Causality and propagation in the Wigner, Husimi, Glauber, and Kirkwood phase-space representations

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Time evolution is considered in phase space in terms of evolution kernels for various phase-space quasidistributions. The propagators for the Wigner function, the standard-ordered function, the Kirkwood (antistandard-ordered) function, the Glauber P and Q functions, and the Husimi function are explicitly written as bilinear transforms of the evolution operator. Free propagation, propagation in dispersive media, and scattering, are studied, and manifestations of causality and interference are analyzed. It is shown that free propagation and scattering in the Husimi, Glauber, and Kirkwood representations with the underlying dynamics of the Schrödinger equation involve divergent evolution kernels connecting distant phase-space points at all times. The time evolution is much simpler in the Wigner representation where (i) free propagation is a simple classical translation involving no interference, and (ii) analytical properties of the scattering matrix restrict the velocities of propagation so that no information can travel due to scattering faster than free motion. As an example, a correlation is found between the coordinate and momentum of particles detected after they are released from a box. Propagators with relativistic dispersion relations of free photons or Klein-Gordon particles are briefly discussed in an Appendix.

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I. INTRODUCTION

The subject of this paper is to study the nature and properties of time evolution or dynamics of a quantum system in phase space. Recent advances in atom optics and molecular physics require a quantum treatment of the time evolution of wave packets, which are partially localized both in momentum and in coordinate. It is often useful to formulate the dynamics of these systems in phase space. Formulation of quantum dynamics in phase space is also interesting from a pure theoretical or fundamental perspective. One may wonder how must we change our concepts of evolution in the quantum regime.

Different representations of quantum mechanics in phase space were introduced over the years. These include the Wigner function, the standard-ordered function, the Kirkwood (antistandard-ordered) function, the Glauber P and Q functions, the Husimi function, and other representations that will not be considered here. For a large body of information regarding these different quasidistributions the reader is referred to several seminal studies as well as excellent and comprehensive reviews [1–16,37,38].

Phase-space representations of quantum systems were often used in the past to analyze the classical limit or classical quantum correspondence. The different phase-space representations were introduced in this context as different quantum descriptions with a common classical limit. Comparisons of varied properties of these representations in the quantum regime, where they differ, were given before with an emphasis on the properties of stationary quantum states [10,11,14]. Our focus here is complementary to the previous studies: We analyze quantum dynamics by considering and comparing the propagators for the various phase-space quasidistribution functions. We do not consider here time evolution in terms of differential equations [11,17,18], Wigner trajectories [8,19–22], or superoperators [23], nor do we discuss the advantages and difficulties of these approaches. Instead, we focus on discussing dynamics in terms of evolution kernels. The different phase-space quasidistributions are different transforms of the density matrix and the time-evolution of the density matrix is given by the time-evolution operator. The propagators for the quasidistributions are therefore given by different bilinear transforms of the evolution operator. We derive these transforms and study their properties.

Quantum propagators can be quite different from classical propagators. The classical equations of motion define classical trajectories in phase space for the coordinates $q(t)$ and momenta $p(t)$, as a function of the time t. The classical propagators in phase space are therefore δ functions over the classical trajectories defining a one-to-one mapping between single points of the initial and final distributions. In quantum mechanics, on the other hand, a single point of the phase-space distribution at one time can, in principle, be causally connected to many points of this phase-space distribution at another time.

We will use the following definitions. When several (or many) initial points contribute to the value at a single point of a phase-space distribution at a later time this is the purely quantum effect of interference. Likewise, when the value at an initial point influences the value at a final point and these two phase-space points are not connected by a trajectory, this is the purely quantum effect of tunneling. The evolution kernels that we study are essentially influence functionals. When the evolution kernel connecting one phase-space point at the initial time $t=0$ to another phase-space point at a later time $t=\tau$ vanishes we conclude that these two points are causally disconnected for this time difference τ.

We will see that properties of the time evolution can assume very different forms in the different phase-space representations. Quantum mechanics can be studied in many different representations. The physical results of an experiment or the theoretical predictions for an observable effect do not depend on the representation chosen but a clever choice often simplifies the analysis and sometimes helps our physical intuition. In the same way, a poor choice of a representation can unnecessarily complicate simple things, obscuring an underlying elegance of a process. We will see that time evolution with the underlying dynamics of the Schrödinger equation assumes the simplest most transparent form in the Wigner representation. Only in the Wigner phase-space representation free propagation involves no tunneling or interference. A causality condition can be easily defined and proven for scattering of the Wigner function, but it has no simple analog in the other phase-space representations. Likewise, in cases where momentum conservation applies, it is explicitly manifested in the propagators of some quasidistributions but not in others.

The paper is organized in the following way: in Sec. II the phase-space propagators for several well known and widely used quasidistributions are explicitly written for a general time evolution as transforms of off-diagonal matrix elements of the evolution operator and as transforms of an evolution function of the momentum in the special case of conserved momentum. Free propagation is discussed in Sec. III with an application to the simple example of releasing a particle from a box where a correlation is found between the coordinate and momentum of particles detected after they are released. The discussion of scattering in Sec. IV includes single-channel scattering, a study of causality, and a generalization to multichannel scattering. Conclusions are presented in Sec. VI. Relativistic dispersion relations are briefly discussed in the Appendix.

II. PROPAGATORS IN PHASE SPACE

A. Phase-space quasidistribution functions

Quantum mechanics can be represented in phase space in different ways. Various phase-space representations of a quantum state can be constructed by transforming the density matrix $\hat{\rho}_t$ and any operator $\hat{A}$ into scalar functions $\varrho_t(q,p)$ and $A(q,p)$, respectively, so that $\mathrm{Tr}[\hat{\rho}_t\hat{A}] = (1/2\pi\hbar)\int dp \int dq A(q,p)\varrho_t(q,p)$. A consistent definition for the transform of an operator versus the transform of the density ensures that quantum expectation values, traces, and observables do not depend on the representation. Formally, these transforms replace operators and density matrixes by symbols, i.e., c-number scalar functions. Different orders of $\hat{q}$ and $\hat{p}$ define the different symbols . For example, the standard ordered function introduced by Mehta [6], and its complex conjugate, the Kirkwood function, are defined by conjugate ordering of $\hat{q}$ and $\hat{p}$.

A unified form for the phase-space quasidistributions is given by

$$\varrho_t^{\zeta}(q,p) = \frac{1}{4\pi^2}\int d\xi \int d\eta \int dq' \,\zeta(\xi,\eta)\exp[i\xi(q'-q) - i\eta p]\left\langle q' + \frac{\hbar}{2}\eta \middle| \hat{\rho}_t \middle| q' - \frac{\hbar}{2}\eta \right\rangle, \tag{1}$$

where $\hat{\rho}$ is the density matrix [14]. For a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ but we do not assume that $\hat{\rho}$ is necessarily pure. The function $\zeta(\xi,\eta)$ defines the representation. Restrictions on the choice of functions $\zeta(\xi,\eta)$ were given, for example, in Ref. [10] but lately relaxed by Ref. [24]. In this paper we focus on the best known and most used phase-space representations: the Wigner function, the standard-ordered function, the Kirkwood (antistandard-ordered) function, the P function, and the Husimi function, for which $\zeta(\xi,\eta)$ is, respectively, given by

$$\zeta^{\mathrm{W}}(\xi,\eta) = 1, \tag{2}$$

$$\zeta^{\mathrm{S}}(\xi,\eta) = \exp\left[-i\frac{\hbar}{2}\xi\eta\right], \tag{3}$$

$$\zeta^{\mathrm{K}}(\xi,\eta) = \exp\left[i\frac{\hbar}{2}\xi\eta\right], \tag{4}$$

$$\zeta^{\mathrm{P}}(\xi,\eta) = \exp\left[\frac{\hbar}{4m\kappa}\xi^2\right]\exp\left[\frac{\hbar m\kappa}{4}\eta^2\right], \tag{5}$$

$$\zeta^{\mathrm{H}}(\xi,\eta) = \exp\left[-\frac{\hbar}{4m\kappa}\xi^2\right]\exp\left[-\frac{\hbar m\kappa}{4}\eta^2\right], \tag{6}$$

where κ and m are constants defining the representation. The Glauber Q function is a special case of the Husimi function. Other phase-space distributions, e.g., with singular kernels, are not considered here but can be considered as well [24].

Transformations between the phase-space representations are given by the integrals [14]:

$$\varrho_t^{\zeta}(q,p) = \int dq' \int dp' g_{\lambda}^{\zeta}(q'-q,p'-p)\varrho_t^{\lambda}(q',p'), \tag{7}$$

$$g_{\lambda}^{\zeta}(q,p) \equiv \frac{1}{4\pi^2}\int d\xi \int d\eta \exp[i(\xi q + \eta p)]\frac{\zeta(\xi,\eta)}{\lambda(\xi,\eta)}. \tag{8}$$

Simple examples include

$$g_{\mathrm{W}}^{\mathrm{S/K}}(q,p) = g_{\mathrm{K/S}}^{\mathrm{W}}(q,p) = \frac{1}{\hbar\pi}\exp\left[\pm\frac{i}{\hbar}2qp\right], \tag{9}$$

$$g_{\mathrm{W}}^{\mathrm{H}}(q,p) = g_{\mathrm{P}}^{\mathrm{W}}(q,p) = \frac{1}{\pi\hbar}\exp\left[-\frac{m\kappa}{\hbar}q^2\right]\exp\left[-\frac{1}{m\kappa\hbar}p^2\right], \tag{10}$$

while divergent integrals are encountered when considering

$$g_{\rm W}^{\rm P}(q,p)=g_{\rm H}^{\rm W}(q,p)=\frac{1}{2\pi}\int d\xi \exp[i\xi q]\exp\left[\frac{\hbar}{4m\kappa}\xi^2\right]\times\frac{1}{2\pi}\int d\eta \exp[i\eta p]\exp\left[\frac{\hbar m\kappa}{4}\eta^2\right]. \tag{11}$$

We will use this unified notation and reversible transforms throughout this paper.

B. Propagators

Time evolution of the density matrix is given by the time-evolution operator

$$\hat{\rho}_t=\hat{U}(t)\hat{\rho}_0\hat{U}^\dagger(t). \tag{12}$$

For the phase-space distributions, a phase-space propagator is defined as the evolution kernel in the following way:

$$\varrho_t^\zeta(q,p)=\int dq_0\int dp_0\mathcal{L}_t^\zeta(q,p;q_0,p_0)\varrho_0^\zeta(q_0,p_0). \tag{13}$$

This definition was considered in Refs. [3,25–30] for the Wigner function. Here, we extend it to all the phase-space representations. The propagators are normalized and integrable

$$\int dq\int dp\mathcal{L}_t^\zeta(q,p;q_0,p_0)=\int dq_0\int dp_0\mathcal{L}_t^\zeta(q,p;q_0,p_0)=1. \tag{14}$$

The propagator for the Wigner phase-space distribution has a simple expression in terms of matrix elements of the evolution operator,

$$\mathcal{L}_t^{\rm W}(q,p;q_0,p_0)=\frac{1}{2\pi\hbar}\int dq'\int dp' \exp\left[\frac{i}{\hbar}(q'p+q_0p')\right]\times\left\langle q-\frac{1}{2}q'\middle|\hat{U}(t)\middle|p_0-\frac{1}{2}p'\right\rangle\times\left\langle p_0+\frac{1}{2}p'\middle|\hat{U}^\dagger(t)\middle|q+\frac{1}{2}q'\right\rangle. \tag{15}$$

Explicit examples can be found in Refs. [25,26,30]. All other propagators can be expressed in terms of this Wigner propagator and the transforming functions g,

$$\mathcal{L}_t^\zeta(q,p;q_0,p_0)=\int dq_1\int dp_1\int dq_2\int dp_2\mathcal{L}_t^{\rm W}(q_1,p_1;q_2,p_2)\times g_{\rm W}^\zeta(q_1-q,p_1-p)g_\zeta^{\rm W}(q_0-q_2,p_0-p_2). \tag{16}$$

Formally, these propagators are equivalent to transforming into the Wigner representation first, propagating with the Wigner propagator second, and transforming back into the original representation at the end.

C. Momentum conservation

In the following sections propagation with different dispersion relations and scattering are considered. It is useful to study first the case of conserved momentum in which the momentum operator commutes with the Hamiltonian, and the momentum eigenstates are also eigenstates of the evolution operator. Namely,

$$\hat{U}(t)|p\rangle=U_t(p)|p\rangle, \tag{17}$$

and $\langle q|\hat{U}(t)|p\rangle=(2\pi\hbar)^{-1/2}U_t(p)\exp[iqp/\hbar]$. The propagator for the Wigner function is then the Wigner transform of the *evolution function* $U(p)$,

$$\mathcal{L}_t^{\rm W}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi \exp[i\xi(q-q_0)]\times U_t\left(p_0+\frac{\hbar}{2}\xi\right)U_t^*\left(p_0-\frac{\hbar}{2}\xi\right). \tag{18}$$

Just as simple are the propagators for the standard-ordered function and the Kirkwood function,

$$\mathcal{L}_t^{\rm S}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi \exp[i\xi(q-q_0)]\times U_t(p_0)U_t^*(p_0-\hbar\xi), \tag{19}$$

$$\mathcal{L}_t^{\rm K}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi \exp[i\xi(q-q_0)]\times U_t(p_0+\hbar\xi)U_t^*(p_0). \tag{20}$$

The propagators for the P function and the Husimi function are more complicated,

$$\mathcal{L}_t^{\rm H}(q,p;q_0,p_0)=\int dp_1\frac{1}{\sqrt{\pi m\kappa\hbar}}\exp\left[-\frac{1}{m\kappa\hbar}(p-p_1)^2\right]\times\frac{1}{2\pi}\int d\eta \exp\left[i\eta(p_0-p_1)+\frac{\hbar m\kappa}{4}\eta^2\right]\times\frac{1}{2\pi}\int d\xi \exp[i\xi(q-q_0)]\times U_t\left(p_1+\frac{\hbar}{2}\xi\right)U_t^*\left(p_1-\frac{\hbar}{2}\xi\right) \tag{21}$$

$$=-\mathcal{L}_t^{\rm P}(q,p_0;q_0,p), \tag{22}$$

where the propagator for the P function is obtained from the propagator for the Husimi function by exchanging initial and final momenta p_0 and p, and changing the overall sign. While the integrals diverge, these propagators are well-defined distributions, as they obey Eq. (14).

The propagators for the Wigner, standard-ordered, and Kirkwood functions explicitly display momentum conservation with $\delta(p-p_0)$ and a symmetry between initial and final coordinates and momenta. The propagators for the P function and the Husimi function do not have these properties. In fact the propagation of the Husimi function seems to be carried out with a Gaussian distribution of momenta around the final momentum, while the relation to the initial momentum is ill defined, and the reverse applies to the propagation of the P function. Clearly momentum is conserved regardless of the representation chosen to describe the process. Integration of the time evolution in all these representations would give the same result for any observable consequences of momentum conservation. In some representations, however, momentum is conserved explicitly in the evolution kernel: the value of these quasidistributions at a phase-space point propagates only into phase-space points with the same momentum. In other representations the intrinsic property of momentum conservation is obscured. The point to point influence functional does not display it and it is recovered only after integration.

III. FREE PROPAGATION

Free propagation is defined by a dispersion relation $\omega(p)$ which gives the evolution function

$$U_t(p) = \exp[-i\omega(p)t]. \tag{23}$$

Introducing the auxiliary velocity functions

$$u^{\mathrm{W}}(p,\xi) \equiv \frac{\hbar\omega\left(p+\frac{\hbar}{2}\xi\right)-\hbar\omega\left(p-\frac{\hbar}{2}\xi\right)}{\hbar\xi}, \tag{24}$$

$$u^{\mathrm{K}}(p,\xi) \equiv \frac{\hbar\omega(p+\hbar\xi)-\hbar\omega(p)}{\hbar\xi}, \tag{254}$$

$$u^{\mathrm{S}}(p,\xi) \equiv \frac{\hbar\omega(p)-\hbar\omega(p-\hbar\xi)}{\hbar\xi}, \tag{26}$$

the free propagators of the Wigner, standard-ordered, and Kirkwood functions are given by the transforms

$$\mathcal{L}_t^{\mathrm{W/K/S}}(q,p;q_0,p_0) = \delta(p-p_0)\frac{1}{2\pi}\int d\xi \exp\{i\xi[q-q_0 - tu^{\mathrm{W/K/S}}(p_0,\xi)]\}, \tag{27}$$

while the propagators for the P function and the Husimi function are

$$\begin{aligned}\mathcal{L}_t^{\mathrm{H}}&(q,p;q_0,p_0)\\ &= \int dp_1 \frac{1}{\sqrt{\pi m\kappa\hbar}} \exp\left[-\frac{1}{m\kappa\hbar}(p-p_1)^2\right]\\ &\quad\times\frac{1}{2\pi}\int d\eta \exp\left[i\eta(p_0-p_1)+\frac{\hbar m\kappa}{4}\eta^2\right]\\ &\quad\times\frac{1}{2\pi}\int d\xi \exp[i\xi(q-q_0-tu^{\mathrm{W}}(p_1,\xi))]\\ &= -\mathcal{L}_t^{\mathrm{P}}(q,p_0;q_0,p).\end{aligned} \tag{28}$$

The integrals over ξ often diverge but reduce to simple δ functions in cases where $u(p,\xi)$ does not depend on ξ.

These propagators do not define in general a one-to-one correspondence between initial and final phase-space points. They can involve both tunneling and interference. The deviations of $\mathcal{L}_t^{\mathrm{W/K/S}}$ from a one-to-one correspondence are the result of a dependence of the velocity functions $u^{\mathrm{W/K/S}}$ on ξ. In the limit of $\hbar\to 0$ the dependence on ξ and these deviations vanish. For some specific dispersion relations u does not depend on ξ also in the quantum regime and the exact quantum propagator is equal to its classical limit. In these cases a simple one-to-one correspondence between initial and final phase-space points is obtained: $p=p_0$ and $q=q_0+tu(p_0)$. Things are essentially more complicated for the Husimi and P functions. Here, even when u does not depend on ξ and the integration over ξ gives $\delta(q-q_0-tu^{\mathrm{W}}(p_1))$, one still has to integrate over p_1, where p_1 is neither equal to the initial momentum p_0 nor to the final momentum p.

A. Free nonrelativistic massive particles

The dispersion relation of free massive particles, whose wave functions obey the Schrödinger equation, is

$$\hbar\omega(p) = \frac{p^2}{2m}. \tag{29}$$

The auxiliary u functions derived from it are

$$u^{\mathrm{W}}(p,\xi) = \frac{p}{m}, \tag{30}$$

$$u^{\mathrm{K}}(p,\xi) = \frac{p}{m}+\frac{\hbar\xi}{2m}, \tag{31}$$

$$u^{\mathrm{S}}(p,\xi) = \frac{p}{m}-\frac{\hbar\xi}{2m}. \tag{32}$$

Note that while $u^{\mathrm{W}}(p,\xi)$ does not depend on ξ, $u^{\mathrm{K/S}}(p,\xi)$ do. Based on the previous considerations, we therefore expect the propagator in the Wigner representation to uniquely display a one-to-one correspondence between initial and final phase-space points. For this dispersion, the general form of all the phase-space propagators is given by

$$\mathcal{L}_t^{\zeta}(q,p;q_0,p_0)$$
$$=\frac{1}{2\pi}\int d\xi\int dp'\exp[i\xi(q_0+tp'/m-q)]$$
$$\times\frac{1}{2\pi}\int d\eta\zeta(\xi,\eta)\exp[i\eta(p'-p)]$$
$$\times\frac{1}{2\pi}\int d\mu[\zeta(\xi,\mu)]^{-1}\exp[i\mu(p_0-p')]. \quad (33)$$

This time, the free propagator depends on the representation. In the classical limit of $\hbar\to 0$ we recover for all the representations the simple propagator

$$\lim_{\hbar\to 0}\mathcal{L}_t^{\zeta}(q,p;q_0,p_0)=\delta(p-p_0)\,\delta(q_0+tp/m-q), \quad (34)$$

but for finite $\hbar$ the propagators are not the simple δ function of free classical propagation. Unique in this respect is the free propagator for the Wigner function, whose simple form is exact in the quantum regime as well as in the classical one,

$$\mathcal{L}_t^{\mathrm{W}}(q,p;q_0,p_0)=\delta(p-p_0)\,\delta(q_0+tp/m-q). \quad (35)$$

In all the other representation the integrals defining the free propagator by Eq. (33) diverge for finite $\hbar$. Nevertheless, the propagators defined by these divergent integrals are well-defined distributions. In particular, they are normalized and integrable, obeying Eq. (14).

The free propagators for the standard-ordered function and the Kirkwood (antistandard-ordered) functions are, respectively,

$$\mathcal{L}_t^{\mathrm{S/K}}(q,p;q_0,p_0)$$
$$=\delta(p_0-p)\frac{1}{2\pi}\int d\xi\exp\left[i\xi\left(q_0+t\frac{p}{m}-q\right)\pm i\frac{\hbar t}{2m}\xi^2\right] \quad (36)$$
$$=\delta(p_0-p)\sqrt{\frac{\pm im}{2\pi t\hbar}}\exp\left[\frac{\mp im}{2t\hbar}(q_0+tp/m-q)^2\right], \quad (37)$$

where the integral representation of Eq. (36) is exact while the expression in Eq.thinspace (37) was obtained after a regularization. The free propagators for the P function and the Husimi distribution are

$$\mathcal{L}_t^{\mathrm{H}}(q,p;q_0,p_0)$$
$$=\frac{-1}{2\pi}\sqrt{\frac{1}{\pi m\hbar\kappa}}\exp\left[-\frac{m}{\hbar\kappa t^2}(q_0+tp/m-q)^2\right]$$
$$\times\int d\eta\exp\left[-i\eta(q_0+tp_0/m-q)+\frac{\hbar\kappa t^2}{4m}\eta^2\right] \quad (38)$$
$$=-\mathcal{L}_t^{\mathrm{P}}(q,p_0;q_0,p). \quad (39)$$

In all the representations, excluding Wigner, there is a competition between the limits of large time and small $\hbar$ and deviations of the free propagators from their classical limit grow with time.

The difference between the Wigner representation and other representations here is striking. Free propagation in all other representations is based on interference. The value of each initial point contributes to the values of many final points and the value of each final point is determined by the values of many initial points. The Wigner function is the only phase-space function freely propagating in a completely classical way. Each point of the freely propagating Wigner function moves on the classical trajectory $q=q_0+tp/m$, $p=p_0$. This is but a manifestation of the well-known fact that the differential equation for the free Wigner function is identical to the classical Liouville equation while no such equivalence exists for the other phase-space quasidistributions.

B. Example: releasing a particle from a box

The problem of releasing a particle from a box in quantum mechanics can be considered for photons, for relativistic or nonrelativistic massive particles, or using model-independent arguments [31–33]. Here we limit the discussion to nonrelativistic particles. Relativistic dispersion relations are considered in the Appendix.

Suppose that at $t=0$ we release a particle from a box and that the particle wave function was confined within the box at $t\leqslant 0$: $\Psi(q)=0$ unless $-a<q<a$. The particle is propagating in free space according to the Schrödinger equation and $\hbar\omega(p)=p^2/2m$. The free propagator depends on the representation, and as a result, the physical picture of the release from the box is different in the different representations. The Husimi and the P distributions are not even confined to the box at $t=0$. The standard ordered and the Kirkwood functions are confined to the box initially but when released propagate in a complicated manner, exhibiting, for example, what seems like interference. In contrast, a remarkably simple picture is obtained for the Wigner function's free propagation. The particle Wigner function at $t=0$ is confined within the box: $\varrho_0^{\mathrm{W}}(q,p)=0$ unless $-a<q<a$ (as a result of the confinement of the wave function and the definition of the Wigner function). Free propagation in the Wigner representation is trivial and we immediately get the Wigner function of the released particle at any later time t,

$$\varrho_t^{\mathrm{W}}(q,p)=\varrho_0^{\mathrm{W}}(q-tp/m,p). \quad (40)$$

Unlike the wave function that is initially confined and instantly fills all space when released, and unlike the Husimi distribution that even initially extends over all space, the Wigner function is confined to the box at $t=0$, freely propagates by a simple classical translation when released, and does not fill phase space even at infinite times. In fact, it is zero almost everywhere in phase space and differs from zero only in the constant-width strip defined by the inequality $-a+tp/m<q<a+tp/m$. For each momentum the release

is a pure translation of the initial Wigner function at a constant velocity defined by this momentum. The projection into coordinate space does not vanish for any q, $\int dp\varrho_t^{\mathrm{W}}(q,p) \neq 0$. This reflects an instantaneous filling of coordinate space, which is indeed the correct description in the nonrelativistic regime. One could equivalently follow the released particle from the wave function and not from the momentum-projected Wigner function. Once the wave function is initially confined to a finite region of space the momentum is unbounded and the released wave function instantaneously spreads to all space. This is well known and is not our concern here. The new result obtained here in the Wigner representation and demonstrated below for an explicit example is a simple correlation between the coordinate and momentum of a particle detected after its release. Suppose that we use a physical detector to detect the released particle, then in the Wigner representation the detector would also be characterized by a phase-space excepting function [28,34]. This analysis shows that for particles measured by a physical detector at time t and at a large distance from the initial location of the particle, there will be a correlation between the approximate position q of the detected particle and this particle's approximate momentum p.

As an explicit example for the release in the Wigner representation, we consider here as an initial state the ground-state wave function of a particle in an infinite box. The initial wave function is

$$\psi(q)=0, \quad |q|>a,$$

$$\psi(q)=\frac{1}{\sqrt{a}}\sin\left[\frac{\pi}{2a}(q+a)\right], \quad |q|<a \tag{41}$$

and the freely propagating Wigner function at any time $t \geqslant 0$ is $\varrho_t^{\mathrm{W}}(q,p)=0$ for $|q-pt|>a$ and

$$\begin{aligned}\varrho_t^{\mathrm{W}}(q,p)=&\frac{\cos[\pi(q-pt)/a]}{\pi\hbar a}\,\frac{\sin[(2p/\hbar)(a-|q-pt|)]}{2p/\hbar}\\&+\frac{\sin[(2p/\hbar+\pi/a)(a-|q-pt|)]}{2\pi\hbar a(2p/\hbar+\pi/a)}\\&+\frac{\sin[(2p/\hbar+\pi/a)(a-|q-pt|)]}{2\pi\hbar a(2p/\hbar+\pi/a)}\end{aligned} \tag{42}$$

for $|q-pt|<a$, where we have used Eq. (40) and results from Refs. [14,8]. This Wigner function at three different times is depicted in Figs. 1 and 2.

The envelope of the spreading wave packet is given by projection into coordinate space i.e., by integrating Eq. (42) over p. The result of numerically integrating $\int dp\varrho_t^{\mathrm{W}}(q,p)$ at different times is also depicted in Fig. 2. As expected, an instantaneous spreading of the wave packet in coordinate space occurs.

If, instead of projecting into coordinate space, we take into account the experimental fact that the released particle can only be detected by a physical detector, an interesting result follows. The probability for a detection is given by

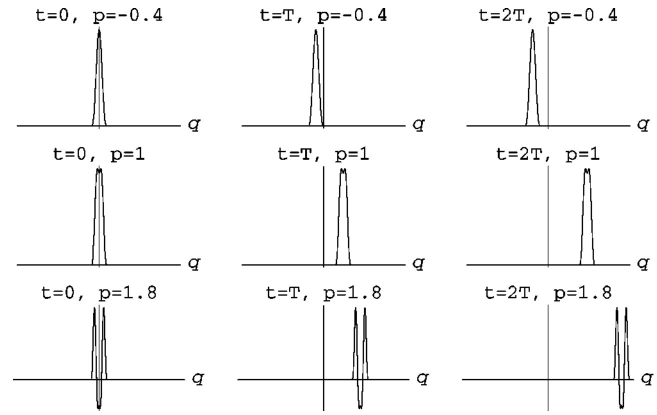

FIG. 1. At $t \leqslant 0$ a massive and nonrelativistic particle is confined to a box $-a<q<a$. At $t=0$ the particle, initially at the ground state of the box, is released. Three slices of the Wigner function of this particle are shown. Each slice is depicted at three different times. At $t \leqslant 0$ the Wigner function is confined to the box $-a<q<a$ and all momentum slices overlap. At $t>0$ each momentum slice is shifted unchanged at a constant velocity proportional to its momentum. Different slices propagate with different velocities. Slices with negative momentum propagate to the left and slices with positive momentum propagate to the right. Thus, at $t=T>0$ the slices spread out and even more so at a later time $t=2T$.

$$\int\int dp\,dq\,\varrho^{\mathrm{D}}(q,p)\varrho_0^{\mathrm{W}}(q-tp/m,p), \tag{43}$$

where $\varrho^{\mathrm{D}}(q,p)$ is the excepting function of the detector [28].

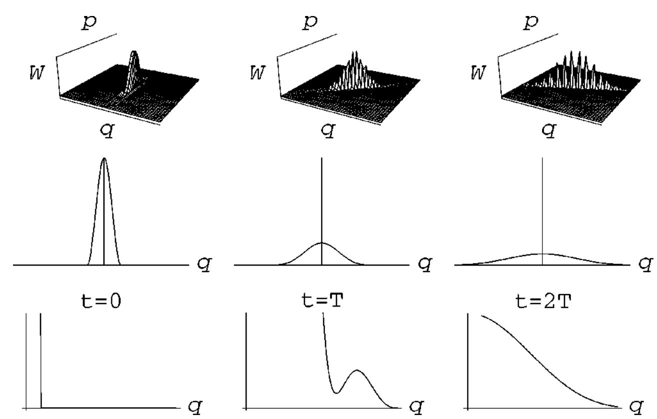

FIG. 2. The complete Wigner function whose slices were depicted in Fig. 1 is shown with its projection into coordinate space at the three different times $t=0,T,2T$. The coordinate probability distribution shown in the middle line instantly fills up space upon the release. A close look at its tail, depicted at the bottom line, reveals that this tail looks the same up to a scale as one approach larger and larger distances (thus no labeling was put on the axes). Only at $t \leqslant 0$ the envelope vanishes abruptly at the wall of the box while at any later time it has an infinite tail that never vanishes. This features of instantaneous spreading are well known. Here we demonstrate that the instantaneous spreading is but a projection of the Wigner function shown in the first line, which is confined to a narrow strip in phase space $-a+pt<q<a+pt$ at all times. This simple observation immediately leads to a rather strong physical result: a correlation between the approximate coordinate and the approximate momentum of a particle detected after its release is obtained.

If the detection is given by the action of an operator then $\varrho^{\mathrm{D}}(q,p)$ is the Weyl transform of this operator, but real detectors can be more complicated than that. For our discussion here it is sufficient to assume that the detector is physically placed at some location, so that a particle can be detected only at a given coordinate range $Q_0 \pm A$. Using only the facts that $\varrho_0^{\mathrm{W}}(q-pt,p)$ is zero unless $-a<q-pt<a$ and $\varrho_0^{\mathrm{D}}(q,p)$ is zero unless $Q_0-A<q<Q_0+A$, we immediately find that the probability to detect the particle with momentum bigger than $(Q_0+A+a)/t$ or smaller than $(Q_0-A-a)/t$ is identically zero.

We conclude that as long as the Schrödinger equation governs the dynamics, particles released from this box are detected at approximate positions and approximate momenta consistent with finite velocities. The release of a particle from a box involves instantaneous spreading in coordinate space but this instantaneous spreading is just the projection of a combination of simple translations with finite velocities in phase space. We note that a measurement by its very nature is always conducted in phase space [28,34].

Clearly, the result for the correlation between the approximate location of the detector and the approximate measured momentum of the detected particle can be proven by an analysis in any representation (Kirkwood, Husimi, Glauber, coordinate or momentum space, or any other) but it is in the Wigner representation that the simple nature of this process is most apparent.

IV. SCATTERING

A. Single-channel scattering

If the propagation is asymptotically free yet includes a small region of interaction, a scattering matrix can be defined. For single-channel scattering, as well as for the elastic channel of multichannel scattering, a single amplitude $A(p)$ and the dispersion relation $\omega(p)$ define the propagation:

$$U_t(p)=A(p)\exp[-i\omega(p)t]. \tag{44}$$

The propagators in the different representations are

$$\mathcal{L}_t^{\mathrm{W}}(q,p;q_0,p_0) = \delta(p-p_0)\frac{1}{2\pi}\int d\xi A\left(p+\frac{\hbar}{2}\xi\right)A^*\left(p-\frac{\hbar}{2}\xi\right) \times\exp\{i\xi[q-q_0-tu^{\mathrm{W}}(p,\xi)]\}, \tag{45}$$

$$\mathcal{L}_t^{\mathrm{S}}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi A(p)A^*(p-\hbar\xi) \times\exp[i\xi(q-q_0-tu^{\mathrm{S}}(p,\xi))], \tag{46}$$

$$\mathcal{L}_t^{\mathrm{K}}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi A(p+\hbar\xi)A^*(p) \times\exp[i\xi(q-q_0-tu^{\mathrm{K}}(p,\xi))]. \tag{47}$$

$$\begin{aligned}\mathcal{L}_t^{\mathrm{P}}&(q,p;q_0,p_0)\\ &=\int dp_1\frac{1}{\sqrt{\pi m\kappa\hbar}}\exp\left[-\frac{1}{m\kappa\hbar}(p_0-p_1)^2\right]\\ &\quad\times\frac{1}{2\pi}\int d\eta\exp\left[i\eta(p_1-p)+\frac{\hbar m\kappa}{4}\eta^2\right]\\ &\quad\times\frac{1}{2\pi}\int d\xi A\left(p_1+\frac{\hbar}{2}\xi\right)A^*\left(p_1-\frac{\hbar}{2}\xi\right)\\ &\quad\times\exp[i\xi(q-q_0-tu^{\mathrm{W}}(p_1,\xi))] \qquad (48)\\ &=\mathcal{L}_t^{\mathrm{H}}(q,p_0;q_0,p). \qquad (49)\end{aligned}$$

The propagators for single-channel scattering in the different phase-space representations are given by bilinear integral transforms of the scattering amplitude. They provide different mappings from the energy domain where the amplitudes are defined to the time domain where the propagators act. Mappings between the energy and time domains are particularly interesting for the discussion of causality.

B. Causality

The scattering amplitude $A(p)$ and its properties depend on the specific problem considered. It often has the following properties when analytically continued into the complex momentum plane:

(i) $A^*(p)=A(-p)$, or $A^*(p)=-A(-p)$.

(ii) $A(p)$ is analytic in the upper half of the complex p plane.

(iii) $A(p)\to 1$ as $|p|\to\infty$.

These properties were proven in Ref. [35] for the Schrödinger equation with a positive potential, but are more general. Whenever the transition amplitude has these properties, the propagators are given by

$$\mathcal{L}_t^{\mathrm{W}}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi A\left(p+\frac{\hbar}{2}\xi\right)A\left(-p+\frac{\hbar}{2}\xi\right)\exp[i\xi(q-q_0-tu^{\mathrm{W}}(p,\xi))], \tag{50}$$

$$\mathcal{L}_t^{\mathrm{S}}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi A(p)A(-p+\hbar\xi) \times\exp[i\xi(q-q_0-tu^{\mathrm{S}}(p,\xi))], \tag{51}$$

$$\mathcal{L}_t^{\mathrm{K}}(q,p;q_0,p_0)=\delta(p-p_0)\frac{1}{2\pi}\int d\xi A(p+\hbar\xi)A(-p) \times\exp[i\xi(q-q_0-tu^{\mathrm{K}}(p,\xi))], \tag{52}$$

$$\mathcal{L}_t^{\mathrm{P}}(q,p;q_0,p_0)$$

$$=\int dp_1 \frac{1}{\sqrt{\pi m\kappa\hbar}}\exp\left[-\frac{1}{m\kappa\hbar}(p_0-p_1)^2\right]$$

$$\times\frac{1}{2\pi}\int d\eta \exp\left[i\eta(p_1-p)+\frac{\hbar m\kappa}{4}\eta^2\right]$$

$$\times\frac{1}{2\pi}\int d\xi A\left(p_1+\frac{\hbar}{2}\xi\right)A\left(-p_1+\frac{\hbar}{2}\xi\right)$$

$$\times\exp[i\xi(q-q_0-tu^{\mathrm{W}}(p_1,\xi))] \tag{53}$$

$$=\mathcal{L}_t^{\mathrm{H}}(q,p_0;q_0,p). \tag{54}$$

The integral over ξ can now be considered as a contour integral in the complex ξ plane. Because $A(p\pm\hbar\xi)$ and $A(p\pm\hbar\xi/2)$ are analytic in the upper ξ plane it is possible to deform the contour of $\int_{\infty}^{\infty}d\xi$ to the arc at infinity of the upper ξ plane, where $A\rightarrow 1$. In general, this will not give any simple result or insight. The integral over the arc at infinity can assume very complicated forms and would often diverge for an arbitrary dependence of the auxiliary u functions on ξ. Only for cases in which $u(p,\xi)=v(p)$ does not depend on ξ, a simple result follows. In these cases the integral over ξ would vanish identically for $q>q_0+tv(p)$. For $q<q_0+tv(p)$ different dispersions, different scattering amplitudes, and different representations would all give different propagators but under the assumptions specified above all these different propagators will vanish for coordinates too far apart at times that are too short. The limit on the propagation is then locally set by the velocity $v(p)$.

A very interesting situation occurs for massive particles in the non-relativistic limit for which $\hbar\omega(p)=p^2/2m$. The manifestation in phase-space of the analytical properties of the scattering matrix then depends on the representation. In previous work we have shown that in the Wigner representation

$$\mathcal{L}_t^{\mathrm{W}}(q>q_0+tp/m)=0. \tag{55}$$

This property of the Wigner function tells us that no information can be transferred faster than free motion as long as the assumptions regarding the analytical properties of the scattering amplitude hold [26,29]. Here we have checked for this property in the other phase-space representations and found that no such simple restriction applies to the other phase-space propagators. In them, the velocity functions u depend on ξ, and the contributions from the arcs at infinity diverge. We note, however, that the reason for the lost simplicity is not in the scattering process but rather in the free propagation which is so simple for the Wigner function yet so complicated for the other phase-space quasidistributions. Note that the Husimi function propagator as was noticed before does not explicitly display momentum conservation. As a result, while it is confined by classical free propagation, this free propagation can be with any momentum, hence the propagation velocity is not really confined.

C. Multichannel scattering

In the general case of multichannel scattering the S matrix is defined in the following way:

$$S_k^p \equiv \lim_{t\to\infty}\langle k|\hat{U}_0^\dagger(t)\hat{U}(t)|p\rangle. \tag{56}$$

The asymptotic time evolution of a momentum eigenfunction is then given by

$$\hat{U}(t)|p\rangle=\int dk\hat{U}_0(t)|k\rangle\langle k|\hat{U}_0^\dagger(t)\hat{U}(t)|p\rangle$$

$$\equiv\int dk S_k^p\exp[-it\omega(k)]|k\rangle. \tag{57}$$

The substitution of Eq. (57) in Eq. (15) gives the propagator for the Wigner function for multichannel scattering with a general dispersion relation as a bilinear transform of the scattering matrix,

$$\mathcal{L}_t^{\mathrm{W}}(q,p;q_0,p_0)$$

$$=\frac{1}{(2\pi\hbar)^2}\int dk\int dq'\int dp'\int dk' S_{k+k'/2}^{p_0-p'/2}S_{p_0+p'/2}^{\dagger k-k'/2}$$

$$\times\exp\left[\frac{i}{\hbar}[p'q_0+k'q+q'(p-k)]\right]$$

$$\times\exp\left[it\omega\left(k-\frac{k'}{2}\right)-it\omega\left(k+\frac{k'}{2}\right)\right]. \tag{58}$$

As in the preceding sections, this expression simplifies considerably for the dispersion relations of free nonrelativistic massive particles,

$$\mathcal{L}_t^{\mathrm{W}}(q,p;q_0,p_0)=\frac{1}{2\pi\hbar}\int dp'\int dk' S_{p+k'/2}^{p_0-p'/2}S_{p_0+p'/2}^{\dagger p-k'/2}$$

$$\times\exp\left[\frac{i}{\hbar}p'q_0+k'\left(q-t\frac{p}{m}\right)\right]. \tag{59}$$

Note that the diagonal part of the scattering matrix reproduces the single-channel case analyzed in the preceding sections. The propagators for the other phase-space functions are obtained from Eqs. (58) or (59) by the transform of Eq. (16) and involve more complicated expressions. Equation (59) is simple enough to be useful. The application of Eq. (59) to specific physical systems remains to be done.

V. CONCLUSIONS

Consider three statements regarding time evolution.

(i) Free propagation involves no tunneling or interference.

(ii) Causality limits velocities of propagation.

(iii) Momentum is conserved if the eigenfunctions of the momentum operator are also eigenfunctions of the Hamiltonian.

Are these statements correct? In this paper we have studied these statements using different phase-space representations. We have found some conditions and premises for which these statements formalized in some representations are indeed correct. Clearly this depends on the underlying dynamics. We have shown that the choice of a representation can also be essential because while the physical content of these statement does not depend on it, the ability to unambiguously define ''tunneling,'' ''interference,'' ''momentum,'' ''velocity,'' and ''causality'' can strongly depend on the representation. For example, for nonrelativistic massive particles these properties of quantum time evolution are simply defined and explicitly correct in the Wigner representation, while in other representations the simplicity of definition is lost and with it these properties of the time evolution, while still true, are obscured.

We have first considered free propagation. We have shown that in general, free propagation is not restricted to a one-to-one correspondence between initial and final phase-space points. The value of a time-evolved quasidistribution at a single phase-space point is usually determined by the values of this quasidistribution at many phase-space points at a previous time. We say that the propagation involves in these cases the quantum phenomena of interference. In free propagation with the underlying dynamics of the Schrödinger equation no initial quasidistribution is shifted unchanged. Each point of the initial Husimi function, for example, influences each and every point of the final Husimi function. Free propagation of the Husimi, Kirkwood, and P functions is based in this case on interference. Unique in this respect is the free propagation of the Wigner function which does define a one-to-one correspondence between initial and final phase-space points. A semiclassical approximation recovers the one-to-one correspondence between initial and final phase-space points in all the representations but deviations of the actual free propagation from its semiclassical limit grow with time for all the quasidistribution except for the Wigner function whose free propagator is identical to its semiclassical limit. While the time evolution of the other quasidistributions is a complicated one, involving, for example, divergent kernels, the time evolution of the Wigner function is simple: each of its points is moving in phase space on a well-defined trajectory and with a well-defined velocity, as if a classical point particle was propagating there. Since each momentum slice is being shifted unchanged at a different velocity, the complete distribution is reshaped, and so is its projection into coordinate space. This was demonstrated for the release of a particle from a box. For other dispersion relations free propagation will in general involve interference in all the phase-space representations.

Single-channel and multichannel scattering was considered next. It was shown that the propagators are bilinear transforms of the scattering matrix. Causality conditions determine the analytical properties of the scattering matrix in the complex momentum plane. These analytical properties then manifest themselves as restrictions on the phase-space propagator, limiting the velocity of propagation. It was shown that the propagator for the Wigner function connecting two phase-space points vanishes if these two points have different momenta or if the coordinate distance between them is too big. The limit on the propagation of information contained in the value of the Wigner function at a single phase-space point is set by free propagation of a classical particle initially at this phase-space point. The analytical properties of the scattering amplitude manifest themselves in phase-space as restrictions on the propagator for the Wigner function. Given these analytical properties, no information can be transferred by scattering faster than by free motion. In other representation this property is obscured. Divergent kernels connect in these representations phase-space points of different momenta or of arbitrary large distances at arbitrary short times.

The particular simple nature of the dynamics of nonrelativistic massive particles in the Wigner representation suggest that this representation may be a useful representation for analyzing actual systems and experiments, in particular in atom-optics as was done, for example, in Refs. [39,40]

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APPENDIX: RELATIVISTIC DISPERSION RELATIONS

In this appendix we make several observations as to the results obtained from naively replacing the dispersion relation $\hbar\omega(p)=p^2/2m$ above by the dispersion relation $\hbar\omega(p)=cp$ or by $\hbar\omega(p)=c\sqrt{p^2+m^2c^2}$. The first is the dispersion relation for free photons or light waves, and the second for Klein-Gordon particles.

There is no position operator that is covariant and Hermitian. Thus, it is not clear whether quasidistribution functions in phase space for photons, light waves, or relativistic massive particles with the underlying dynamics of the Klein-Gordon equation can be rigorously defined. Furthermore, a new definition for the quasidistribution may require a modified definition for the propagators. Nevertheless, different phase-space quasidistributions, in particular the Glauber P and Q functions, are widely used in quantum optics with definitions based on second quantization of the radiation field. It was shown in Ref. [13] that a covariant Wigner phase-space representation can be rigorously applied to localized light waves by using the light-cone coordinate system. These localized light waves, however, cannot represent photons. A recent discussion of the difficulties in defining a Wigner representation for massive particles obeying Lorenz invariance was given in Ref. [36].

The difficulties pertaining to the definition of these quasidistributions are beyond the scope of this paper and are not the subject of this appendix. Here we simply examine the properties of the propagator as it was defined above with the changed dispersion relations. Our attempts to treat in this way relativistic propagation with the dispersion relations of Klein-Gordon particles have failed due to difficulties with branch cuts. Even free propagation with the dispersion relations of Klein-Gordon particles is not well defined in phase space. In contrast, we find that the propagators with the dispersion relations of light waves are well defined, simple, and causal in all the phase-space representations considered here.

Properties that were unique to the Wigner representation for the Schrödinger dispersion apply just as well in all the other phase-space representations as soon as one (naively) replaces the dispersion relation $\hbar\omega(p)=p^2/2m$ by $\hbar\omega(p)=cp$. All the propagators describing free motion in the different representations become trivial,

$$\mathcal{L}_t^\zeta(q,p;q_0,p_0)=\delta(p-p_0)\,\delta(|q_0-q|-ct). \tag{A1}$$

The propagators for free evolution define in this case a one-to-one correspondence between initial and final points, and the initial quasidistribution, whatever it was, is shifted unchanged at the speed of light. Propagation with the dispersion relations of free photons of all the phase-space quasidistributions involves no interference and no tunneling and the free propagator does not depend on the representation. These features are not true for other dispersion relations. In general, free propagation is not restricted to a one-to-one correspondence between initial and final phase-space points, and different representations require different propagators.

For the Schrödinger equation, we have proven a restriction on the propagator in the Wigner representation for single-channel scattering under certain assumptions on the analytical and asymptotic properties of the scattering amplitude. This restriction was unique to the Wigner representation. If, however, we replace $\hbar\omega(p)=p^2/2m$ by $\hbar\omega(p)=cp$ a similar restriction applies as well to all the different phase-space representations here considered, namely,

$$\mathcal{L}_t^{\mathrm{W/S/K/P/H}}(|q_0-q|>tc)=0. \tag{A2}$$

Even though the propagators inside the lightcone ($|q_0-q|<tc$) may differ considerably depending on the representation, the phase-space propagators considered here are confined to within the lightcones, regardless of the representation chosen. Under the above assumptions, the phase-space point-to-point propagation of quasidistributions with the dispersion relations of asymptotically free photons is limited by the speed of light.

The asymptotic limit $\hbar\omega(p)\rightarrow cp$ as $|p|\rightarrow\infty$, which universally holds for relativistic systems, is not a sufficient condition to ensure $\mathcal{L}(|q_0-q|>tc)=0$. If $u(p,\xi)$ has singularities at the upper half of the complex ξ plane, as it does, for example, for $\hbar\omega(p)=c\sqrt{p^2+m^2c^2}$ the integral does not vanish even though the integral over the arc at infinity does. Difficulties with the causality of free time evolution of relativistic particles due to branch cuts in the dispersion relations were considered in Ref. [31] not in phase space but otherwise in a similar context, while difficulties with consistently defining a Wigner function for relativistic free particles were considered in Ref. [36].

Comparing the expressions for both free propagators and single-channel scattering we recognize a similarity between the propagators with $\hbar\omega(p)=cp$ and these propagators with $\hbar\omega(p)=p^2/2m$ in the Wigner representation. No such similarity exits in the other phase-space representations.

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