. (8). One way to achieve this is to coherently couple the levels $|1\rangle$ and $|2\rangle$ to N read-out levels $\{|3\rangle, \ldots, |N+2\rangle\}$ and then to observe in which of these the atom resides. This will succeed if we can transform $\hat{\rho}$ according to

$$\hat{\rho} \rightarrow \hat{\rho}' = \sum_{j=1}^{N} |j+2\rangle \langle \Psi_j | \hat{\rho} | \Psi_j \rangle \langle j+2|. \tag{16}$$

The probability to find the atom in level $j+2$ corresponds then to the jth measurement outcome,

$$\langle j+2 | \hat{\rho}' | j+2 \rangle = P_j. \tag{17}$$

In the following we identify the unitary transformation $\hat{T}$ that converts $\hat{\rho}$ into $\hat{\rho}' = \hat{T}\hat{\rho}\hat{T}^{\dagger}$.

The action of $\hat{T}$ on the states $|1\rangle$ and $|2\rangle$ is given by the requirement that $\hat{\rho}'$ provides the correct probabilities P_j. Hence only the action of $\hat{T}$ on the states $|3\rangle, \ldots, |N+2\rangle$ remains to be determined. We write $\hat{T}$ in the form

$$\hat{T} = \sum_{j=1}^{N} \left(|j+2\rangle \langle \Psi_j| + |\Phi_{j+2}\rangle \langle j+2| \right) = \sum_{j=1}^{N+2} |\Phi_j\rangle \langle j|. \tag{18}$$

In going from the first to the second line we have introduced the states

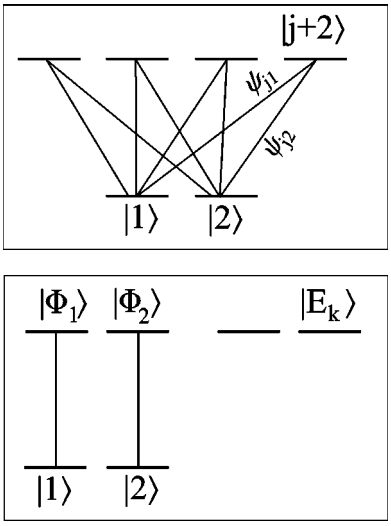

FIG. 4. Top: Quantum information is initially stored in the atomic levels $|1\rangle$ and $|2\rangle$, which are subsequently coupled to N read-out levels. The population probability in the read-out level $|j+2\rangle$ corresponds to the measurement outcome j. Bottom: dressed-state picture of the required coupling between the levels $|1\rangle, |2\rangle, |\Phi_1\rangle, |\Phi_2\rangle$. The remaining $N-2$ states $|E_k\rangle$ do not interact with the rest of the system.

$$|\Phi_1\rangle = \sum_{j=1}^{N} \psi_{j1}^{*} |j+2\rangle,$$
$$|\Phi_2\rangle = \sum_{j=1}^{N} \psi_{j2}^{*} |j+2\rangle. \tag{19}$$

The POM conditions, Eq. (9), guarantee that these states are orthonormal, and we note that they are also orthonormal to $|1\rangle$ and $|2\rangle$. We choose the remaining N states $|\Phi_3\rangle, \ldots, |\Phi_{N+2}\rangle$ so that they, together with $|\Phi_1\rangle$ and $|\Phi_2\rangle$, span the $N+2$-dimensional Hilbert space. The time evolution Eq. (18) corresponds to a transformation of the basis set $\{|j\rangle\}, j = 1, \ldots, N+2$ into $\{|\Phi_j\rangle\}, j = 1, \ldots, N+2$. The unitarity of $\hat{T}$ follows directly from the orthonormality of these sets of states.

As only the states $|1\rangle$ and $|2\rangle$ are initially populated, the required time evolution is fully determined by the coupling $|1\rangle \leftrightarrow |\Phi_1\rangle$ and $|2\rangle \leftrightarrow |\Phi_2\rangle$, while the other states $|3\rangle, \ldots, |N+2\rangle$ may be transformed into any N states that together with $|\Phi_1\rangle$ and $|\Phi_2\rangle$ form a basis set of the $N+2$ dimensional Hilbert space, see Fig. 4.

The essential part of the interaction-picture Hamiltonian that effects the desired transformation is therefore given by

$$\hat{H} = \Omega \left(|\Phi_1\rangle \langle 1| + |\Phi_2\rangle \langle 2| \right) + \mathrm{H.c.} \tag{20}$$

Because all states involved are orthogonal, this Hamiltonian implies the time-evolution operator

$$\hat{T}=\cos\Omega t(|\Phi_1\rangle\langle\Phi_1|+|1\rangle\langle 1|+|\Phi_2\rangle\langle\Phi_2|+|2\rangle\langle 2|) - i\sin\Omega t(|\Phi_1\rangle\langle 1|+|1\rangle\langle\Phi_1|+|\Phi_2\rangle\langle 2|+|2\rangle\langle\Phi_2|) + \sum_{k=1}^{N-2}|E_k\rangle\langle E_k|. \quad (21)$$

The states $|E_k\rangle$ are the eigenstates of the Hamiltonian (20), supplementing the four eigenstates $(|\Phi_1\rangle\pm|1\rangle)/\sqrt{2}$ and $(|\Phi_2\rangle\pm|2\rangle)/\sqrt{2}$. Thus, the states $|E_k\rangle$ span the subspace corresponding to the projector $\hat{\mathbf{1}}-|1\rangle\langle 1|-|2\rangle\langle 2|-|\Phi_1\rangle\langle\Phi_1|-|\Phi_2\rangle\langle\Phi_2|$. At the time $\Omega t=\pi/2$, the desired transformation is effected. Equation (20) gives the simplest Hamiltonian that produces the required time evolution in Eq. (18). We may add arbitrary couplings between the states $|E_k\rangle$ without changing the essential part of the time evolution.

The Hamiltonian Eq. (20) can also be written in terms of the atomic levels $|j\rangle$ by inserting Eq. (19),

$$\hat{H}=\Omega\sum_{j=1}^{N}(\psi_{j1}^*|j+2\rangle\langle 1|+\psi_{j2}^*|j+2\rangle\langle 2|)+\text{H.c.} \quad (22)$$

The parameters ψ_{j1}^* and ψ_{j2}^* that characterize the POM elements, as defined in Eq. (8), specify the relative phase and magnitude of the required laser coupling between the states $|1\rangle$, $|2\rangle$ and the read-out levels.

A. Effects of decay

We will now modify the previously described scheme by allowing decay from the read-out levels $|3\rangle,\ldots,|N+2\rangle$. We denote the nondecaying part of the wave function of our system by $|\Psi(t)\rangle=\Sigma_{j=1}^{N+2}c_j(t)|j\rangle$, and the decay rate from the read-out level $|j+2\rangle$ by Γ_{j+2}. Our aim is to associate the total probability for a decay of the state $|j+2\rangle$ with the measurement outcome P_j,

$$\int_0^\infty dt\Gamma_{j+2}|c_{j+2}(t)|^2=P_j. \quad (23)$$

The time evolution of the atomic system is given by the equations

$$\dot{c}_1(t)=-i\sum_{j=1}^{N}\tilde{\psi}_{j1}c_{j+2}(t),$$

$$\dot{c}_2(t)=-i\sum_{j=1}^{N}\tilde{\psi}_{j2}c_{j+2}(t), \quad (24)$$

$$\dot{c}_{j+2}(t)=-i[\tilde{\psi}_{j1}^*c_1(t)+\tilde{\psi}_{j2}^*c_2(t)]-\frac{\Gamma_{j+2}}{2}c_{j+2}(t),$$

where $\tilde{\psi}_{j1}$ and $\tilde{\psi}_{j2}$ describe the coupling of the levels $|1\rangle$ and $|2\rangle$ to the N read-out levels $|j+2\rangle$. We have neglected off-diagonal damping, assuming that the bare states are well enough separated in energy for this to be unimportant. In general, we now have to determine the coupling rates $\tilde{\psi}_{j1}$ and $\tilde{\psi}_{j2}$ from the set of $N+2$ differential equations so that Eq. (23) is fulfilled. In the following we will do this for two special cases, first, assuming identical decay rates $\Gamma_{j+2}=\Gamma$, and second, for decay rates that exceed all coupling rates.

When all read-out states decay with the same probability Γ, the required coupling rates must be proportional to the coupling in the absence of decay and we set $\tilde{\psi}_{j1}=\Omega_0\psi_{j1}$ and $\tilde{\psi}_{j2}=\Omega_0\psi_{j2}$. We expect that the amplitude of the read-out state $|j+2\rangle$ depends on the initial population in states $|1\rangle$ and $|2\rangle$ and on the respective coupling rates. This inspires the Ansatz

$$c_{j+2}(t)=[\psi_{j1}^*c_1(0)+\psi_{j2}^*c_2(0)]c(t). \quad (25)$$

The overall decay from this level can then be written as

$$\int_0^\infty dt\Gamma|c_{j+2}(t)|^2=P_j\int_0^\infty dt\Gamma|c(t)|^2, \quad (26)$$

where we have expressed the jth measurement outcome probability as $P_j=|\psi_{j1}^*c_1(0)+\psi_{j2}^*c_2(0)|^2$. Comparing this with Eq. (23) we require

$$\int_0^\infty dt\Gamma|c(t)|^2=1. \quad (27)$$

Inserting the Ansatz (25) into Eq. (24) we find

$$\dot{c}_1(t)=-i\Omega_0c_1(0)c(t),$$

$$\dot{c}_2(t)=-i\Omega_0c_2(0)c(t), \quad (28)$$

$$\dot{c}(t)=-i\Omega_0\frac{\psi_{j1}^*c_1(t)+\psi_{j2}^*c_2(t)}{\psi_{j1}^*c_1(0)+\psi_{j2}^*c_2(0)}-\frac{\Gamma}{2}c(t).$$

Here we have used the POM conditions (9). Eliminating c_1 and c_2 we find the second-order equation

$$\ddot{c}(t)+\frac{\Gamma}{2}\dot{c}(t)+\Omega_0^2c(t)=0, \quad (29)$$

which is solved by

$$c(t)=e^{-\Gamma t/4}[A\cos(\Omega t/2)+B\sin(\Omega t/2)], \quad (30)$$

with the Rabi frequency $\Omega=\sqrt{4\Omega_0^2-\Gamma^2/4}$. The assumption that initially only the levels $|1\rangle$ and $|2\rangle$ are populated and $c_{j+2}(0)=0$ requires that $A=0$, and $\dot{c}_{j+2}(0)=-i\Omega_0[\psi_{j1}^*c_1(0)+\psi_{j2}^*c_2(0)]$ determines the coefficient B as $B=-i2\Omega_0/\Omega$. For this choice of coefficients also the requirement $\int_0^\infty\Gamma|c(t)|^2=1$ from Eq. (27) is satisfied. The amplitude of the read-out level $|j+2\rangle$ undergoes damped Rabi oscillations,

$$c_{j+2}(t)=-i2\frac{\Omega_0}{\Omega}[\psi_{j1}^*c_1(0)+\psi_{j2}^*c_2(0)]e^{-\Gamma t/4}\sin(\Omega t/2). \quad (31)$$

The population probability in level $|j+2\rangle$ is given by $|c_{j+2}(t)|^2=P_j|c(t)|^2$ and the overall decay from this level is indeed equal to the measurement outcome P_j.

In general, for differing decay rates Γ_{j+2} in Eq. (24), the coupling rates have to be changed from the ones we used in the absence of decay. In the following we will consider the case of strong decay, when $\Gamma_{j+2}\gg|\tilde{\psi}_{i1,i2}|$ for all j and i. In this case, the population of the levels $|j+2\rangle$ immediately follows the population of levels $|1\rangle$ and $|2\rangle$. This enables us to use adiabatic elimination, setting $\dot{c}_{j+2}=0$. We note that this approximation is correct only over larger time intervals and $c_{j+2}(0)=0$ is not satisfied. The population in level $|j+2\rangle$ is then determined by

$$c_{j+2}(t)=-\frac{2i}{\Gamma_{j+2}}[\tilde{\psi}_{j1}^{*}c_1(t)+\tilde{\psi}_{j2}^{*}c_2(t)]. \tag{32}$$

Substituting this into Eq. (24) we obtain the equation system

$$\begin{aligned}\dot{c}_1(t)&=-2\sum_{j=1}^{N}\frac{|\tilde{\psi}_{j1}|^2c_1(t)+\tilde{\psi}_{j1}\tilde{\psi}_{j2}^{*}c_2(t)}{\Gamma_{j+2}},\\ \dot{c}_2(t)&=-2\sum_{j=1}^{N}\frac{\tilde{\psi}_{j1}^{*}\tilde{\psi}_{j2}c_1(t)+|\tilde{\psi}_{j2}|^2c_2(t)}{\Gamma_{j+2}}.\end{aligned} \tag{33}$$

We now set the coupling rates proportional to ψ_{j1} and ψ_{j2} defining the POM elements and to the square root of the decay rates,

$$\tilde{\psi}_{j1}=\sqrt{\frac{\Omega_0\Gamma_{j+2}}{2}}\psi_{j1},\quad \tilde{\psi}_{j2}=\sqrt{\frac{\Omega_0\Gamma_{j+2}}{2}}\psi_{j2}. \tag{34}$$

For this choice of coupling rates we can apply the POM conditions (9) so that the equations of motion for c_1 and c_2 decouple,

$$\begin{aligned}\dot{c}_1(t)&=-\Omega_0c_1(t),\\ \dot{c}_2(t)&=-\Omega_0c_2(t).\end{aligned} \tag{35}$$

By inserting the solutions of these equations, $c_{1,2}(t)=\exp[-\Omega_0t]c_{1,2}(0)$, into Eq. (32) we find for the amplitudes of the read-out levels,

$$c_{j+2}(t)=-i\sqrt{\frac{2\Omega_0}{\Gamma_{j+2}}}[\psi_{j1}^{*}c_1(0)+\psi_{j2}^{*}c_2(0)]e^{-\Omega_0t}. \tag{36}$$

The corresponding population probability is then

$$|c_{j+2}(t)|^2=P_j\frac{2\Omega_0}{\Gamma_{j+2}}e^{-\Omega_0t}, \tag{37}$$

so that the probability for decay out of the system via level $|j+2\rangle$ is equal to the probability P_j as required in Eq. (23). Thus, we expect that, in the case of (strong) decay, we are able to correct for the decay by weighing the couplings between the levels as described in Eq. (34).

We note that the set of equations (24) describes a system, where a cw laser couples the read-out levels to a continuum and induces ionization. The measurement outcome P_j would then be contained in the properties of the liberated electron.

B. Examples

To illustrate the modified measurement scheme, we will consider the same examples as before. We want to unambiguously distinguish between the two nonorthogonal states given in Eq. (4) with a minimal probability of obtaining an inconclusive result. The POM elements $\hat{\Pi}_1$, $\hat{\Pi}_2$, and $\hat{\Pi}_?$ corresponding to the three measurement outcomes are defined by the states given in Eq. (5). In order to associate the respective probabilities with the population in the three atomic levels $|1\rangle$, $|2\rangle$, and $|3\rangle$, we apply a Hamiltonian of the form in Eq. (22)

$$\begin{aligned}\hat{H}=\Omega\Bigg(&\frac{\tan(\theta)}{\sqrt{2}}(|3\rangle\langle1|+|4\rangle\langle1|)+\frac{1}{\sqrt{2}}(|3\rangle\langle2|-|4\rangle\langle2|)\\ &-\sqrt{1-\tan^2(\theta)}|5\rangle\langle1|\Bigg)+\text{H.c.},\end{aligned} \tag{38}$$

where we have inserted the values ψ_{j1} and ψ_{j2} as defined by the POM elements in Eq. (5).

Experimentally this Hamiltonian may be realized by driving the specified transitions with resonant laser pulses with the given amplitudes. The population of the auxiliary Rydberg states may then be determined by ionization.

With decay, the couplings need to be modified according to Eq. (34) for the probability to decay out of the system via level $j+2$ to be equal to P_j as required. In this case the observation of the fluorescence constitutes the projective measurement. In order to be able to tell which transition the photon came from, the bare read-out levels need to be well separated in energy. The measurement outcome will be encoded in the frequency of the fluorescent light and for a suitable choice of levels also in its polarization.

The trine states of Eq. (6) are optimally distinguished by the same Hamiltonian as the one derived above Eq. (38) if the angle θ is set to $\pi/3$.

VI. CONCLUSIONS

We have investigated the possibility of accessing quantum information which is stored as a superposition of atomic levels. Measurements of populations or transition amplitudes within this atomic system correspond to von Neumann measurements and are limited in principle. In order to perform generalized measurements beyond von Neumann projections we suggest employing additional atomic levels and associating every possible measurement outcome with the population of a different atomic read-out level. We have devised several experimentally feasible methods to perform such a generalized measurement. The initial atomic population may be transferred by controlled Raman pulses into a higher-

dimensional Hilbert space, and the resulting population probability can then be tested via the quantum jump technique. In another scheme the atomic population is transferred by laser pulses from stable information storage levels into a different set of atomic read-out levels which may be Rydberg levels or fast decaying levels. The measurement outcomes may then be obtained by ionization, or by directly observing the decay photons. We expect that generalized measurements on atomic qubits will become important, e.g., in the context of quantum computation.

ACKNOWLEDGMENTS

This work was supported by the TMR program of the Commission of the European Union through the Quantum Structure Network. Partial support from the ESF QIT program is gratefully acknowledged.

[1] *The Physics of Quantum Information*, edited by D. Bouwmeester, A. Ekert, and A. Zeilinger (Springer, Berlin, 2000).
[2] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
[3] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1993).
[4] K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
[5] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, 1955).
[6] E. Arthurs and J. L. Kelly, Bell Syst. Tech. J. **44**, 725 (1965).
[7] S. Stenholm, Ann. Phys. (N.Y.) **218**, 233 (1992).
[8] C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).
[9] M. Sasaki and O. Hirota, Phys. Rev. A **54**, 2728 (1996).
[10] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).
[11] D. Dieks, Phys. Lett. A **126**, 303 (1988).
[12] A. Peres, Phys. Lett. A **128**, 19 (1988).
[13] A. Chefles and S. M. Barnett, J. Mod. Opt. **45**, 1295 (1998).
[14] A. Chefles, Phys. Lett. A **239**, 339 (1998).
[15] A. Chefles and S. M. Barnett, Phys. Lett. A **250**, 223 (1998).
[16] B. Huttner, A. Müller, J. D. Gautier, H. Zbinden, and N. Gisin, Phys. Rev. A **54**, 3783 (1996).
[17] S. M. Barnett and E. Riis, J. Mod. Opt. **44**, 1061 (1997).
[18] R. B. M. Clarke, A. Chefles, S. M. Barnett, and E. Riis, J. Phys. A (to be published), e-print quant-ph/0007063.
[19] R. B. M. Clarke, V. Kendon, A. Chefles, S. M. Barnett, E. Riis, and M. Sasaki, J. Phys. A (to be published), e-print quant-ph/0008028.
[20] E. B. Davies, IEEE Trans. Inf. Theory **IT-24**, 596 (1978).
[21] A. S. Holevo, Probl. Peredachi Inf. **9**, 31 (1973); Probl. Inf. Transm. **9**, 110 (1973).
[22] A. Peres and W. K. Wootters, Phys. Rev. Lett. **66**, 1119 (1992).
[23] P. Hausladen and W. K. Wootters, J. Mod. Opt. **41**, 2385 (1994).
[24] A. S. Holevo, J. Multivariate Anal. **3**, 337 (1973).
[25] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, Int. J. Theor. Phys. **36**, 1269 (1997).
[26] M. A. Neumark, Izv. Akad. Nauk. SSSR, Ser. Mat. **4**, 53 277 (1940).
[27] S. J. D. Phoenix, S. M. Barnett, and A. Chefles, J. Mod. Opt. **47**, 507 (2000).
[28] M. Sasaki, S. M. Barnett, R. Jozsa, M. Osaki, and O. Hirota, Phys. Rev. A **59**, 3325 (1999).
[29] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).
[30] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995); T. Pellizzari, S. A. Gardiner, J. I. Cirac *et al.*, **75**, 3788 (1995); D. F. V. James, Fortschr. Phys. **48**, 823 (2000), and references therein.
[31] Similar Raman pulses are frequently used in experiments on cold ions; see, for example, D. Leibfried, D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, J. Mod. Opt. **44**, 2485 (1997).
[32] The detection efficiency for the electronic states is almost 100%; see W. Nagourney, J. Sandberg, and H. Dehmelt, Phys. Rev. Lett. **56**, 2797 (1986); Th. Sauter, W. Neuhauser, R. Blatt, and P. E. Toschek, *ibid.* **57**, 1696 (1986); J. C. Bergquist, R. G. Hulet, W. M. Itano, and D. J. Wineland, *ibid.* **57**, 1699 (1986).