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Quantum transitions and dressed unstable states

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We consider the problem of the meaning of quantum unstable states including their dressing. According to both Dirac and Heitler this problem has not been solved in the usual formulation of quantum mechanics. A precise definition of excited states is still needed to describe quantum transitions. We use our formulation given in terms of density matrices outside the Hilbert space. We obtain a dressed unstable state for the Friedrichs model, which is the simplest model that incorporates both bare and dressed quantum states. The excited unstable state is derived from the stable states through analytic continuation. It is given by an irreducible density matrix with broken time symmetry. It can be expressed by a superposition of Gamow density operators. The main difference from previous studies is that excited states are not factorizable into wave functions. The dressed unstable state satisfies all the criteria that we can expect: it has a real average energy and a nonvanishing trace. The average energy differs from Green's function energy by a small effect starting with fourth order in the coupling constant. Our state decays following a Markovian equation. There are no deviations from exponential decay neither for short nor for long times, as is the case for the bare state. The dressed state satisfies an uncertainty relation between energy and lifetime. We can also define dressed photon states and describe how the energy of the excited state is transmitted to the photons. There is another very important aspect: deviations from exponential decay would be in contradiction with indiscernibility as one could define, e.g., old mesons and young mesons according to their lifetime. This problem is solved by showing that quantum transitions are the result of two processes: a dressing process, discussed in a previous publication, and a decay process, which is much slower for electrodynamic systems. During the dressing process the unstable state is prepared. Then the dressed state decays in a purely exponential way. In the Hilbert space the two processes are not separated. Therefore it is not astonishing that we obtain for the unstable dressed state an irreducible density matrix outside the Liouville-Hilbert-space. This is a limit of Hilbert space states that are arbitrarily close to the decaying state. There are experiments that could verify our proposal. A typical one would be the study of the line shape, which is due to the superposition of the short-time process and the long-time process. The long-time process taken separately leads to a much sharper line shape, and avoids the divergence of the fluctuation predicted by the Lorentz line shape.

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I. INTRODUCTION

In particle theory or atomic physics the concepts of bare and dressed stable states are naturally introduced when the Hamiltonian H is split into a free part H_0 and and interaction λV. For example, for an isolated atom the bare stable state is the ground state. If the atom is coupled to an electromagnetic field, a new dressed ground state is obtained, corresponding to the atom surrounded by a cloud of virtual photons. Dressed stable states are eigenstates of the Hamiltonian and are related to bare states through a unitary transformation (we assume the interaction has a suitable form factor that avoids ultraviolet divergences). The eigenstates of H can be obtained by perturbation theory. The transition from bare to dressed ground states has been studied in a recent paper [1]. Here we consider excited states, which involve lifetimes.

For excited atomic states or unstable particles the situation changes dramatically. The usual perturbation expansion of the eigenstates of H is not applicable due to resonances [2,3]. Unstable particles emit decay products. No eigenstates of the Hamiltonian in the Hilbert space can describe this time-dependent behavior.

For this reason unstable particles and excited states have been regarded as approximate concepts. Unstable states are often identified with bare states, and are considered to be ''approximately stationary.'' As pointed out by Dirac [4]: '' . . . The fact that we had to use the word ''approximately'' in stating the conditions required for the phenomena of emission and absorption to be able to occur shows that these conditions are not expressible in exact mathematical language. One can give meaning to these phenomena only with reference to a perturbation method. They occur when the unperturbed system (of scatterer plus particle) has stationary states that are closed. The introduction of the perturbation spoils the stationary property of these states and gives rise to spontaneous emission and its converse absorption.''

While it is possible to speak about the bare states derived from the unperturbed Hamiltonian H_0, the difficulty is to introduce the dressing, which is necessary to have a consistent theory that incorporates, for example, the Van der Waals or Casimir-Polder forces between unstable atoms. In Ref. [5] Heitler has concluded that it is impossible to distinguish virtual photons (the dressing) from emitted photons: ''In fact, no exact definition of an isolated excited atomic state with a finite lifetime can be given at all.'' This is true in the usual quantum mechanics associated to a Hilbert space description, but we shall show that this definition can be obtained in the Liouville space associated to functions outside the Hilbert space. The dressed excited state we shall define is a part of a complete set of states that includes dressed photons as well as correlations.

In experimental situations the initial conditions that can be prepared belong to the Hilbert space. Therefore a dressed excited state outside the Hilbert space can indeed not exist in isolation. On the other hand in the Hilbert space there is necessarily a Zeno time [6] that leads to deviations from exponential. Nonexponential behavior, no matter how small, is in contradiction with indiscernibility. The main purpose of this paper is to lift this contradiction. Our main aim is therefore to obtain a dressed unstable state decaying in an exponential way. An initial condition belonging to the Hilbert space may then be written as a superposition of this decaying state plus additional components, which are associated with the preparation conditions and are responsible for the Zeno effect.

We require the dressed excited states to be a natural extension of the dressed stable states. In addition we require the following properties:

(1) Dressed unstable states, as well as dressed photons, are generated through a transformation operator Λ, starting from the corresponding bare states. This operator is obtained by analytic continuation of the unitary operator U that generates dressed stable states.

(2) The transformation Λ preserves the trace of the transformed density matrices.

(3) The transformation Λ preserves Hermiticity of density matrices.

(4) The transformation Λ is analytic with respect to the coupling constant λ at $\lambda = 0$.

(5) The dressed unstable state obeys a Markovian time evolution corresponding to the irreversible energy transfer from the dressed state to the decay products.

(6) The dressed unstable state has an energy uncertainty of the order of the inverse lifetime.

In this paper we shall show that one can construct unstable states that fulfill all these requirements.

In recent years some progress towards the construction of dressed unstable states has been achieved by considering analytic continuations of the wave-function space outside the Hilbert space [7,8] leading to Gamow vectors [9]. The dressed wavefunctions are related to the bare wave functions by a *nonunitary* transformation [2].[1] The dressed particle wave functions are eigenstates of the total Hamiltonian, with complex eigenvalues (the eigenvalues being poles of Green's energy function). These eigenstates, together with the dressed photon states, form a complex spectral representation of the Hamiltonian. This representation breaks time symmetry, as the particle states decay in a fixed direction of time [2]. The Gamow vector formulation has also been applied to second quantization [11,12] with creation and annihilation operators extended outside the usual Fock space.

Still, the decaying states obtained in this way have undesirable features. They have either a vanishing or a complex constant average energy. Since they are eigenstates of the Hamiltonian, their time evolution is independent of the dressed photon states. They cannot describe the energy transfer from particle to emitted photons.

These difficulties are avoided if we consider the extension of density operators (rather than wave functions) outside the Hilbert space.[2] Our unstable state is indeed given by a nonfactorizable density operator. Its construction is based on the complex-spectral representation of the Liouville operator $L_H \equiv [H,]$, whose eigenstates are generally not products of eigenstates of the Hamiltonian [15]. The Liouville operator, in spite of being Hermitian, can have complex eigenvalues corresponding to eigenstates outside the Hilbert space. Other applications of this method have been given in Refs. [15–17]. The idea of extending quantum mechanics on the level of density operators was introduced by the Brussels school, led by one of the authors (I.P.) [10,18–20]. This approach was already applied to the study of unstable particles [21–25], but there remained ambiguities, which we can now overcome.[3]

In Sec. II we introduce the Friedrichs model [2,3]. This model is a simplified version of a two-level atom interacting with a scalar field. In this model the interactions are simplified by the dipole approximation and by neglecting virtual transitions (the so-called ''rotating wave'' approximation[4]). In the Friedrichs model the discrete state may be either stable or unstable depending on whether the energy of this state is below or above a certain threshold. An advantage of this model is that it is exactly solvable, i.e., the eigenstates of the Hamiltonian can be explicitly found for both stable and unstable cases. This model may also be used as a simple model of unstable particles, radioactive nuclei, electron wave guides with resonant cavities [29], and other systems that may be described as a discrete state coupled to a continuum.

In Sec. III we present the Liouville-space formulation for the stable case of the Friedrichs model with no decay. We introduce the unitary transformation U in the Liouville space that relates bare states to dressed stable states. The main point is that we can write this transformation in terms of operators whose analytic continuation to the complex energy plane can be performed explicitly [15,30].

In Sec. IV we present the complex spectral representation of L_H. Using this representation, in Sec. V we consider the extension Λ of the unitary transformation U. We use analytic continuation to obtain a well-defined extension of the unitary operator.

In Sec. VI we apply conditions (1)–(6) to complete the

[1]This is a *star-unitary* transformation. Star-unitary transformations have been introduced in Ref. [10] and will be defined in Sec. IV in the Liouville-space formulation.

[2]The space of density operators (the Liouville space) has a richer structure than the space of wave functions [13,14].

[3]In Ref. [26] a heuristic construction of unstable states was introduced by imposing the block diagonality of the evolution operator. The unstable state given in Ref. [26] coincides up to fourth order in the coupling constant with the one obtained in this paper. See also Ref. [27].

[4]The model that includes virtual transitions is still solvable [28]. However we shall not consider it here because the essential points of our discussion can be made without virtual transitions.

extension of U. As we shall show the Λ transformation we obtain is ''star unitary.'' In Sec. VII we enumerate the new properties of the dressed excited state. An interesting result is that the average energy is different from the real part of the pole of the S-matrix (the so-called Green-function energy). The difference appears at fourth order in the coupling constant and is proportional to the decay rate of the particle. We derive the line shape of emission of photons and give the explicit formulation of the dressed unstable state and the dressed photon states. The line shape we obtain is generated by the relatively slow decay process. In contrast, the line shape associated with the bare state is dominated by the fast dressing process. Correspondingly, the energy uncertainty of our dressed state is of the order of the decay rate, while the energy uncertainty of the bare state is much larger, of the order or the ultaviolet cutoff of the form factor.

In this paper we consider global quantities such as the total energy or trace, which can be obtained from local densities by integration. In a subsequent paper [31] we shall discuss local quantities and we shall consider the space-time description of the emission process. This will allow us to identify the virtual photons involved in the dressing of the bare state and discuss the possibility of experiments where the line shape of the dressed excited state may be observed.

II. THE FRIEDRICHS MODEL

The Hamiltonian of the Friedrichs model is given by [2]

$$\begin{aligned} H &= H_0 + \lambda V \\ &= \omega_1 |1\rangle\langle 1| + \sum_k \omega_k |k\rangle\langle k| \\ &\quad + \lambda \sum_k V_k (|k\rangle\langle 1| + |1\rangle\langle k|), \end{aligned} \tag{2.1}$$

where

$$|1\rangle\langle 1| + \sum_k |k\rangle\langle k| = 1, \quad \langle \alpha|\alpha'\rangle = \delta_{\alpha\alpha'} \tag{2.2}$$

for $\alpha = 1$ or k, and λ is a dimensionless coupling constant. We assume the dispersion relation of the scalar field is given by

$$\omega_k = |k| \tag{2.3}$$

and V_k is real with the relation $V_k = V_{-k}$. As a convention we call the quanta of the field ''photons.'' The state $|1\rangle$ represents the atom (or particle) in its bare excited level, and no photons present, and $|k\rangle$ represents a bare photon of momentum k together with the atom in its ground level.

We consider a one-dimensional system enclosed in a box of size L with usual periodic boundary conditions, in the continous spectrum limit $L \to \infty$. Extension to three-dimensional cases is straightforward. We assume that

$$V_k = \left(\frac{2\pi}{L}\right)^{1/2} v_k \tag{2.4}$$

with $v_k \sim L^0$. This gives a consistent volume dependence in the limit $L \to \infty$. In this limit the field modes form a continuum and we have

$$\frac{2\pi}{L} \sum_k \to \int dk, \quad \frac{L}{2\pi} \delta_{k,0} \to \delta(k). \tag{2.5}$$

In this paper we assume v_k has a suitable form factor that eliminates ultraviolet divergences.

There are several distinct situations depending on the value of the energy ω_1 of the bare particle. Suppose at $t = 0$ the system is in the state $|1\rangle$. For $\lambda \neq 0$ this state will evolve in time. Then

(1) for $\omega_1 < 0$ the state $|1\rangle$ evolves towards a stable dressed state that represents the bare atom surrounded by a cloud of photons. The energy of the dressed state is lower than the energy of the bare state, and the excess energy is emitted away by an off-resonance process [1]. The eigenstates are analytic in the coupling constant at $\lambda = 0$.

(2) For $0 < \omega_1 < \omega_1^0$, where

$$\omega_1^0 \equiv \lambda^2 \int_{-\infty}^{\infty} dk \frac{v_k^2}{\omega_k} \tag{2.6}$$

in the continous limit, the bare state also evolves towards a dressed stable state, as in the previous case. However the eigenstates of the Hamiltonian are not analytic in the coupling constant at $\lambda = 0$. This situation has been discussed in the literature (see, for example, Ref. [32]) and we shall not consider it in this paper.

(3) For $\omega_1 > \omega_1^0$ the state $|1\rangle$ becomes unstable due to the resonance interaction and decays obeying an approximately exponential law, with the emission of photons. At the same time it also creates a cloud of virtual photons around the bare state [31].

Since the Hamiltonian is a bilinear form of bra and ket states, it is possible to find exact eigenstates and eigenvalues of the total Hamiltionian H [2] for both the stable and unstable cases. Let us briefly summarize the results.

A. Stable case ($\omega_1 < 0$)

For the stable case we have the eigenstates $|\overline{\phi_1}\rangle$ and $|\overline{\phi_k}\rangle$, which for $\lambda \to 0$ reduce to $|1\rangle$ and $|k\rangle$, respectively,

$$H|\overline{\phi_1}\rangle = \overline{\omega}_1 |\overline{\phi_1}\rangle, \quad H|\overline{\phi_k}\rangle = \omega_k |\overline{\phi_k}\rangle, \tag{2.7}$$

where $\overline{\omega}_1$ is the perturbed energy of the particle. Hereafter we use bars to distinguish expressions related to the stable case from the ones related to the unstable case.

The eigenstates $|\overline{\phi_1}\rangle$ and $|\overline{\phi_k}\rangle$ are stationary states, and they correspond to the dressed stable particle and dressed photons, respectively. For $|\overline{\phi_1}\rangle$ we have

$$|\overline{\phi_1}\rangle = \overline{N}_1^{1/2} \left[|1\rangle + \lambda \sum_k |k\rangle \overline{c_k} \right], \tag{2.8}$$

where

$$\bar{c}_k \equiv \frac{V_k}{\bar{\omega}_1 - \omega_k}. \tag{2.9}$$

$\bar{N}_1$ is a normalization constant given by

$$\bar{N}_1 \equiv (1+\bar{\xi})^{-1}, \quad \bar{\xi} \equiv \lambda^2 \sum_k \bar{c}_k^2, \tag{2.10}$$

where $\bar{\xi}$ is a real positive number. The dressed energy $\bar{\omega}_1$ is the solution of the equation

$$\eta(\bar{\omega}_1) = 0, \tag{2.11}$$

where

$$\eta(z) \equiv z - \omega_1 - \sum_k \frac{\lambda^2 V_k^2}{z - \omega_k}. \tag{2.12}$$

For $|\bar{\phi}_k\rangle$ we have

$$|\bar{\phi}_k\rangle = |k\rangle + \frac{\lambda V_k}{\eta^+(\omega_k)}\left[|1\rangle + \sum_l{}' \frac{\lambda V_l}{\omega_k - \omega_l + i\epsilon}|l\rangle\right], \tag{2.13}$$

where the prime on the summation sign denotes that $l \neq k$, the constant ϵ is a positive infinitesimal, $\epsilon \to 0+$, and $\eta^{\pm}(\omega_k) \equiv \eta(\omega_k \pm i\epsilon)$. The states with $+i\epsilon$ correspond to the "in" states in scattering theory.[5] In the limit $L \to \infty$ the denominators are interpreted as distributions, with the condition that the limit $L \to \infty$ is taken first and the limit $\epsilon \to 0+$ is taken later. With this convention we shall use summation signs [cf. Eq. (2.5)] unless the integration has to be explicitly displayed. Moreover, we shall not write the "limit" notation for ϵ to avoid too heavy notations. We have

$$|\bar{\phi}_1\rangle\langle\bar{\phi}_1| + \sum_k |\bar{\phi}_k\rangle\langle\bar{\phi}_k| = 1,$$
$$\langle\bar{\phi}_\alpha|\bar{\phi}_\beta\rangle = \delta_{\alpha\beta} \quad \text{for} \quad \alpha = 1 \quad \text{or} \quad k. \tag{2.14}$$

For sufficiently small λ one can expand $|\bar{\phi}_\alpha\rangle$ around $\lambda = 0$. For example to first order in λ we have

$$|\bar{\phi}_1\rangle = |1\rangle - \sum_k \frac{\lambda V_k}{\omega_k - \omega_1}|k\rangle + O(\lambda^2). \tag{2.15}$$

Since $\omega_1 < 0$, the denominator $\omega_k - \omega_1$ cannot be zero. Namely, there are no resonances and the expansion around $\lambda = 0$ is well defined. This implies that the system is integrable in the sense of Poincaré [2].

The dressing states can be generated from the bare states by a unitary transformation in the Hilbert space as

$$|\bar{\phi}_\alpha\rangle = u^{-1}|\alpha\rangle \quad \text{for} \quad \alpha = 1 \text{ or } k, \tag{2.16}$$

[5]We may also define a different set of states with $-i\epsilon$ corresponding to the "out" states.

where

$$u = \sum_\alpha |\alpha\rangle\langle\bar{\phi}_\alpha|, \quad u^{-1} = \sum_\alpha |\bar{\phi}_\alpha\rangle\langle\alpha|. \tag{2.17}$$

B. Unstable case

For the case $\omega_1 > 0$ there appear "Poincaré resonances" at $\omega_k = \omega_1$ in the perturbation expansion of the state $|\bar{\phi}_1\rangle$ in the continous limit [see Eq. (2.15)]. For the case $\omega_1 > \omega_1^0$ this state "disappears" and the continous states $|\bar{\phi}_k\rangle$ alone form a complete orthonormal set in the limit $L \to \infty$ [3]

$$\sum_k |\bar{\phi}_k\rangle\langle\bar{\phi}_k| \to 1. \tag{2.18}$$

In other words, there is no dressed unstable state in the Hilbert space that can be obtained by a unitary transformation acting on the bare state $|1\rangle$. This is consistent with the stationary character of the eigenstates of H in the Hilbert space, which cannot represent an unstable state that decays.

The dissapearence of $|\bar{\phi}_1\rangle$ may also be interpreted as the disappearance of the invariant of motion $|\bar{\phi}_1\rangle\langle\bar{\phi}_1|$. The other invariants $|\bar{\phi}_k\rangle\langle\bar{\phi}_k|$ become nonanalytic at $\lambda = 0$, due to the appearance of the absolute value squared of the factor $[\eta^+(\omega_k)]^{-1}$ in these invariants. The disappearance of one invariant of motion and the nonanalyticity of the other invariants indicates that the system is nonintegrable in the sense of Poincaré [26,33]. The states $|\bar{\phi}_k\rangle$ constitute the so-called "Friedrichs representation" [2].

We note that in the continuous spectrum limit $L \to \infty$ Green's function $[\eta^+(\omega)]^{-1}$ has a resonance pole on the lower half-complex plane at $\omega = z_1$, i.e.,

$$\eta^+(z_1) = z_1 - \omega_1 - \int_{-\infty}^{+\infty} dk \frac{\lambda^2 v_k^2}{(z-\omega_k)_{z_1}^+} = 0, \tag{2.19}$$

where

$$z_1 \equiv \tilde{\omega}_1 - i\gamma \tag{2.20}$$

with $\tilde{\omega}_1$ and γ real, $2\gamma > 0$ being the decay rate. Here, we have used the abbreviated notation

$$\frac{1}{(z-\omega_k)_{z_1}^+} \equiv \frac{1}{(z-\omega_k)^+}\Big|_{z=z_1} \tag{2.21}$$

to indicate that the propagator is evaluated on the second Riemann energy sheet of z (i.e., z is continued from the upper to the lower half-plane). In previous publications [2] this has been referred to as "delayed analytic continuation," as we first evaluate the integration on the upper half-plane of z and then substitute $z = z_1$ on the lower half-plane.

While the resonance pole gives the decaying contributions of the system, there is no quantum state in the Hilbert space corresponding to the resonance. However, the states associated with the resonance poles may be found outside the Hilbert space by analytic continuation of the Hamiltonian [2,7–9]. This corresponds to complex spectral representations of this operator. A detailed analysis can be found in Ref. [2]. Here we present only the main results. In the complex representation of H the eigenstates are not self-dual, and we have distinct right and left eigenstates,

$$H|\phi_1\rangle = z_1|\phi_1\rangle, \quad \langle\tilde{\phi}_1|H = \langle\tilde{\phi}_1|z_1, \tag{2.22a}$$

$$H|\phi_k\rangle = \omega_k|\phi_k\rangle, \quad \langle\tilde{\phi}_k|H = \langle\tilde{\phi}_k|\omega_k, \tag{2.22b}$$

where the eigenstates form a complete biorthonormal set

$$\sum_{\alpha=1,k}|\phi_\alpha\rangle\langle\tilde{\phi}_\alpha| = 1, \quad \langle\tilde{\phi}_\alpha|\phi_\beta\rangle = \delta_{\alpha\beta}. \tag{2.23}$$

Corresponding to the branch of the ‘‘in’’ states Eq. (2.13), the particle eigenstates are given by

$$|\phi_1\rangle = N_1^{1/2}\left[|1\rangle + \lambda\sum_k |k\rangle c_k\right], \tag{2.24}$$

$$\langle\tilde{\phi}_1| = N_1^{1/2}\left[\langle 1| + \lambda\sum_k c_k\langle k|\right], \tag{2.25}$$

where

$$c_k \equiv \frac{V_k}{(z-\omega_k)^+_{z_1}} \tag{2.26}$$

and

$$N_1 \equiv (1+\xi)^{-1}, \quad \xi \equiv \lambda^2\sum_k c_k^2. \tag{2.27}$$

In contrast to the stable case, ξ is a complex number. The states (2.24) and (2.25) are also called ‘‘Gamow vectors’’ [9].

In the complex spectral representation the photon eigenstates are given by

$$|\phi_k\rangle = |k\rangle + \frac{\lambda V_k}{\eta_d^+(\omega_k)}\left[|1\rangle + \sum_l{}' \frac{\lambda V_l}{\omega_k - \omega_l + i\epsilon}|l\rangle\right] \tag{2.28}$$

and [see Eq. (2.13)]

$$\langle\tilde{\phi}_k| = \langle\overline{\phi_k}| \neq \langle\phi_k|, \tag{2.29}$$

where

$$\frac{1}{\eta_d^+(\omega_k)} \equiv \frac{1}{\eta^+(\omega_k)}\,\frac{\omega_k - z_1}{(\omega_k - z)^+_{z1}}. \tag{2.30}$$

Using the state $|\phi_1\rangle$ and its dual we have the following four possibilites to construct factorizable density operators in terms of Gamow vectors

$$\rho_a = |\tilde{\phi}_1\rangle\langle\phi_1|, \quad \rho_b = |\phi_1\rangle\langle\tilde{\phi}_1|, \tag{2.31a}$$

$$\rho_c = |\phi_1\rangle\langle\phi_1|, \quad \rho_d = |\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|. \tag{2.31b}$$

Although these factorizable states bring us closer to the definition of a dressed excited state, they still do not satisfy our requirements stated in the Introduction. For example, the states ρ_a and ρ_b are invariants of motion and hence they are not good candidates to describe an unstable state that decays. Furthermore they have a complex expectation value z_1 of the Hamiltonian.[6] The other density operators ρ_c and ρ_d decay for $t>0$ and $t<0$, respectively, which is a characteristic of unstable states (the choice of either ρ_c or ρ_d depends on whether we want a state that decays in the future or in the past, respectively). However, these states have a vanishing trace and consequently they have a zero average energy [2]

$$\langle\phi_1|\phi_1\rangle = \langle\tilde{\phi}_1|\tilde{\phi}_1\rangle = 0, \quad \langle\phi_1|H|\phi_1\rangle = \langle\tilde{\phi}_1|H|\tilde{\phi}_1\rangle = 0. \tag{2.32}$$

These are the reasons why we seek the unstable dressed state in a more general space of density operators, i.e., a generalized Liouville space that is spanned by eigenstates with complex eigenvalues of the Liouville-von Neumann operator.

III. LIOUVILLE-SPACE FORMULATION FOR THE STABLE CASE

We first introduce the Liouville space for the stable case of the Friedrichs model (see Ref. [15] and references therein for the general formulation). The Liouville space is a vector space formed by the ordinary quantum-mechanical linear operators that act on wave functions. The Liouville-von Neumann superoperator (or ‘‘Liouvillian’’) is given by

$$L_H \equiv [H,\,] = H\times 1 - 1\times H, \tag{3.1}$$

where the operation $\times$ is defined by $(A\times B)\rho = A\rho B$ for arbitrary linear operators A, B, and ρ. We use the term ‘‘superoperator’’ to emphasize that L_H acts on operators. For example, the density operators evolve according to the Liouville equation

$$i\frac{\partial\rho}{\partial t} = L_H\rho. \tag{3.2}$$

Superoperators of the form $(A\times B)$ are called ‘‘factorizable superoperators.’’

Corresponding to the decomposition of the Hamiltonian in Eq. (2.1), we split L_H into a free part and an interaction part, $L_H = L_0 + \lambda L_V$. The dyads $|\alpha\rangle\langle\beta|$, which consist of the eigenvectors of the unperturbed Hamiltonian H_0 with $H_0|\alpha\rangle = \omega_\alpha|\alpha\rangle$, are eigenstates of the unperturbed Liouvillian L_0. Introducing the notation [15]

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[6] To these density operators we could associate both the well-defined real ‘‘energy’’ $\tilde{\omega}_1 = \mathrm{Re}(z_1)$ and the decay rate $2\gamma = 2|\mathrm{Im}(z_1)|$ at the same time. But this is not satisfactory from the point of view of the energy-time uncertainty relation. We shall come back to this uncertainty relation in Sec. VII.

$$|\alpha;\beta\rangle\rangle \equiv |\alpha\rangle\langle\beta| = [\langle\langle\alpha;\beta|]^\dagger \quad (3.3)$$

we have

$$L_0|\alpha;\beta\rangle\rangle = (\omega_\alpha - \omega_\beta)|\alpha;\beta\rangle\rangle. \quad (3.4)$$

These dyadic operators form a complete orthonormal set in the Liouville space,

$$\langle\langle\alpha';\beta'|\alpha;\beta\rangle\rangle = \delta_{\alpha',\alpha}\delta_{\beta'\beta}, \quad \sum_{\alpha;\beta} |\alpha;\beta\rangle\rangle\langle\langle\alpha;\beta| = 1 \quad (3.5)$$

with the inner product defined by

$$\langle\langle A|B\rangle\rangle \equiv \mathrm{Tr}(A^\dagger B). \quad (3.6)$$

Using this inner product, we can define Hermitian superoperators and unitary superoperators in the usual way. The matrix elements of a linear operator such as A are given by

$$A_{\alpha\beta} \equiv \langle\alpha|A|\beta\rangle = \langle\langle\alpha;\beta|A\rangle\rangle \quad (3.7)$$

while the matrix elements of a superoperator S are denoted by

$$S_{\alpha'\beta';\alpha\beta} \equiv \langle\langle\alpha';\beta'|S|\alpha;\beta\rangle\rangle. \quad (3.8)$$

For factorizable superoperators we have

$$(A\times B)_{\alpha'\beta';\alpha\beta} = A_{\alpha'\alpha}B_{\beta\beta'}. \quad (3.9)$$

For density matrices the diagonal elements give the probability to find the particle in the state $|1\rangle$ or the field in a mode $|k\rangle$, while the off-diagonal elements give information on the quantum correlations between particle and field, or among fields. The interaction changes the state of the correlations. Hence, in the density-matrix formulation, there appears naturally a ‘‘dynamics of correlations’’ [18].

To formulate this more precisely, let us first introduce the concept of the ‘‘vacuum-of-correlations subspace’’ that is the set of diagonal dyads $|\alpha\rangle\langle\alpha|$. We then introduce an integer d that specifies the degree of correlation. This is defined as the minimum number d of successive interactions λL_V by which a given dyadic state can reach the vacuum of correlation. For example, the dyadic states $|1\rangle\langle k|$ and $|k\rangle\langle 1|$ corresponding to particle-field correlations have $d=1$, while the dyads $|k\rangle\langle k'|$ corresponding to field-field correlations have $d=2$. For the Friedrichs model $d=2$ is the maximum value of the degree of correlation.

We introduce the projection operators $P^{(\nu)}$,

$$P^{(0)} \equiv \sum_{\alpha=1,k} |\alpha;\alpha\rangle\rangle\langle\langle\alpha;\alpha|,$$

$$P^{(\alpha\beta)} \equiv |\alpha;\beta\rangle\rangle\langle\langle\alpha;\beta| \quad (\alpha\neq\beta), \quad (3.10)$$

which are orthogonal and complete [cf. Eq. (3.5)]:

$$P^{(\mu)}P^{(\nu)} = P^{(\mu)}\delta_{\mu\nu}, \quad \sum_\nu P^{(\nu)} = 1 \quad (3.11)$$

with $(\nu)=(0)$ or $(\alpha\beta)$. The projector $P^{(0)}$ corresponds to the vacuum of correlations subspace, while the projectors $P^{(k1)}$ and $P^{(1k)}$ correspond to the $d=1$ subspace and $P^{(kk')}$ to the $d=2$ subspace. The complement projectors $Q^{(\nu)}$ are defined by

$$P^{(\nu)} + Q^{(\nu)} = 1. \quad (3.12)$$

They are orthogonal to $P^{(\nu)}$, i.e., $Q^{(\nu)}P^{(\nu)} = P^{(\nu)}Q^{(\nu)} = 0$, and satisfy $[Q^{(\nu)}]^2 = Q^{(\nu)}$. The bare projectors $P^{(\nu)}$ commute with L_0 and they are eigenprojectors of L_0,

$$[P^{(\nu)}, L_0] = 0, \quad L_0P^{(\nu)} = w^{(\nu)}P^{(\nu)}, \quad (3.13)$$

where $w^{(\nu)}$ are the eigenvalues

$$w^{(0)} = 0, \quad w^{(\alpha\beta)} = \omega_\alpha - \omega_\beta. \quad (3.14)$$

The interaction L_V leads to a transition between two different correlations,

$$(L_V)_{\alpha\beta;\alpha'\beta'} = V_{\alpha\alpha'}\delta_{\beta'\beta} - \delta_{\alpha\alpha'}V_{\beta'\beta}. \quad (3.15)$$

For the interaction in Eq. (2.1) only one index is changed in this transition.

We note that the subspace $P^{(0)}$ is a degenerate subspace as any state $|\alpha;\alpha\rangle\rangle$ has the same eigenvalue $w^{(0)}=0$. In general we may write a degenerate subspace as

$$P^{(\nu)} = \sum_j |\nu_j\rangle\rangle\langle\langle\nu_j|, \quad (3.16)$$

where j is a degeneracy index. In our case we have $P^{(0)} = \Sigma_{\alpha=1,k}|0_\alpha\rangle\rangle\langle\langle 0_\alpha|$, where $|0_\alpha\rangle\rangle \equiv |\alpha;\alpha\rangle\rangle$. The subspaces with $\nu\neq 0$, being nondegenerate, are simply given by $P^{(\nu)} = |\nu\rangle\rangle\langle\langle\nu|$. To simplify the notations we may write the bare dyads as $|\nu_j\rangle\rangle$, with the understanding that the index $j=\alpha$ appears only for the $\nu=0$ subspace.[7] For example, the eigenvalue equation Eq. (3.4) is written as $L_0|\nu_j\rangle\rangle = w^{(\nu)}|\nu_j\rangle\rangle$.

We now turn to the eigenstates of L_H. They form the basis used to perform the analytic continuation from the stable case to the unstable case. The eigenstates of L_H are given by the dyads of dressed states $|\bar{\phi}_\alpha, \bar{\phi}_\beta\rangle\rangle = |\bar{\phi}_\alpha\rangle\langle\bar{\phi}_\beta|$. We denote them as [15]

$$|\bar{F}^0_\alpha\rangle\rangle \equiv |\bar{\phi}_\alpha;\bar{\phi}_\alpha\rangle\rangle, \quad |\bar{F}^{\alpha\beta}\rangle\rangle \equiv |\bar{\phi}_\alpha;\bar{\phi}_\beta\rangle\rangle \quad (\alpha\neq\beta). \quad (3.17)$$

We have

$$L_H|\bar{F}^\nu_j\rangle\rangle = \bar{\omega}^{(\nu)}|\bar{F}^\nu_j\rangle\rangle, \quad (3.18)$$

[7]In the Friedrichs model, the other subspaces have accidental degeneracies, such as $\omega_k - \omega_{k'} = \omega_l - \omega_{l'}$ with $l\neq k$ and $l'\neq k'$. However, these degeneracies are negligible as they give higher-order contributions in powers of L^{-1} in the limit $L\to\infty$.

where $\overline{w^{(\alpha\beta)}} \equiv \overline{\omega}_\alpha - \overline{\omega}_\beta$, with $\overline{\omega}_k \equiv \omega_k$ and $\overline{w^{(0)}} \equiv 0$. The eigenstates of L_H form a complete orthonormal set in the Liouville space

$$\sum_\nu \sum_j |\overline{F_j^\nu}\rangle\rangle\langle\langle\overline{F_j^\nu}| = 1, \quad \langle\langle\overline{F_{j'}^{\nu'}}|\overline{F_j^\nu}\rangle\rangle = \delta_{jj'}\delta_{\nu\nu'}. \tag{3.19}$$

Associated with these eigenstates we have the projectors $\overline{\Pi}^{(\nu)}$,

$$\overline{\Pi}^{(\nu)} \equiv \sum_j |\overline{F_j^\nu}\rangle\rangle\langle\langle\overline{F_j^\nu}|, \tag{3.20}$$

which satisfy the relations

$$\overline{\Pi}^{(\mu)}\overline{\Pi}^{(\nu)} = \overline{\Pi}^{(\mu)}\delta_{\mu\nu}, \quad \sum_\nu \overline{\Pi}^{(\nu)} = 1, \tag{3.21}$$

$$[\overline{\Pi}^{(\nu)}, L_H] = 0, \quad \lim_{\lambda\to 0}\overline{\Pi}^{(\nu)} = P^{(\nu)}. \tag{3.22}$$

The bare and dressed dyadic states are related by

$$|\overline{\phi_\alpha}\rangle\langle\overline{\phi_\beta}| = u^{-1}|\alpha\rangle\langle\beta|u, \tag{3.23}$$

where u is given in Eq. (2.17). Using the Liouville-space notations we write this relation as

$$|\overline{\phi_\alpha};\overline{\phi_\beta}\rangle\rangle = U^{-1}|\alpha;\beta\rangle\rangle \tag{3.24}$$

or

$$|\overline{F_j^\nu}\rangle\rangle = U^{-1}|\nu_j\rangle\rangle, \tag{3.25}$$

where U^{-1} is a factorizable unitary superoperator in the Liouville space,

$$U^{-1} = u^{-1}\times u \tag{3.26}$$

that transforms bare dyads into dressed ones. Equation (3.25) together with Eq. (3.16) and Eq. (3.20) lead to the similitude relation

$$\overline{\Pi}^{(\nu)} = U^{-1}P^{(\nu)}U. \tag{3.27}$$

The eigenstates of L_H may be written in terms of ''kinetic'' operators. This will allow us to obtain their analytic continuation in the unstable case. We first decompose the unitary operator U^{-1} in two components,

$$\begin{aligned}\overline{\chi}^{(\nu)} &\equiv P^{(\nu)}U^{-1}P^{(\nu)},\\ \overline{C}^{(\nu)}\overline{\chi}^{(\nu)} &\equiv Q^{(\nu)}U^{-1}P^{(\nu)}.\end{aligned} \tag{3.28}$$

We have as well the Hermitian conjugate components

$$\begin{aligned}[\overline{\chi}^{(\nu)}]^\dagger &\equiv P^{(\nu)}UP^{(\nu)},\\ [\overline{\chi}^{(\nu)}]^\dagger\overline{D}^{(\nu)} &\equiv P^{(\nu)}UQ^{(\nu)},\end{aligned} \tag{3.29}$$

where $\overline{D}^{(\nu)} \equiv [\overline{C}^{(\nu)}]^\dagger$. The superoperator $\overline{C}^{(\nu)}$ is an ''off-diagonal'' superoperator, as it describes off-diagonal transitions $\overline{C}^{(\nu)} = Q^{(\nu)}\overline{C}^{(\nu)}P^{(\nu)}$ from the $P^{(\nu)}$ correlation subspace to the $Q^{(\nu)}$ subspace. By operating $\overline{C}^{(\nu)}$ on the ν correlation subspace $P^{(\nu)}$, this operator creates correlations other than the ν correlation. In particular $\overline{C}^{(0)}$ creates higher correlations from the vacuum of correlations. For this reason the $\overline{C}^{(\nu)}$ are generally called ''creation-of-correlations'' superoperators [15,18], or creation operators in short. Conversely, the $\overline{D}^{(\nu)}$ are called destruction operators.

The superoperator $\overline{\chi}^{(\nu)}$ is ''diagonal,'' as it describes a diagonal transition between states belonging to the same subspace $P^{(\nu)}$.

Using Eq. (3.12) we have

$$\begin{aligned}U^{-1}P^{(\nu)} &= (P^{(\nu)} + \overline{C}^{(\nu)})\overline{\chi}^{(\nu)},\\ P^{(\nu)}U &= [\overline{\chi}^{(\nu)}]^\dagger(P^{(\nu)} + \overline{D}^{(\nu)}).\end{aligned} \tag{3.30}$$

The eigenstates of L_H are then written in the form

$$|\overline{F_j^\nu}\rangle\rangle = (P^{(\nu)} + \overline{C}^{(\nu)})|f_j^\nu\rangle\rangle, \quad \langle\langle\overline{F_j^\nu}| = \langle\langle f_j^\nu|(P^{(\nu)} + \overline{D}^{(\nu)}), \tag{3.31}$$

where $|f_j^\nu\rangle\rangle \equiv \overline{\chi}^{(\nu)}|\nu_j\rangle\rangle$. Note that $P^{(\nu)}|\overline{F_j^\nu}\rangle\rangle = |f_j^\nu\rangle\rangle$ and $Q^{(\nu)}|\overline{F_j^\nu}\rangle\rangle = \overline{C}^{(\nu)}|f_j^\nu\rangle\rangle$. Hence the $Q^{(\nu)}$ component of $|\overline{F_j^\nu}\rangle\rangle$ is a functional of the $P^{(\nu)}$ component,

$$Q^{(\nu)}|\overline{F_j^\nu}\rangle\rangle = \overline{C}^{(\nu)}P^{(\nu)}|\overline{F_j^\nu}\rangle\rangle. \tag{3.32}$$

Similarly the for the left eigenstates of L_H we have

$$\langle\langle\overline{F_j^\nu}|Q^{(\nu)} = \langle\langle\overline{F_j^\nu}|P^{(\nu)}\overline{D}^{(\nu)}. \tag{3.33}$$

The $\overline{\Pi}^{(\nu)}$ projectors in Eq. (3.27) can also be written in terms of the kinetic operators as

$$\overline{\Pi}^{(\nu)} = (P^{(\nu)} + \overline{C}^{(\nu)})\overline{A}^{(\nu)}(P^{(\nu)} + \overline{D}^{(\nu)}), \tag{3.34}$$

where

$$\overline{A}^{(\nu)} \equiv \overline{\chi}^{(\nu)}[\overline{\chi}^{(\nu)}]^\dagger. \tag{3.35}$$

From the relation $\overline{\Pi}^{(\nu)}U^{-1} = U^{-1}P^{(\nu)}$ we obtain

$$\begin{aligned}\overline{A}^{(\nu)} &= P^{(\nu)}[(P^{(\nu)} + \overline{D}^{(\nu)})(P^{(\nu)} + \overline{C}^{(\nu)})]^{-1}P^{(\nu)}\\ &= P^{(\nu)}[P^{(\nu)} + \overline{D}^{(\nu)}\overline{C}^{(\nu)}]^{-1}P^{(\nu)}.\end{aligned} \tag{3.36}$$

While in the stable case the dressed states and the eigenstates of L_H are the same, we shall see that in the unstable case the analytic continuation of U leads to a distinction between the dressed states and the eigenstates of L_H. For this reason it is convenient to introduce the notation $|\overline{\rho_j^\nu}\rangle\rangle \equiv U^{-1}|\nu_j\rangle\rangle$. The dressed particle and dressed photon states are, respectively, given by

$$|\overline{\rho_\alpha^0}\rangle\rangle = U^{-1}|\alpha;\alpha\rangle\rangle, \quad \alpha = 1,k \tag{3.37}$$

while the states $|\overline{\rho^{\alpha\beta}}\rangle\rangle = U^{-1}|\alpha;\beta\rangle\rangle$ with $\alpha\neq\beta$ are dressed correlations. These states satisfy

$$\overline{\Pi}^{(\nu)}|\overline{\rho_j^\nu}\rangle\rangle = |\overline{\rho_j^\nu}\rangle\rangle. \tag{3.38}$$

The identity $|\overline{\rho_j^\nu}\rangle\rangle = |\overline{F_j^\nu}\rangle\rangle$ holds only in the stable case. For the extended states that will be defined in the next sections we have in general $|\rho_j^\nu\rangle\rangle \neq |F_j^\nu\rangle\rangle$.

The explicit forms of the matrix elements of $\overline{C}^{(\nu)}$, $\overline{D}^{(\nu)}$, and $\overline{\chi}^{(\nu)}$ may be easily constructed from $|\overline{\phi}_1\rangle$ in Eq. (2.8) and $|\overline{\phi}_k\rangle$ in Eq. (2.13). For example, let us consider the relation

$$\begin{aligned}\langle\langle\alpha;\alpha|\overline{\rho_1^0}\rangle\rangle &= \langle\alpha|\overline{\phi}_1\rangle\langle\overline{\phi}_1|\alpha\rangle\\ &= \langle\langle\alpha;\alpha|(P^{(0)}+\overline{C}^{(0)})\overline{\chi}^{(0)}|1;1\rangle\rangle\\ &= \langle\langle\alpha;\alpha|\overline{\chi}^{(0)}|1,1\rangle\rangle.\end{aligned} \tag{3.39}$$

Then, using Eq. (2.8) we obtain

$$\overline{\chi}^{(0)}_{11;11} = \frac{1}{1+\overline{\xi}}, \quad \overline{\chi}^{(0)}_{kk;11} = \frac{\lambda^2\overline{c}_k^2}{1+\overline{\xi}}. \tag{3.40}$$

Similarly, we have

$$\begin{aligned}\langle\langle\alpha;\beta|\overline{\rho_1^0}\rangle\rangle &= \langle\alpha|\overline{\phi}_1\rangle\langle\overline{\phi}_1|\beta\rangle\\ &= \langle\langle\alpha;\beta|\overline{C}^{(0)}\overline{\chi}^{(0)}|1;1\rangle\rangle\\ &= \overline{C}^{(0)}_{\alpha\beta;11}\overline{\chi}^{(0)}_{11;11} + \sum_k \overline{C}^{(0)}_{\alpha\beta;kk}\overline{\chi}^{(0)}_{kk;11}.\end{aligned} \tag{3.41}$$

Due to the volume dependence of the interaction in Eq. (2.4) one can easily verify that the last term in the second line is $O(1/L)$ smaller than the first term. Neglecting the last term in the limit of $L\to\infty$, we obtain

$$\overline{C}^{(0)}_{k1;11} = \overline{C}^{(0)}_{1k;11} = \lambda\overline{c}_k, \quad \overline{C}^{(0)}_{kk';11} = \overline{C}^{(0)}_{k1;11}\overline{C}^{(0)}_{1k';11}. \tag{3.42}$$

The analytic continuation to the unstable case of the kinetic operators $\overline{C}^{(\nu)}$ and $\overline{D}^{(\nu)}$ has already been studied (see, for example, Refs. [10,15,22]), and will be summarized in Sec. IV. The analytic continuation of $\overline{\chi}^{(\nu)}$ will be considered in Secs. V and VI.

In summary the dressed stable states have been represented by factorizable density matrices, generated by a factorizable unitary superoperator. This is a direct consequence of the factorizablility of the eigenstates of the Liouvillian L_H in terms of eigenstates of the Hamiltonian for the stable case. We have established as well a relation between unitary transformations and the operators used in kinetic theory.

IV. COMPLEX SPECTRAL REPRESENTATIONS OF L_H

For the unstable case the system admits complex spectral representations of L_H in a nonHilbert space. In these representations the eigenstates are nonfactorizable into a product of wave functions, and the eigenvalues are generally complex, which breaks time symmetry [15].

As in the stable case we start with the eigenvalue equation for L_H as follows:

$$L_H|F_j^\nu\rangle\rangle = z_j^{(\nu)}|F_j^\nu\rangle\rangle, \quad \langle\langle\tilde{F}_j^\nu|L_H = \langle\langle\tilde{F}_j^\nu|z_j^{(\nu)}. \tag{4.1}$$

For the unstable case the eigenvalues $z_j^{(\nu)}$ are generally complex numbers. Since L_H is Hermitian this is only possible if the corresponding eigenstates have no Hilbert norm. For this reason the eigenvalue problem we consider corresponds to an extension of L_H outside the Hilbert space. In this extension we have $\langle\langle\tilde{F}_j^\nu| \neq \langle\langle F_j^\nu|$, similar to Eq. (2.29).

We assume these eigenstates satisfy the biorthogonality and bicompleteness relations

$$\langle\langle\tilde{F}_i^\nu|F_j^\mu\rangle\rangle = \delta_{\nu,\mu}\delta_{i,j}, \quad \sum_\nu\sum_j |F_j^\nu\rangle\rangle\langle\langle\tilde{F}_j^\nu| = 1. \tag{4.2}$$

For the Friedrichs model, these relations can be explicitly verified.

Similarly to the stable case in Eq. (3.32) we insist on a functional relation between the $Q^{(\nu)}$ and $P^{(\nu)}$ components of the eigenstates as

$$\begin{aligned}Q^{(\nu)}|F_j^\nu\rangle\rangle &= [N_j^{(\nu)}]^{1/2}C^{(\nu)}|u_j^\nu\rangle\rangle,\\ \langle\langle\tilde{F}_j^\nu|Q^{(\nu)} &= \langle\langle\tilde{v}_j^\nu|D^{(\nu)}[N_j^{(\nu)}]^{1/2},\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}|u_j^\nu\rangle\rangle &\equiv [N_j^{(\nu)}]^{-1/2}P^{(\nu)}|F_j^\nu\rangle\rangle,\\ \langle\langle\tilde{v}_j^\nu| &\equiv \langle\langle\tilde{F}_j^\nu|P^{(\nu)}[N_j^{(\nu)}]^{-1/2}\end{aligned} \tag{4.4}$$

and $N_j^{(\nu)}$ is a normalization constant. This leads to the formal expression for the left and right eigenstates of the Liouvillian

$$|F_j^\nu\rangle\rangle = [N_j^{(\nu)}]^{1/2}\Phi_C^{(\nu)}|u_j^\nu\rangle\rangle, \tag{4.5a}$$

$$\langle\langle\tilde{F}_j^\nu| = \langle\langle\tilde{v}_j^\nu|\Phi_D^{(\nu)}[N_j^{(\nu)}]^{1/2}, \tag{4.5b}$$

where

$$\Phi_C^{(\nu)} \equiv P^{(\nu)} + C^{(\nu)}, \tag{4.6a}$$

$$\Phi_D^{(\nu)} \equiv P^{(\nu)} + D^{(\nu)}. \tag{4.6b}$$

We have the relations

$$[\Phi_C^{(\nu)}]^2 = \Phi_C^{(\nu)}, \quad [\Phi_D^{(\nu)}]^2 = \Phi_D^{(\nu)}. \tag{4.7}$$

Substituting Eq. (4.5a) into the first equation in Eq. (4.1), we obtain

$$L_H\Phi_C^{(\nu)}|u_j^\nu\rangle\rangle = z_j^{(\nu)}\Phi_C^{(\nu)}|u_j^\nu\rangle\rangle. \tag{4.8}$$

Multiplying $P^{(\nu)}$ from the left on both sides of Eq. (4.8), we have

$$\theta_C^{(\nu)}|u_j^\nu\rangle\rangle = z_j^{(\nu)}|u_j^\nu\rangle\rangle, \tag{4.9}$$

where

$$\theta_C^{(\nu)} \equiv P^{(\nu)} L_H \Phi_C^{(\nu)} = w^{(\nu)} P^{(\nu)} + P^{(\nu)} \lambda L_V \Phi_C^{(\nu)} P^{(\nu)} \tag{4.10}$$

is the *collision operator* associated with the creation operator $C^{(\nu)}$. The collision operators are generally non-Hermitian dissipative operators. These operators play a central role in nonequilibrium situations. For example, the collision operator associated with the vacuum of correlations $\nu=0$ leads to the collision operator in the well-known Pauli master equation for weakly coupled systems [15].

Equation (4.9) shows that $|u_j^\nu\rangle\rangle$ is an eigenstate of $\theta_C^{(\nu)}$, with the *same eigenvalues* as L_H. This indicates that through the extension of L_H outside the Hilbert space, quantum mechanics can be connected with dissipative dynamics.

The fact that L_H and $\theta_C^{(\nu)}$ share the same eigenvalues also implies that $\Phi_C^{(\nu)}$ satisfies the intertwining relation with L_H and $\theta_C^{(\nu)}$ [see Eq. (4.8)],

$$L_H \Phi_C^{(\nu)} = \Phi_C^{(\nu)} \theta_C^{(\nu)}. \tag{4.11}$$

It can be similarly shown that the states $\langle\langle \tilde{v}_j^{(\nu)}|$ are left eigenstates of the collision operators

$$\theta_D^{(\nu)} \equiv w^{(\nu)} P^{(\nu)} + P^{(\nu)} \Phi_D^{(\nu)} \lambda L_V P^{(\nu)} \tag{4.12}$$

associated to the destruction operator $D^{(\nu)}$. These operators also share the same eigenvalues with L_H and lead to the intertwining relation $\Phi_D^{(\nu)} L_H = \theta_D^{(\nu)} \Phi_D^{(\nu)}$.

As in the stable case, we introduce the projectors

$$\Pi^{(\nu)} = \sum_j |F_j^\nu\rangle\rangle\langle\langle \tilde{F}_j^\nu|. \tag{4.13}$$

These projectors commute with L_H and are complete and orthonormal [cf. Eq. (4.2)]. We require that they are analytic at $\lambda=0$ as $\lim_{\lambda\to 0}\Pi^{(\nu)} = P^{(\nu)}$ [see Eq. (3.22)].

Similar to the stable case we can write the projectors in the form [10]

$$\Pi^{(\nu)} = (P^{(\nu)} + C^{(\nu)}) A^{(\nu)} (P^{(\nu)} + D^{(\nu)}), \tag{4.14}$$

where

$$A^{(\nu)} = P^{(\nu)} [P^{(\nu)} + D^{(\nu)} C^{(\nu)}]^{-1} P^{(\nu)}. \tag{4.15}$$

Equation (4.13) shows that the $\Pi^{(\nu)}$ are not Hermitian operators, i.e.,

$$(\Pi^{(\nu)})^\dagger \neq \Pi^{(\nu)}, \tag{4.16}$$

which is in contrast to the stable case. In fact, $\Pi^{(\nu)}$ are star Hermitian, as will be defined below.

All the above expressions are still formal, since $\Phi_C^{(\nu)}$ and $\Phi_D^{(\nu)}$ in Eq. (4.6) are not yet determined. One can find their explicit form from the intertwining relation Eq. (4.11). Multiplying $\Phi_C^{(\nu)}$ from the left on the both sides of this relation and using Eq. (4.7) we obtain

$$\Phi_C^{(\nu)} L_H \Phi_C^{(\nu)} = L_H \Phi_C^{(\nu)}. \tag{4.17}$$

This is a closed nonlinear equation for $\Phi_C^{(\nu)}$, which leads to

$$(\omega^{(\nu)} - L_0)\Phi_C^{(\nu)} = \lambda L_V \Phi_C^{(\nu)} - \Phi_C^{(\nu)} \lambda L_V \Phi_C^{(\nu)}. \tag{4.18}$$

Multiplying $(\omega^{(\nu)} - L_0)^{-1}$ from the left on the both sides of this equation, and adding a suitable infinitesimal $-i\epsilon_{\mu\nu}$ to regularize the denominator, we obtain a nonlinear integral equation for $\Phi_C^{(\nu)}$, i.e., the so-called "nonlinear Lippmann-Schwinger equation," [15,35]

$$\Phi_C^{(\nu)} = P^{(\nu)} + \sum_{\mu(\neq\nu)} P^{(\mu)} \frac{-1}{w^{(\mu)} - w^{(\nu)} - i\epsilon_{\mu\nu}} \times [\lambda L_V \Phi_C^{(\nu)} - \Phi_C^{(\nu)} \lambda L_V \Phi_C^{(\nu)}] P^{(\nu)}. \tag{4.19}$$

The nonlinear term (the second term inside brackets) is related to the collision operator through Eq. (4.10). As we have seen the collision operators have complex eigenvalues, associated with dissipative effects. The complex eigenvalues are incorporated into the solutions of Eq. (4.19) through the nonlinear term.

In order to determine the sign of the infinitesimals $\epsilon_{\mu\nu}$ for the unstable case, we require that the $\Phi_C^{(\nu)}$ are analytic at $\lambda=0$ and that the complex eigenvalues of L_H have imaginary parts with a definite sign (so that the corresponding eigenstates decay for either $t>0$ or $t<0$). This leads to

$$\begin{aligned} \epsilon_{\mu\nu} &= +\epsilon \quad \text{if } d_\mu \geqslant d_\nu, \\ \epsilon_{\mu\nu} &= -\epsilon \quad \text{if } d_\mu < d_\nu, \end{aligned} \tag{4.20}$$

where ϵ is an infinitesimal [20,30,36]. Here, d_ν is the degree of correlation of the $P^{(\nu)}$ subspace, which has been defined in Sec. III. For $\epsilon>0$ the eigenstates will decay for $t>0$ while for $\epsilon<0$ they decay for $t<0$.

For the stable case one can derive nonlinear equations similar to Eq. (4.19). In this case the imaginary parts of the eigenvalues of the collision operator vanish and the nonlinear terms lead simply to the energy shift of the particle.

Similar to Eq. (4.19) one can obtain nonlinear integral equations for the destruction operators

$$\Phi_D^{(\nu)} = P^{(\nu)} + P^{(\nu)} [\Phi_D^{(\nu)} \lambda L_V - \Phi_D^{(\nu)} \lambda L_V \Phi_D^{(\nu)}] \times \sum_{\mu(\neq\nu)} P^{(\mu)} \frac{1}{w^{(\nu)} - w^{(\mu)} - i\epsilon_{\nu\mu}}. \tag{4.21}$$

The choice of analytic continuation in Eq. (4.20) leads to a biorthogonal and bicomplete set of eigenstates of L_H.

Recall that we have the relation $\overline{D}^{(\nu)} = [\overline{C}^{(\nu)}]^\dagger$ for the stable case. For the unstable case, due to the different analytic continuation from the stable case [cf. Eqs. (4.19) and (4.21)], these operators are no more related by Hermitian conjugation. However, we can introduce a different type of conjugation called *star conjugation* denoted by "*," which is obtained by Hermitian conjugation plus the change $\epsilon_{\mu\nu} \Rightarrow \epsilon_{\nu\mu}$. Then we have

$$D^{(\nu)} = [C^{(\nu)}]^*, \quad A^{(\nu)} = [A^{(\nu)}]^* \tag{4.22}$$

and

$$\Pi^{(\nu)} = [\Pi^{(\nu)}]^*, \tag{4.23}$$

i.e., the destruction operator is star-conjugate to the creation operator, and $A^{(\nu)}$ and $\Pi^{(\nu)}$ are star-Hermitian operators. Correspondingly, the star conjugation in the eigenstates of the Liouvillian leads to

$$(|F_j^\nu\rangle\rangle)^* = \langle\langle \tilde{F}_j^\nu|. \tag{4.24}$$

In Appendix A we give a solution of the nonlinear equation Eq. (4.19) for the Friedrichs model. For the components $C^{(0)}_{\alpha\beta;11}$ we obtain

$$C^{(0)}_{kk';11} = C^{(0)}_{k1;11} C^{(0)}_{1k';11}, \tag{4.25a}$$

$$C^{(0)}_{k1;11} = [C^{(0)}_{1k;11}]^{\text{c.c.}} = \lambda c_k, \tag{4.25b}$$

where c_k is defined in Eq. (2.26). This coincides with the results obtained by de Haan and Henin by an alternative approach based on the resolvent formulation of the Liouville operator [30]. Other matrix elements of the creation operators presented in Ref. [30] are shown in Appendix B. By a direct substitution, one can verify these components satisfy our nonlinear Lippmann–Schwinger equations.

Expressions for matrix elements of $D^{(\nu)}$ are obtained by using the star-Hermiticity relation $D^{(\nu)} = C^{(\nu)*}$. As the operators L_0, L_V that appear in $C^{(\nu)}$ and $D^{(\nu)}$ are hermitian, the star conjugation is obtained by replacing $i\epsilon_{\mu\nu} \Rightarrow (i\epsilon_{\mu\nu})^* = -i\epsilon_{\nu\mu}$ and taking the transpose. For example, for $\nu=0$ we have $\epsilon_{0\mu} = -\epsilon_{\mu 0}$ and $(i\epsilon_{\mu 0})^* = i\epsilon_{\mu 0}$. This leads to

$$D^{(0)}_{11;\alpha\beta} = C^{(0)}_{\alpha\beta;11}. \tag{4.26}$$

In Appendix B we give the explicit form of eigenstates of the collision operators as well as L_H for the Friedrichs model. Here we only write the eigenvalues

$$L_H|F_1^0\rangle\rangle = -2i\gamma|F_1^0\rangle\rangle, \quad L_H|F_k^0\rangle\rangle = O(1/L) \to 0, \tag{4.27a}$$

$$L_H|F^{\alpha\beta}\rangle\rangle = z^{(\alpha\beta)}|F^{\alpha\beta}\rangle\rangle, \quad \alpha \neq \beta, \tag{4.27b}$$

where

$$z^{(1k)} = z_1 - \omega_k,$$
$$z^{(k1)} = \omega_k - z_1^{\text{c.c.}}, \tag{4.28}$$
$$z^{(kk')} = \omega_k - \omega_{k'}.$$

The eigenvalues are obtained from Eq. (B8), together with the explicit forms of the creation operators given in Eq. (B11). In Eq. (4.27) we have neglected terms of order L^{-1} that vanish in the continuous spectrum limit $L \to \infty$.

Since $\text{Tr}(L_H\rho) = \text{Tr}(H\rho) - \text{Tr}(\rho H) = 0$ for any ρ, for the eigenstates of L_H we have $\text{Tr}(L_H F_\alpha^\nu) = z_\alpha^{(\nu)} \text{Tr}(F_\alpha^\nu) = 0$. This means that the eigenstates of L_H with nonzero eigenvalue are traceless. Hence, we cannot identify an isolated decaying eigenstate of L_H with a dressed unstable particle state [cf. condition (2) in the Introduction].

To end this section we now show that the eigenstates of the Liouvillian lead to a Markovian time evolution of the system. Indeed Eq. (4.14) together with Eq. (4.11) leads to the relation

$$e^{-iL_H t}\Pi^{(\nu)} = \Pi^{(\nu)} e^{-iL_H t}$$
$$= (P^{(\nu)} + C^{(\nu)}) e^{-i\theta_C^{(\nu)} t} A^{(\nu)} (P^{(\nu)} + D^{(\nu)}). \tag{4.29}$$

We call the component $P^{(\nu)}\rho^{(\nu)}(t)$ the ''privileged'' component of

$$\rho^{(\nu)}(t) \equiv \Pi^{(\nu)} e^{-iL_H t}\rho(0). \tag{4.30}$$

Taking the time derivative of the privileged component and using Eq. (4.29), we obtain the Markovian kinetic equation for each $\Pi^{(\nu)}$ subspace,

$$i\frac{\partial}{\partial t} P^{(\nu)}\rho^{(\nu)}(t) = \theta_C^{(\nu)} P^{(\nu)} \rho^{(\nu)}(t). \tag{4.31}$$

The decay rates of the processes in each subspace are given by $|\text{Im}\, z_j^{(\nu)}|$ [see Eq. (4.9)] which are generally finite nonvanishing numbers. Equation (4.29) shows that the nonprivileged component is a functional of the privileged component,

$$Q^{(\nu)}\rho^{(\nu)}(t) = C^{(\nu)} P^{(\nu)} \rho^{(\nu)}(t), \tag{4.32}$$

i.e., the nonprivileged component is driven by the privileged component. These relations show that the evolution of any density states in a single $\Pi^{(\nu)}$ subspace are *Markov processes.*

It is well-known that the time evolution of any matrix element $\langle\langle \alpha;\beta|\rho(t)\rangle\rangle$ of a density matrix in the Hilbert space obeys a non-Markovian equation with memory effects (i.e., deviations from the exponential decay for short-time scales associated to the quantum Zeno effect [6], and for long-time scales associated to the long-time tails [34]). Due to the completeness relation of $\Pi^{(\nu)}$, we see that the non-Markovian process is represented as a superposition of Markov processes in each $\Pi^{(\nu)}$ subspace,

$$\langle\langle \alpha;\beta|\rho(t)\rangle\rangle = \sum_\nu \langle\langle \alpha;\beta|\Pi^{(\nu)}|\rho(t)\rangle\rangle. \tag{4.33}$$

V. THE DRESSED UNSTABLE STATE

We come to our main problem, i.e., to identify the dressed states for the unstable case. We strictly follow the results obtained for the stable case in Sec. III. Therefore we introduce a nonunitary transformation Λ that satisfies [see Eq. (3.30)]

$$\Lambda^{-1} P^{(\nu)} = (P^{(\nu)} + C^{(\nu)})\chi^{(\nu)},$$
$$P^{(\nu)}\Lambda = [\chi^{(\nu)}]^* (P^{(\nu)} + D^{(\nu)}), \tag{5.1}$$

where $\chi^{(\nu)}$ are diagonal operators to be determined.

We identify the dressed unstable particle and photon states, respectively, as [see Eq. (3.37)]

$$|\rho_\alpha^0\rangle\rangle \equiv \Lambda^{-1}|\alpha;\alpha\rangle\rangle = (P^{(0)}+C^{(0)})\chi^{(0)}|\alpha;\alpha\rangle\rangle, \quad \alpha=1,k. \tag{5.2}$$

For the limit $\mathrm{Im}\, z_1 \to 0$ corresponding to the stable case with $\omega_1<0$, we should have $|\rho_\alpha^0\rangle\rangle \to |\overline{\rho_\alpha^0}\rangle\rangle = U^{-1}|\alpha;\alpha\rangle\rangle$.

To determine $\chi^{(\nu)}$ we look for the analytic continuation of the unitary operator U in Eq. (3.30). Therefore we extend the relation Eq. (3.35) to

$$A^{(\nu)} = \chi^{(\nu)}[\chi^{(\nu)}]^* \tag{5.3}$$

for the diagonal operator $A^{(\nu)}$, which is star-Hermitian. The projection operators in Eq. (3.34) now become

$$\Pi^{(\nu)} = \Lambda^{-1}P^{(\nu)}\Lambda. \tag{5.4}$$

Note that from Eq. (5.1) we have

$$P^{(\nu)}\Lambda\Lambda^{-1}P^{(\nu)} = P^{(\nu)}. \tag{5.5}$$

Equation (5.5) and the summation of Eq. (5.4) over ν show that Λ^{-1} is indeed the inverse operator of Λ. Due to the relation (4.22), we have

$$\Lambda^{-1} = \Lambda^*, \tag{5.6}$$

i.e., Λ is a star-unitary operator. Star-unitary operators correspond to the extension of unitarity to dissipative systems [10].

We assume the analyticity of Λ at $\lambda=0$ [condition (4) in the Introduction]. Then, $\chi^{(\nu)}$ is also analytic at $\lambda=0$,

$$\lim_{\lambda\to 0}\chi^{(\nu)} = P^{(\nu)} \tag{5.7}$$

and we have

$$\lim_{\lambda\to 0}\Lambda = 1. \tag{5.8}$$

We now focus on the $\nu=0$ subspace. A brief comment on the dressed states for $\nu\neq 0$ is made at the end of Sec. VI. From the relation $\Lambda^{-1}P^{(0)} = \Pi^{(0)}\Lambda^{-1}$ we conclude that the states $|\rho_\alpha^0\rangle\rangle$ are entirely in the $\Pi^{(0)}$ subspace

$$\Pi^{(0)}|\rho_\alpha^0\rangle\rangle = |\rho_\alpha^0\rangle\rangle, \quad \alpha=1,k. \tag{5.9}$$

Therefore, the time evolution of the unstable states obeys a Markovian process with a finite decay rate [condition (5); see Eq. (4.31)].

The ''kinetic'' operators $C^{(0)}$, $D^{(0)}$, and $A^{(0)}$, as well as $\Pi^{(0)}$, have already been defined the previous section. However for $\chi^{(0)}$ there still remains an ambiguity. Indeed, for any $\chi^{(0)}$ we can associate the operator $\chi_\sigma^{(0)} \equiv \chi^{(0)}\sigma^{(0)}$ that also satisfies Eq. (5.3). Here, $\sigma^{(0)}$ is an arbitrary star-unitary operator in the $P^{(0)}$ subspace,

$$[\sigma^{(0)}]^*\sigma^{(0)} = P^{(0)}. \tag{5.10}$$

This problem appears only in the $\nu=0$ subspace because of the degeneracy of the eigenstates of L_0 in this subspace. In the other subspaces the $\sigma^{(\nu)}$ are simply numbers given by arbitrary unitary phase factors.

Actually, the same problem appears in the stable case in the Liouville space formalism. However, this ambiguity has been removed by using the factorizability of the unitary superoperator U [see Eqs. (3.26) and (3.40)]. In contrast, the factorizable formulation of the unstable states leads to unsatisfactory results as discussed in Sec. II [see Eq. (2.31)]. The ambiguity of $\chi^{(0)}$ is the main difficulty already encountered in previous work in the study of unstable states [25].

Let us first show that among the various components $\chi^{(0)}_{\alpha\alpha;\beta\beta}$ there is only one component $\chi^{(0)}_{kk;11}$ that presents the ambiguity. Indeed, from the relation

$$\langle\langle 1;1|\Lambda\Lambda^{-1}|1;1\rangle\rangle = 1 \tag{5.11}$$

we have

$$[\chi^{(0)}_{11;11}]^2(1+\langle\langle 1,1|D^{(0)}C^{(0)}|1,1\rangle\rangle) = 1, \tag{5.12}$$

where we have used $[\chi^{(0)}_{11;11}]^* = \chi^{(0)}_{11;11}$. Using Eqs. (4.25b), (4.26), and (2.27), we see that the matrix element of $D^{(0)}C^{(0)}$ is given by

$$\langle\langle 1,1|D^{(0)}C^{(0)}|1,1\rangle\rangle = \xi + \xi^{\text{c.c.}} + \xi\xi^{\text{c.c.}}. \tag{5.13}$$

Inserting this in Eq. (5.12) we obtain

$$\chi^{(0)}_{11;11} = \frac{1}{\sqrt{(1+\xi)(1+\xi^{\text{c.c.}})}} = \frac{1}{|1+\xi|}. \tag{5.14}$$

We have chosen the plus branch of the square root to be consistent with the stable case [see Eq. (3.40)].

We also have

$$\chi^{(0)}_{kk;k'k'} = \delta_{k,k'} + O(L^{-2}), \tag{5.15}$$

which is proved by noting that the perturbation expansion of $A^{(0)}$ for the Friedrichs model is given by

$$A^{(0)} = P^{(0)} + \lambda^2 A_2^{(0)} + \lambda^4 A_4^{(0)} + \ \ldots\ . \tag{5.16}$$

Absence of odd order terms in this expansion is a consequence of the Friedrichs Hamiltonian [Eq. (2.1)]. In the perturbation expansion, the transition $A^{(0)}_{kk;k'k'}$ between the two states $|k;k\rangle\rangle$ and $|k';k'\rangle\rangle$ comes from successive interactions in Eq. (3.15). As we have seen there, in each interaction only one index can change, as, e.g., $k'\Rightarrow 1$, or $1\Rightarrow k$. Therefore, to achieve the transition between the two states $|k;k\rangle\rangle$ and $|k';k'\rangle\rangle$ we need at least four interactions $(\lambda V)^4$ which is proportional to L^{-2} [see Eq. (2.4)]. Combining this fact with the star-Hermiticity relation

$$(\chi^{(0)*})_{\alpha\alpha;\beta\beta} = \chi^{(0)}_{\beta\beta;\alpha\alpha} \tag{5.17}$$

and with Eq. (5.3) we obtain the result (5.15).

Furthermore, the relation (5.3) with Eq. (5.17) leads to

$$\chi^{(0)}_{11;kk}=A^{(0)}_{kk;11}-\chi^{(0)}_{kk;11}\chi^{(0)}_{11;11}, \tag{5.18}$$

where we have again neglected L^{-2} order terms. This equation shows that if $\chi^{(0)}_{kk;11}$ is known, then $\chi^{(0)}_{11;kk}$ is automatically determined, as $A^{(0)}_{kk;11}$ is known.

From condition (2) on the trace conservation of the dressed unstable particle state we have

$$\mathrm{Tr}(\rho^0_1)=1. \tag{5.19}$$

This leads to a restriction on the form of $\chi^{(0)}_{kk;11}$ as

$$\chi^{(0)}_{11;11}+\sum_k \chi^{(0)}_{kk;11}=1. \tag{5.20}$$

In summary we have already used conditions (1), (2), and (4) and (5) stated in the Introduction to come to this stage. In Sec. VI we shall determine $\chi^{(0)}_{kk;11}$.

VI. DETERMINATION OF $\chi^{(0)}_{KK;11}$

In order to obtain the form of $\chi^{(0)}_{kk;11}$, we now use the remaining conditions (3) and (6). By a direct calculation, one can see that condition (3) on the preservation of the hermiticity leads to the relation

$$(\Lambda^{-1})^{\mathrm{c.c.}}_{\alpha\beta;\alpha'\beta'}=(\Lambda^{-1})_{\beta\alpha;\beta'\alpha'}. \tag{6.1}$$

This relation was introduced previously in Ref. [10]. It expresses the ''adjoint'' symmetry of Λ. It leads to

$$\chi^{(0)\mathrm{c.c.}}_{kk;11}=\chi^{(0)}_{kk;11}, \tag{6.2}$$

i.e, $\chi^{(0)}_{kk;11}$ is real. Let us now rewrite $\overline{\chi}^{(0)}_{kk;11}$ for the stable case [cf. Eq. (3.40)],

$$\overline{\chi}^{(0)}_{kk;11}=\frac{\lambda^2\overline{c}^2_k}{1+\overline{\xi}}=\overline{\chi}^{(0)}_{11;11}\lambda^2\overline{c}^2_k, \tag{6.3}$$

which is also real. The analytic continuation of $\overline{\chi}^{(0)}_{11;11}$ is written in Eq. (5.14), while the analytic continuation of $\overline{c}^2_k$ is either c^2_k or its complex conjugate [compare Eqs. (3.42) and (4.25)]. Taking into account the condition (6.2) we are led to the expression

$$\chi^{(0)}_{kk;11}=\frac{\lambda^2}{|1+\xi|}(rc^2_k+\mathrm{c.c.}), \tag{6.4}$$

where r is a complex constant to be determined, with the boundary value $r=1/2$ in the stable case. As we shall see this constant plays an important role in the time-energy uncertainty relation. Of course one could always add to Eq. (6.4) nonanalytic quantities that vanish in the stable case. However, this would be contrary to the main assumptions of this paper [see property (1)].

Substituting Eq. (6.4) into the relation (5.20), we obtain

$$\frac{1+r\xi+\mathrm{c.c.}}{|1+\xi|}=1. \tag{6.5}$$

Because r is a complex number, it is necessary to introduce one more condition to determine it. Here, we use the last condition (6) on the mean energy fluctuation ΔE_1 that is defined as usual by

$$(\Delta E_1)^2\equiv\mathrm{Tr}(H^2\rho^0_1)-[\mathrm{Tr}(H\rho^0_1)]^2=\langle\langle H^2|\rho^0_1\rangle\rangle-[\langle\langle H|\rho^0_1\rangle\rangle]^2. \tag{6.6}$$

From Eq. (5.2) we have

$$|\rho^0_1\rangle\rangle=|1;1\rangle\rangle\chi^{(0)}_{11,11}+\sum_k |k;k\rangle\rangle\chi^{(0)}_{kk;11}+\sum_k [|k;1\rangle\rangle C^{(0)}_{k1;11}+|1;k\rangle\rangle C^{(0)}_{k1;11}]\chi^{(0)}_{11;11}+\sum_{k,k'}{}' |k,k;\rangle\rangle C^{(0)}_{kk';11}\chi^{(0)}_{11;11}, \tag{6.7}$$

where the prime in the last summation indicates the restriction $k\neq k'$. To obtain this expression, we have neglected terms of the form $\Sigma_k C^{(0)}_{\alpha\beta;kk}\chi^{(0)}_{kk;11}$, which are L^{-1} smaller than the other terms. Substituting the explicit expressions (4.25), (5.14), and (6.4) into Eq. (6.7), we get

$$|\rho^0_1\rangle\rangle=\frac{1}{|1+\xi|}\Bigg[|1;1\rangle\rangle+\lambda^2\sum_k |k;k\rangle\rangle(c^2_kr+\mathrm{c.c.})+\lambda\sum_k (|k;1\rangle\rangle c_k+|1;k\rangle\rangle c^{\mathrm{c.c.}}_k)+\lambda^2\sum_{k,k'}{}'|k;k'\rangle\rangle c_kc^{\mathrm{c.c.}}_{k'}\Bigg]. \tag{6.8}$$

This expression shows that $|\rho^0_1\rangle\rangle$ consists of the bare state $|1;1\rangle\rangle$ plus a dressing. Using Eq. (2.1), Eq. (6.8) leads to

$$\langle\langle H|\rho^0_1\rangle\rangle=\frac{1}{|1+\xi|}\Bigg[\omega_1+\lambda^2\sum_k \omega_k(c^2_kr+\mathrm{c.c.})+\lambda^2\sum_k V_k(c_k+\mathrm{c.c.})\Bigg], \tag{6.9a}$$

$$\langle\langle H^2|\rho^0_1\rangle\rangle=\frac{1}{|1+\xi|}\Bigg[\omega^2_1+\lambda^2\sum_k \omega^2_k(c^2_kr+\mathrm{c.c.})+\Delta E^2_{\mathrm{bare}}+\lambda^2\sum_k V_k(\omega_1+\omega_k)(c_k+\mathrm{c.c.})+\lambda^4\sum_{k,k'}{}' V_kV_{k'}c_kc^{\mathrm{c.c.}}_{k'}\Bigg], \tag{6.9b}$$

where

$$\Delta E_{\mathrm{bare}}\equiv\lambda\left(\sum_k V^2_k\right)^{1/2} \tag{6.10}$$

is the energy fluctuation of the bare excited state. Both expressions in Eq. (6.9) are real. This is a consequence of condition (3). From $\eta^+(z_1)=0$ in Eq. (2.19), we have

$$\lambda^2 \sum_k V_k c_k = z_1 - \omega_1 . \tag{6.11a}$$

Since c_k is a function of $z_1 = \tilde{\omega}_1 - i\gamma$ [cf. Eq. (2.26)], this relation implicitly determines $\tilde{\omega}_1$ and γ. An iterative use of Eq. (6.11a) leads to the relations (the proofs are given in Appendix C)

$$\lambda^2 \sum_k \omega_k c_k^2 = z_1(\xi - 1) + \omega_1 \tag{6.11b}$$

and

$$\lambda^2 \sum_k V_k \omega_k c_k = z_1(z_1 - \omega_1) - \Delta E_{\text{bare}}^2, \tag{6.12a}$$

$$\lambda^2 \sum_k \omega_k^2 c_k^2 = -2 z_1(z_1 - \omega_1) + z_1^2 \xi + \Delta E_{\text{bare}}^2 . \tag{6.12b}$$

Substituting Eqs. (6.11) into Eq. (6.9a) and using Eq. (6.5), we obtain

$$\langle\langle H | \rho_1^0 \rangle\rangle = \tilde{\omega}_1 + \delta\tilde{\omega}_1 + \frac{1}{|1+\xi|}(\omega_1 - \tilde{\omega}_1)(r + r^{\text{c.c.}} - 1), \tag{6.13}$$

where

$$\delta\tilde{\omega}_1 \equiv -\frac{i\gamma}{|1+\xi|}[r(\xi - 1) - \text{c.c.}]. \tag{6.14}$$

Similary, substituting Eqs. (6.12) into Eq. (6.9b), we obtain

$$\langle\langle H^2 | \rho_1^0 \rangle\rangle = \frac{1}{|1+\xi|}[\,|z_1|^2 + ([z_1^2(r\xi - r + r^{\text{c.c.}}) + \omega_1 z_1 (r - r^{\text{c.c.}})] + \text{c.c.}) + (r + r^{\text{c.c.}} - 1)\Delta E_{\text{bare}}^2]. \tag{6.15}$$

The term ΔE_{bare} is of the order of the ultraviolet cutoff of the interaction and is generally much larger than the decay rate 2γ. This term would destroy the time-energy uncertainty relation, i.e., it would lead to $\Delta E_1 \sim \Delta E_{\text{bare}}$ instead of $\Delta E_1 \sim \gamma$ [see Eq. (7.19)]. Hence, to satisfy condition (6), we choose r that eliminates this term in Eq. (6.15), i.e.,

$$r + r^{\text{c.c.}} = 1. \tag{6.16}$$

As a result, the last term in Eq. (6.13) also vanishes. Notice that a state $|\rho_1^0\rangle\rangle'$ that would differ from $|\rho_1^0\rangle\rangle$ by a small change $r \to r'$ with $r' + (r')^{\text{c.c.}} \neq 1$ would have a small variation in the average energy [Eq. (6.13)], but it would have a considerably larger change in the energy fluctuation, due to the term ΔE_{bare}.

Combining Eq. (6.16) with Eq. (6.5), we determine the complex constant r as

$$r = \frac{1}{2} + \frac{|1+\xi| - 1 - (\xi + \xi^{\text{c.c.}})/2}{\xi - \xi^{\text{c.c.}}}. \tag{6.17}$$

Using polar coordinates $1+\xi = |1+\xi| \exp(ia)$ we have

$$r = \frac{1}{2\cos(a/2)} e^{-ia/2} \tag{6.18}$$

and Eq. (6.4) takes the form

$$\chi_{kk;11}^{(0)} = \frac{1}{2}\frac{\lambda^2}{|1+\xi|}[(c_k^2 + \text{c.c.}) - i(c_k^2 - \text{c.c.})\tan(a/2)]. \tag{6.19}$$

In the stable case with $\omega_1 < 0$ we have $\text{Im}\, z_1 \to 0$, $a \to 0$ and we recover Eq. (6.3).

As mentioned below Eq. (2.6) the system may also become stable when the coupling λ is relatively strong. In this case we again have $\text{Im}\, z_1 = 0$ and $a = 0$.

It is interesting to see the relation of our unstable dressed particle state to the Gamow vectors discussed in Sec. II. The Gamow vector dyads $|\phi_1;\phi_1\rangle\rangle$, $|\tilde{\phi}_1;\phi_1\rangle\rangle$ and $|\phi_1;\tilde{\phi}_1\rangle\rangle$ have the following matrix elements:

$$\langle\langle k;1|\phi_1;\phi_1\rangle\rangle = \langle\langle 1;k|\phi_1;\phi_1\rangle\rangle^{\text{c.c.}} = \frac{1}{|1+\xi|}\lambda c_k ,$$

$$\langle\langle k;k'|\phi_1;\phi_1\rangle\rangle = \frac{1}{|1+\xi|}\lambda^2 c_k c_{k'}^{\text{c.c.}}, \tag{6.20}$$

$$\langle\langle 1;1|\phi_1;\tilde{\phi}_1\rangle\rangle = \frac{1}{1+\xi}; \quad \langle\langle k;k|\phi_1;\tilde{\phi}_1\rangle\rangle = \frac{1}{1+\xi}\lambda^2 c_k^2 ,$$

$$\langle\langle \alpha;\alpha|\tilde{\phi}_1;\phi_1\rangle\rangle = \langle\langle \alpha;\alpha|\phi_1;\tilde{\phi}_1\rangle\rangle^{\text{c.c.}}.$$

Comparing these matrix elements with the matrix elements of $|\rho_1^0\rangle\rangle$ in Eq. (6.8) and noting that $r/|1+\xi| = r^{\text{c.c.}}/(1+\xi)$ we obtain the relation

$$|\rho_1^0\rangle\rangle = Q^{(0)}|\phi_1;\phi_1\rangle\rangle + P^{(0)}[r^{\text{c.c.}}|\phi_1;\tilde{\phi}_1\rangle\rangle + r|\tilde{\phi}_1;\phi_1\rangle\rangle]. \tag{6.21}$$

This shows that the dressed particle state can only be expressed as a superposition of dyads of Gamow vectors. The dressed photons are also given by a superposition of dyads, as will be shown in Eq. (7.7). We shall study the time dependence of these states later.

Through the Λ transformation we can define general dressed states $|\rho_j^\nu\rangle\rangle$ and their duals as

$$|\rho_j^\nu\rangle\rangle \equiv \Lambda^{-1}|\nu_j\rangle\rangle, \quad \langle\langle \tilde{\rho}_j^\nu| \equiv \langle\langle \nu_j|\Lambda = (|\rho_j^\nu\rangle\rangle)^*. \tag{6.22}$$

For $\nu = \alpha\beta \neq 0$ these are dressed correlations. Similar to Eq. (5.9) we have

$$\Pi^{(\alpha\beta)}|\rho^{\alpha\beta}\rangle\rangle = |\rho^{\alpha\beta}\rangle\rangle. \tag{6.23}$$

From the completeness and orthogonality relations of the unperturbed states $|\nu_j\rangle\rangle$, one can conclude that the dressed states also form a bicomplete and biorthogonal basis in the Liouville space

$$\sum_{\nu,j} |\rho_j^\nu\rangle\rangle\langle\langle\tilde{\rho}_j^\nu| = 1, \quad \langle\langle\tilde{\rho}_i^\mu|\rho_j^\nu\rangle\rangle = \delta_{\mu\nu}\delta_{ij}, \tag{6.24}$$

and Eq. (6.22) leads to

$$\Lambda = \sum_{\nu,j} |\nu_j\rangle\rangle\langle\langle\tilde{\rho}_j^\nu|, \quad \Lambda^{-1} = \sum_{\nu,j} |\rho_j^\nu\rangle\rangle\langle\langle\nu_j|. \tag{6.25}$$

The construction of the dressed particle state is now completed. Section VII is devoted to some interesting properties of the states we have introduced.

VII. SOME PROPERTIES OF THE DRESSED STATES

A. Nonfactorizability

The dressed particle state $|\rho_1^0\rangle\rangle$ is not factorizable into a product of wave functions [see Eq. (6.21)]. This is welcome, since as discussed in Sec. II, factorizable density matrices are not adequate to identify the dresssed unstable state.

B. Hilbert norm of the unstable state

The unstable dressed state lies outside the Hilbert space. Indeed, as shown in Appendix D the Hilbert norm vanishes

$$\langle\langle\rho_1^0|\rho_1^0\rangle\rangle = 0. \tag{7.1}$$

Nevertheless, $|\rho_1^0\rangle\rangle$ belongs to the trace class [Eq. (5.19)] and one can calculate the expectation value of a given observable for the unstable dressed state. The states $|\rho_k^0\rangle\rangle$ also belong to the trace class and in contrast to $|\rho_1^0\rangle\rangle$ they have a nonvanishing Hilbert norm (see Appendix D).

C. Dressed states vs eigenstates of L_H

As mentioned in Eq. (5.10), $\sigma^{(\nu)}$ (hence $\chi^{(\nu)}$) for $\nu\neq 0$ is just a number. Combining this fact with Eqs. (5.1) and (4.5a), we see that

$$|\rho^{\alpha\beta}\rangle\rangle = |F^{\alpha\beta}\rangle\rangle \quad \text{for} \quad \alpha\neq\beta, \tag{7.2}$$

i.e., each dressed correlation is an eigenstate of the Liouvillian L_H.

However, this is not the case for $\nu=0$, due to the multiplicity of the eigenstates of L_H belonging to the $\Pi^{(0)}$ subspace. Indeed, as indicated in Eq. (4.27a), the eigenstates $|F_k^0\rangle\rangle$ have all the same eigenvalue zero. Since the state $|\rho_1^0\rangle\rangle$ belongs to this subspace, it can be written as the linear superposition

$$|\rho_1^0\rangle\rangle = b_1|F_1^0\rangle\rangle + \sum_k b_k|F_k^0\rangle\rangle. \tag{7.3}$$

One can find the coefficients b_α as follows. First we note that in order to obtain finite expectation values of arbitrary observables in the continous limit the coefficients b_k must be of order $1/L$. From the relation $Q^{(0)}|\rho_1^0\rangle\rangle = Q^{(0)}|\phi_1;\phi_1\rangle\rangle = Q^{(0)}|F_1^0\rangle\rangle$ [cf. Eqs. (6.21) and (B18)] we find that $b_1 = 1 + O(1/L)$. Taking the diagonal $\langle\langle k;k|$ component on both sides of Eq. (7.3) we then obtain [cf. Eqs. (B4) and (B9)]

$$b_k = \chi_{kk;11}^{(0)} - \frac{\theta_{kk;11}^{(0)}}{\theta_{11;11}^{(0)}}\chi_{11;11}^{(0)}. \tag{7.4}$$

Substituting the explicit forms of the matrix elements presented in Eqs. (5.14), (6.4), and (B14) into Eq. (7.4), we have

$$\begin{aligned} b_k &= \frac{\lambda^2}{|1+\xi|}[(rc_k^2 + \text{c.c.}) - c_k c_k^{\text{c.c.}}] \\ &= \frac{1}{2}\frac{\lambda^2}{|1+\xi|}[(c_k - \text{c.c.})^2 - i(c_k^2 - \text{c.c.})\tan(a/2)]. \end{aligned} \tag{7.5}$$

Using Eq. (5.20) one can see these coefficients satisfy the relation

$$\sum_k b_k = 1. \tag{7.6}$$

The dressed photon states are also given by a superposition of eigenstates of L_H in the $\Pi^{(0)}$ subspace. From Eq. (5.15) and the relation $\langle\langle\tilde{\rho}_1^0|\rho_k^0\rangle\rangle = 0$ we find these states as

$$|\rho_k^0\rangle\rangle = |F_k^0\rangle\rangle - b_k|F_1^0\rangle\rangle. \tag{7.7}$$

Note that, from the relations $\text{Tr}(F_1^0)=0$, $\text{Tr}(F_k^0)=1$ as well as $\text{Tr}(HF_1^0)=0$ and $\text{Tr}(HF_k^0)=\omega_k$, [see Eq. (B18)] we have

$$\text{Tr}(\rho_\alpha^0) = 1 \quad \text{for} \quad \alpha = 1,k,$$

$$\text{Tr}(H\rho_1^0) = \sum_k b_k \text{Tr}(H\rho_k^0) = \sum_k b_k\omega_k. \tag{7.8}$$

D. Time evolution and line shape

The time evolution of the dressed states is given by [see Eq. (4.27)]

$$e^{-iL_H t}|\rho^{\alpha\beta}\rangle\rangle = e^{-iz^{(\alpha\beta)}t}|\rho^{\alpha\beta}\rangle\rangle \quad \text{for} \quad \alpha\neq\beta \tag{7.9}$$

and

$$\begin{aligned} e^{-iL_H t}|\rho_1^0\rangle\rangle &= e^{-2\gamma t}|F_1^0\rangle\rangle + \sum_k b_k|F_k^0\rangle\rangle \\ &= e^{-2\gamma t}|\rho_1^0\rangle\rangle + (1-e^{-2\gamma t})\sum_k |\rho_k^0\rangle\rangle b_k, \end{aligned} \tag{7.10a}$$

$$e^{-iL_H t}|\rho_k^0\rangle\rangle = |\rho_k^0\rangle\rangle + (1-e^{-2\gamma t})b_k\left[|\rho_1^0\rangle\rangle - \sum_l |\rho_l^0\rangle\rangle b_l\right]. \tag{7.10b}$$

The dressed particle state has strictly exponential decay. This was suggested since long in Ref. [19]. Memory effects (deviations from exponential decay) would need a distinction between young and old particles and destroy indiscernibility.

From Eq. (7.8) one can verify that the Markovian equations [Eq. (7.10)] preserve the trace and energy. The time evolution of the dressed particle state gives rise to dressed photons, as shown in Eq. (7.10a),

$$\langle\langle\tilde{\rho}_k^0|e^{-iL_H t}|\rho_1^0\rangle\rangle=(1-e^{-2\gamma t})b_k. \tag{7.11}$$

This approaches b_k in the asymptotic limit $t\to+\infty$. Therefore, b_k may be interpreted as the line shape of dressed photons emitted from the dressed unstable state. The relation (7.6) is consistent with this interpretation. In the weak-coupling case $\lambda\ll 1$ (which leads to $\gamma/\tilde{\omega}_1\ll 1$) we may neglect $a\sim O(\lambda^2)$ and $\xi\sim O(\lambda^2)$ in Eq. (7.5) to obtain

$$\begin{aligned} b_k &\approx \frac{1}{2}(\lambda c_k-\text{c.c.})^2 \\ &= \frac{\lambda^2 V_k^2}{2}\left(\frac{1}{[z-\omega_k]_{z_1}^+}-\frac{1}{[z-\omega_k]_{z_1^{\text{c.c.}}}^-}\right)^2 \\ &= -\frac{2\pi}{L}\frac{2\lambda^2 v_k^2\gamma^2}{([z-\omega_k]_{z_1}^+[z-\omega_k]_{z_1^{\text{c.c.}}}^-)^2}. \end{aligned} \tag{7.12}$$

For $\tilde{\omega}_1\gg\gamma$ we have, under integration with a test function $f(\omega_k)$,

$$\begin{aligned} &-\int_{-\infty}^{\infty} dk \frac{f(\omega_k)}{([z-\omega_k]_{z_1}^+[z-\omega_k]_{z_1^{\text{c.c.}}}^-)^2} \\ &\approx \int_{-\infty}^{\infty} dk \frac{f(\omega_k)}{([z_1-\omega_k][z_1^{\text{c.c.}}-\omega_k])^2}, \end{aligned} \tag{7.13}$$

where we have neglected the branch-point contribution at $k=0$. Using Eq. (7.13) in Eq. (7.12) we may then approximate

$$b_k\approx\frac{2\pi}{L}\frac{2\lambda^2 v_k^2\gamma^2}{[(\omega_k-\tilde{\omega}_1)^2+\gamma^2]^2}. \tag{7.14}$$

This distribution has a sharp peak at $\omega_k=\tilde{\omega}_1$ with a width γ. Hence, we can further approximate b_k as

$$b_k\approx\left(\frac{2\pi}{L}\right)\frac{1}{\pi}\frac{(\lambda^2\gamma_2)^3}{[(\omega_k-\tilde{\omega}_1)^2+\lambda^4\gamma_2^2]^2}, \tag{7.15}$$

where $\lambda^2\gamma_2$ is the lowest-order approximation of γ given by

$$\lambda^2\gamma_2\equiv\pi\lambda^2\int_{-\infty}^{+\infty}dk\, v_k^2\delta(\omega_k-\omega_1)=2\pi\lambda^2 v_{\omega_1}^2. \tag{7.16}$$

Our line shape $(L/2\pi)b_k$ approaches $\delta(\omega_k-\omega_1)/2$ in the limit $\lambda\to 0$, which is consistent with the lowest-order approximation in the λ expansion.

From Eq. (7.15) the average energy of the emitted dressed photons is given by

$$\langle E\rangle\equiv\frac{L}{2\pi}\int_{-\infty}^{+\infty}\omega_k b_k\,dk=\frac{L}{\pi}\int_0^{+\infty}\omega b_\omega\,d\omega=\tilde{\omega}_1+O(\lambda^4), \tag{7.17}$$

where we have approximated the integration by extending the lower bound of the integration over ω to $-\infty$ in Eq. (7.17) to obtain the last equality.

Our line shape has a well-defined mean deviation of the energy. Indeed, we obtain from Eq. (7.15) that

$$\begin{aligned} \langle\Delta E\rangle^2 &\equiv \frac{L}{2\pi}\int_{-\infty}^{+\infty}\omega_k^2 b_k\,dk-\left(\frac{L}{2\pi}\int_{-\infty}^{+\infty}\omega_k b_k\,dk\right)^2 \\ &= \lambda^4\gamma_2^2+O(\lambda^6), \end{aligned} \tag{7.18}$$

which leads to

$$\langle\Delta E\rangle\Delta t=\tfrac{1}{2}+O(\lambda^2), \tag{7.19}$$

where $\Delta t\equiv 1/(2\gamma)$ is the lifetime of the unstable particle. Therefore, our line shape satisfies the energy-time ''uncertainty relation.''

It is interesting to compare our line shape $(L/2\pi)b_k$ with the Lorentzian given by

$$b_k^L\equiv\left(\frac{2\pi}{L}\right)\frac{1}{2\pi}\frac{\lambda^2\gamma_2}{(\omega_k-\tilde{\omega}_1)^2+\lambda^4\gamma_2^2}, \tag{7.20}$$

which also approaches $\delta(\omega_k-\omega_1)/2$ in the limit $\lambda\to 0$. The Lorentzian line shape gives the distribution of photon energies emitted by the *bare* unstable state $|1;1\rangle\rangle$. While the Lorentzian distribution gives the same average energy as Eq. (7.17), the mean deviation of energy diverges. This divergence, associated with the approximate Lorentzian line shape, corresponds to the exact energy fluctuation ΔE_{bare} for the bare state. This is an invariant of motion and also corresponds to the exact mean deviation of energy of emitted photons. As mentioned below Eq. (6.15), ΔE_{bare} is much larger than the decay rate.

The difference between our line shape and the Lorentzian line shape may be understood as follows. As is well know, the time evolution of the bare state involves an initial fast dressing process associated with the quantum Zeno effect. This process occurs during a short-time scale $t_{\text{short}}\sim(\Delta E_{\text{bare}})^{-1}$. The survival probability $|\langle 1|\exp(-iHt)|1\rangle|^2$ of the bare state decreases as t^2 during this initial period. The observation of the large energy fluctuation ΔE_{bare} in the exact Friedrichs solution therefore is a manifestation of this short-time Zeno effect. We obtain the time-energy uncertainty relation $\Delta E_{\text{bare}}t_{\text{short}}\sim 1$. On the other hand, the time scale associated with the dressed state is the relaxation time $(2\gamma)^{-1}$, which is much longer than t_{short}. For this is reason our true dressed state generates a line shape narrower than the Lorentzian shape.

In Fig. 1 we show plots of our line shape $(L/2\pi)b_k$ and of the Lorentzian distribution $(L/2\pi)b_k^L$ as a function of $\omega_k=k>0$. In the figure we put $2\gamma\approx 2\lambda^2\gamma_2=0.1$ and $\tilde{\omega}_1=1$. The two line shapes cross at the points $k-\tilde{\omega}_1=\pm\gamma$. For $|k-\tilde{\omega}_1|<\gamma$ we have $\Delta b_k\equiv b_k-b_k^L>0$, i.e., the dressed particle is more likely to emit resonant photons. On the other hand, for $|k-\tilde{\omega}_1|>\gamma$, the dressed state emits less off-resonance photons than the bare state. This difference

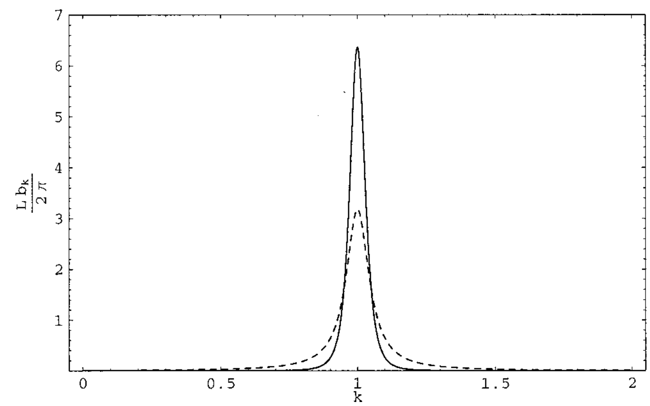

FIG. 1. The line shapes of $(L/2\pi)b_k$ (solid) and the Lorentzian distribution $(L/2\pi)b_k^L$ (dashed) with $2\lambda^2\gamma_2=0.1$ and $\tilde{\omega}_1=1$. The momentum k is measured in units of $\tilde{\omega}_1$, with $\hbar=1$ and $c=1$; $(L/2\pi)b_k$ is measured in units of $\tilde{\omega}_1^{-1}$.

may be attributed to the existence of virtual photons involved in the dressing process of the bare state, which are absent in the dressed state. Therefore, if these virtual photons can be separated experimentally, the line shape generated by the remaining photons should be close to our line shape b_k. This will be further discussed in Ref. [31].

To be consistent with dynamics, we expect that the energy fluctuation associated with the dressed unstable particle $|\rho_1^0\rangle\rangle$ should satisfy an energy-time uncertainty relation similar to Eq. (7.19). In the next paragraph, we shall show indeed this is the case.

E. Energy fluctuation in the dressed unstable state

By substituting Eqs. (6.13) and (6.15) with Eq. (6.16) into Eq. (6.6), we have for the energy fluctuation of the dressed unstable particle state

$$(\Delta E_1)^2=\gamma^2\left(\frac{2}{|1+\xi|}-1\right)-(\delta\tilde{\omega}_1)^2+\frac{2i\gamma}{|1+\xi|}(\tilde{\omega}_1-\omega_1)(r-r^{\text{c.c.}}). \tag{7.21}$$

Note that ξ is of order λ^2 [see Eq. (2.27)]. Expanding the right-hand side in the perturbation series of λ, the second term $(\delta\tilde{\omega}_1)^2$ is of order λ^8 [see Eq. (6.14)], while the last term is of order λ^6. Thus, we obtain

$$(\Delta E_1)^2=\lambda^4\gamma_2^2+O(\lambda^6). \tag{7.22}$$

Similar to Eq. (7.18) this leads to

$$\Delta E_1\Delta t=\tfrac{1}{2}+O(\lambda^2). \tag{7.23}$$

Therefore, the energy-time uncertainty relation is also satisfied for the dressed unstable particle.

The energy fluctuation ΔE_k associated to a dressed photon mode vanishes in the continuous spectrum limit $L\to\infty$,

$$(\Delta E_k)^2\equiv\langle\langle H^2|\rho_k^0\rangle\rangle-[\langle\langle H|\rho_k^0\rangle\rangle]^2\to 0, \tag{7.24}$$

which can be seen by using Eq. (7.7) with Eqs. (2.32) and (B18).

The results [Eqs. (7.10a), (7.18), (7.22), and (7.24)] offer a consistent picture of the transformation process of the energy fluctuation of the dressed unstable particle to the line shape of the dressed photons. Indeed, in this process the energy fluctuation of the system is preserved, as it should be.

F. Energy shift

Because of the relation (6.16), the expectation value of the energy for the unstable particle deviates from Green's function energy $\tilde{\omega}_1$ by $\delta\tilde{\omega}_1$ [see Eqs. (6.13) and (6.14)]. This deviation depends on the decay rate 2γ, and vanishes in the stable case. Expanding $\delta\tilde{\omega}_1$ in the perturbation series of λ and taking the continous limit, we obtain

$$\delta\tilde{\omega}_1=-\frac{3i\lambda^4\gamma_2}{4}\int_{-\infty}^{+\infty}dl\, v_l^2\left[\frac{1}{(\omega_1-\omega_l+i\epsilon)^2}-\text{c.c.}\right]+O(\lambda^6). \tag{7.25}$$

The difference between the average energy of the unstable particle and Green's function energy thus starts at fourth order in λ. Up to this order, this result as well as the uncertainty relation [Eq. (7.23)] coincide with the results obtained in Ref. [26] based on a perturbative approach to construct the Λ transformation.

One can estimate order of magnitude of $\delta\tilde{\omega}_1$ using the form factor for the hydrogen atom with the transition between $2p$ and $1s$ states, which is given in Ref. [37],

$$v_\omega=\frac{i\omega^{1/2}}{[1+(\omega/M)^2]^2}. \tag{7.26}$$

In this form factor the three dimensionality of the atom is already taken into account.[8] For the hydrogen atom

$$\omega_1\approx 1.6\times 10^{16}\ \text{rad/s} \tag{7.27}$$

is the frequency difference between the $2p$ and $1s$ states, and

$$M=\frac{3}{2a_0}\approx 8.5\times 10^{18}\ \text{rad/s} \tag{7.28}$$

is a natural cutoff frequency determined by the Bohr radius a_0. The coupling constant λ is given by

$$\lambda=\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2}{3}\right)^{9/2}\alpha^{3/2}\approx 0.8\times 10^{-4} \tag{7.29}$$

with the fine-structure constant α.

The detailed calculation is presented in Appendix E, and we present here only the result. We have $\delta\tilde{\omega}_1>0$ and

$$\delta\tilde{\omega}_1\approx\lambda^4\frac{3\pi^2\omega_1}{2}\approx 0.6\times 10^{-15}\times\omega_1. \tag{7.30}$$

[8] The factor i does not play a role in the order estimation.

Moreover, expanding Green's function energy in the perturbation series,

$$\tilde{\omega}_1 = \omega_1 + \lambda^2 \omega_2^G + \lambda^4 \omega_4^G + \ \dots \tag{7.31}$$

we have $\lambda^4 \omega_4^G < 0$ and

$$\lambda^4 \omega_4^G \approx -\lambda^4 \frac{5\pi M}{32} [\ln(M/\omega_1) - 1]. \tag{7.32}$$

Because of the factor M, this is much larger than $\delta\tilde{\omega}_1$, and we obtain

$$\frac{\delta\tilde{\omega}_1}{\lambda^4 \omega_4^G} \approx -0.9 \times 10^{-2}. \tag{7.33}$$

So our energy shift is about 1% of the λ^4 contribution to Green's function energy.

G. Asymptotic evolution of the dressed photons

It is interesting to compare the asymptotic time evolution of the dressed photons with the asymptotic evolution of the bare photons. The latter is given by the Möller states of *S*-matrix theory,

$$\lim_{t \to +\infty} e^{+iH_0 t} e^{-iHt} |k\rangle = |\tilde{\phi}_k\rangle. \tag{7.34}$$

In the Liouville space we have $\lim_{t \to +\infty} e^{-iL_H t} |k;k\rangle\!\rangle = |\tilde{\phi}_k;\tilde{\phi}_k\rangle\!\rangle$.

On the other hand Eqs. (7.10b) and (7.7) lead to the asymptotic time evolution of the dressed photon modes $|\rho_k^0\rangle\!\rangle$ as

$$\lim_{t \to +\infty} e^{-iL_H t} |\rho_k^0\rangle\!\rangle = |F_k^0\rangle\!\rangle = |\tilde{\phi}_k;\tilde{\phi}_k\rangle\!\rangle, \tag{7.35}$$

where we have used Eq. (B18) to get the last equality. Thus for $t \to \infty$ (the *S*-matrix regime) one cannot asymptotically distinguish the evolution of dressed photons from the evolution of bare photons.

However for finite time scales, e.g., during the scattering process of a large wave packet, one can see the difference between the evolution of bare and dressed photons [see Eq. (7.10b)]. Scattering for finite time scales (during the collision process) has been studied in Refs. [16,38]. There, it has been shown that there also appear differences between the asymptotic time evolution corresponding to the *S*-matrix regime and the time evolution during the scattering process. This distinction is important in many-particle systems (specially in dense systems) as the particles keep colliding all the time, which is not taken into account in the *S*-matrix approach.

VIII. CONCLUDING REMARKS

Let us summarize our results. We have introduced dressed states in the Friedrichs model, which consists of a discrete state coupled to a continuum. Depending on the parameters of the system, the discrete state can be either stable or unstable. For the stable case it is straightforward to obtain a dressed discrete state, which corresponds to a stable particle or the ground state of an atom. This state is obtained by a diagonalization of the Hamiltonian in the Hilbert space.

On the other hand for the unstable case there appear difficulties. An unstable particle is expected to have several properties analogous to the properties of a stable particle (see Sec. I). As we have seen, for the Friedrichs model there are two known representations of the Hamiltonian in the unstable case. In the Friedrichs representation the photon eigenstates form a complete set and there is no state representing the dressed unstable particle. In the complex representation the factorizable density operators lead to either a complex or a vanishing energy (see Sec. II) and have other features that are not what we expect from an unstable particle.

An alternative approach to solve this problem is to perform analytic continuation directly in the Liouville space of density matrices [15] (see also Ref. [13]). This leads to a *nonfactorizable complex-spectral representation* of the Liouville operator extended outside the Hilbert space. The eigenstates of L_H are generally not products of eigenstates of the Hamiltonian and they break time symmetry. Using this representation we have constructed a star unitary transformation Λ that maps bare states to dressed states. This transformation is star-unitary, which is an extension of unitarity to unstable systems.

We have seen that the unstable state satisfies an uncertainty relation between lifetime and energy. The expectation value of the energy has a deviation compared with Green's function energy. These two effects are related to the fact that the unstable state is not an eigenstate of the evolution operator L_H, and there is an energy transfer from dressed particles to dressed photons. In this way we can avoid the "Hamiltonian dilemma" [19], which would occur if the definition of dressed unstable particles was given in terms of eigenstates of a diagonalized Hamiltonian. We would then obtain noninteracting units. In the Liouville space we have a different possibility. We can define dressed unstable particles and dressed photons that interact.

We have obtained, as well, a new line shape of photon emission [see Eq. (7.15)], associated with the dressed particle. This line shape gives an energy fluctuation of the order of the decay rate. In contrast, the line shape associated with the bare particle gives a much larger energy fluctuation. This may be understood from the fact that the bare state undergoes a rapid dressing process during the quantum Zeno period associated with memory effects. This fast process is responsible for the large energy fluctuation, in accordance to the energy-time uncertainty relation. After a short time the particle starts to decay. This is a slower process associated with our new line shape. For a given initial condition we can substract our line shape from the observed line shape. In this sense we distinguish photons involving the dressing and photons emitted by the unstable particle.

In this paper we have considered global quantities such as the total energy and probability, which lie in the $\Pi^{(0)}$ subspace [see Eq. (7.8)]. In the subsequent paper [31] we shall consider local quantities such as the probability or energy densities. We shall discuss the role of the dressed states in the local evolution. We shall discuss also the possibility of preparing initial conditions belonging to the Hilbert space that can approximate the dressed unstable state we have constructed in this paper.

Quantum theory started from the Einstein-Bohr concept of quantum transitions. It is interesting that there are still aspects of quantum transitions that are worth discussing.

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APPENDIX A: INTEGRAL EQUATIONS FOR THE CREATION OPERATOR

In this appendix we give a solution of the nonlinear Lippmann-Schwinger equation [Eq. (4.19)] for the Friedrichs model. This equation leads to a set of nonlinear integral equations for the various matrix elements of $C^{(\nu)}$. As an example we consider the matrix elements for $\nu=0$ in Eq. (4.25). For brevity we omit here the (0) superscript. Equation (4.19) leads to the set of equations

$$C_{k1;11}=\frac{\lambda}{i\epsilon-w_{k1}}\left[V_k-\sum_{k'}V_{k'}C_{kk';11}-C_{k1;11}\sum_{k'}V_{k'}(C_{k'1;11}-C_{1k';11})\right],\quad (A1)$$

$$C_{1k;11}=\frac{\lambda}{i\epsilon-w_{1k}}\left[-V_k+\sum_{k'}V_{k'}C_{k'k;11}+C_{1k;11}\sum_{k'}V_{k'}(C_{1k';11}-C_{k'1;11})\right],\quad (A2)$$

$$C_{kk';11}=\frac{\lambda}{i\epsilon-w_{kk'}}\left[V_kC_{1k';11}-V_{k'}C_{k1;11}-C_{kk';11}\sum_{l}V_l(C_{l1;11}-C_{1l;11})\right].\quad (A3)$$

Now we show that Eq. (4.25) is a solution of this set of equations. Comparing Eq. (A1) and Eq. (A2) and using Eq. (4.25a) we obtain $C_{1k';11}=C^{\rm c.c.}_{k'1;11}$. Then Eq. (A1) can be written as

$$C_{k1;11}=\frac{\lambda}{i\epsilon-w_{k1}}\left[V_k-\sum_{k'}V_{k'}C_{k1;11}C^{\rm c.c.}_{k'1;11}-C_{k1;11}\sum_{k'}V_{k'}(C_{k'1;11}-C^{\rm c.c.}_{k'1;11})\right].\quad (A4)$$

This leads to

$$\begin{aligned}i\epsilon-w_{k1}&=V_k[C_{k1;11}]^{-1}-\sum_{k'}V_{k'}C^{\rm c.c.}_{k'1;11}-\sum_{k'}V_{k'}(C_{k'1;11}-C^{\rm c.c.}_{k'1;11})\\&=V_k[C_{k1;11}]^{-1}-\sum_{k'}V_{k'}C_{k'1;11}.\end{aligned}\quad (A5)$$

Substituting Eq. (4.25b) in this equation we get in the continous limit

$$z_1=\omega_1+\int dk\,\frac{\lambda^2v_k^2}{(z-\omega_k)^+_{z_1}}.\quad (A6)$$

This is simply the equation $\eta^+(z_1)=0$, which is satisfied by the definition of the complex pole z_1. This demonstrates that Eq. (4.25) is indeed a solution of Eq. (A1). Similarly one can show that Eq. (4.25) satisfies Eq. (A3). The solution Eq. (4.25) coincides with the solution obtained by de Haan and Henin [30]. The solutions for the other components including the $\nu\neq0$ subspaces are presented in Sec. 3 of Appendix B.

APPENDIX B: ON THE NONFACTORIZABLE REPRESENTATION OF L_H

In this Appendix we present the eigenstates of L_H for the Friedrichs model. The eigenstates are presented in Subsection 5. Before reaching the final form of the eigenstates we need some preparations, which are presented in Subsections 1–4.

1. Eigenstates of the collision operators

As a first step to obtain the eigenstates of L_H we solve the eigenvalue problem of the collision operators. Since the collision operators $\theta_C^{(\nu)}$ are non-Hermitian, their left eigenstates are not necessarily the same as the right eigenstates. We denote the left eigenstates by $\langle\langle\tilde{u}_j^\nu|$. Assuming biorthogonality and bicompleteness of the left and right eigenstates we have

$$\langle\langle \tilde{u}_j^{\nu} | u_i^{\mu} \rangle\rangle = \delta_{\nu,\mu}\delta_{j,i}, \quad \sum_j |u_j^{\nu}\rangle\rangle\langle\langle \tilde{u}_j^{\nu}| = P^{(\nu)}. \tag{B1}$$

Similarly the right and left eigenstates of $\theta_D^{(\nu)}$ are denoted by $|v_j^{\nu}\rangle\rangle$ and $\langle\langle \tilde{v}_j^{\nu}|$, respectively, and are assumed to form a complete biorthonormal set.

For the $\nu=0$ subspace we have the eigenvalue equation

$$\theta_C^{(0)}|u_\alpha^0\rangle\rangle = z_\alpha^{(0)}|u_\alpha^0\rangle\rangle. \tag{B2}$$

Since the states $|u_\alpha^0\rangle\rangle$ belong to the $P^{(0)}$ subspace, their general form is a linear superposition of the states $|1;1\rangle\rangle$ and $|k;k\rangle\rangle$. Noting the volume dependencies

$$\theta_{11;11}^{(0)} \sim L^0, \quad \theta_{11;kk}^{(0)} \sim \theta_{kk;11}^{(0)} \sim \theta_{kk;kk}^{(0)} \sim L^{-1} \tag{B3}$$

as well as the relation $\theta_{kk;k'k'}^{(0)} = \delta_{kk'}\theta_{kk;kk}^{(0)} + O(1/L^2)$, where we have abbreviated $\theta^{(0)} \equiv \theta_C^{(0)}$, we obtain the following solutions by neglecting terms of higher order in L^{-1},

$$|u_1^0\rangle\rangle = |1;1\rangle\rangle + \sum_k \frac{\theta_{kk;11}^{(0)}}{\theta_{11;11}^{(0)}}|k;k\rangle\rangle \quad \text{for} \quad z_1^{(0)} = \theta_{11;11}^{(0)}, \tag{B4}$$

$$|u_k^0\rangle\rangle = |k;k\rangle\rangle - \frac{\theta_{11;kk}^{(0)}}{\theta_{11;11}^{(0)}}|1;1\rangle\rangle \quad \text{for} \quad z_k^{(0)} = O(1/L) \to 0. \tag{B5}$$

As will be shown in Eq. (B14a) we have $z_1^{(0)} = -2i\gamma$.

The left eigenstates are similarly found to be given by

$$\langle\langle \tilde{u}_1^0| = \langle\langle 1;1| + \sum_k \frac{\theta_{11;kk}^{(0)}}{\theta_{11;11}^{(0)}}\langle\langle k;k|,$$

$$\langle\langle \tilde{u}_k^0| = \langle\langle k;k| - \frac{\theta_{kk;11}^{(0)}}{\theta_{11;11}^{(0)}}\langle\langle 1;1|. \tag{B6}$$

From the second relation in Eq. (B1), we obtain the inverse relations

$$|1;1\rangle\rangle = |u_1^0\rangle\rangle - \sum_k \frac{\theta_{kk;11}^{(0)}}{\theta_{11;11}^{(0)}}|u_k^0\rangle\rangle,$$

$$|k;k\rangle\rangle = \frac{\theta_{11;kk}^{(0)}}{\theta_{11;11}^{(0)}}|u_1^0\rangle\rangle + |u_k^0\rangle\rangle, \tag{B7a}$$

$$\langle\langle 1;1| = \langle\langle \tilde{u}_1^0| - \sum_k \frac{\theta_{11;kk}^{(0)}}{\theta_{11;11}^{(0)}}\langle\langle \tilde{u}_k^0|,$$

$$\langle\langle k;k| = \frac{\theta_{kk;11}^{(0)}}{\theta_{11;11}^{(0)}}\langle\langle \tilde{u}_1^0| + \langle\langle \tilde{u}_k^0|. \tag{B7b}$$

The eigenstates $|v_\alpha^0\rangle\rangle$ and $\langle\langle \tilde{v}_\alpha^0|$ of the collision operator associated with the destruction operator are given by the expressions (B4)–(B6) with the replacement $\theta_{\alpha\alpha;\beta\beta}^{(0)} \Rightarrow \theta_{\beta\beta;\alpha\alpha}^{(0)}$.

For the other subspaces the collision operators are simply numbers since the subspaces are one dimensional. Then, we have for $\alpha\neq\beta$:

$$|u^{\alpha\beta}\rangle\rangle = |\alpha;\beta\rangle\rangle, \quad \langle\langle \tilde{v}^{\alpha\beta}| = \langle\langle \alpha;\beta|,$$

$$z^{(\alpha\beta)} = \langle\langle \alpha;\beta|\theta_C^{(\alpha\beta)}|\alpha;\beta\rangle\rangle. \tag{B8}$$

2. Matrix elements of the right eigenstates of L_H

We shall mainly consider the right eigenstates of L_H. Similar considerations apply to the left eigenstates.

From the general expression for the right eigenstates [see Eq. (4.5a)], we get

$$\langle\langle \alpha;\alpha|F_1^0\rangle\rangle = \sqrt{N_1^{(0)}}\langle\langle \alpha;\alpha|u_1^{(0)}\rangle\rangle,$$

$$\langle\langle \alpha;\beta|F_1^0\rangle\rangle = \sqrt{N_1^{(0)}}\langle\langle \alpha;\beta|C^{(0)}|u_1^0\rangle\rangle = \sqrt{N_1^{(0)}}C_{\alpha\beta;11}^{(0)},$$

$$\langle\langle \alpha;\alpha|F_k^0\rangle\rangle = \sqrt{N_k^{(0)}}\langle\langle \alpha;\alpha|u_k^0\rangle\rangle,$$

$$\langle\langle \alpha;\beta|F_k^0\rangle\rangle = \sqrt{N_k^{(0)}}\langle\langle \alpha;\beta|C^{(0)}|u_k^0\rangle\rangle$$

$$= \sqrt{N_k^{(0)}}\left[-C_{\alpha\beta;11}^{(0)}\frac{\theta_{11;kk}^{(0)}}{\theta_{11;11}^{(0)}} + C_{\alpha\beta;kk}^{(0)}\right] \tag{B9}$$

and for $\alpha\neq\beta$:

$$\langle\langle \alpha;\beta|F^{\alpha\beta}\rangle\rangle = \sqrt{N^{\alpha\beta}},$$

$$\langle\langle \alpha';\beta'|F^{\alpha\beta}\rangle\rangle = \sqrt{N^{(\alpha\beta)}}C_{\alpha'\beta';\alpha\beta}^{(\alpha\beta)}. \tag{B10}$$

3. Matrix elements of $C^{(\nu)}$

We have presented the explicit forms of two matrix elements of the creation operator for $\nu=0$ in the text [see Eq. (4.25)]. Other matrix elements including other subspaces have been presented in Ref. [30]. One can verify by a direct substitution that the expressions presented in Ref. [30] satisfy the nonlinear Lippmann-Schwinger equation. For the reader's convenience we shall present all these matrix elements in this section. In these elements there are two classes: one consists of the elements that are factorizable in terms of the complex eigenstates of the Hamiltonian presented in Sec. II, and the other consists of nonfactorizable elements.

The factorizable elements are

$$C_{\alpha\beta;11}^{(0)} = |N_1|^{-1}\langle\langle \alpha;\beta|\phi_1;\phi_1\rangle\rangle, \quad \alpha\neq\beta,$$

$$C_{ll';kk}^{(0)} = \langle\langle l;l'|\tilde{\phi}_k;\tilde{\phi}_k\rangle\rangle,$$

$$C_{11;1k}^{(1k)} = N_1^{-1/2}\langle\langle 1;1|\phi_1;\phi_k\rangle\rangle,$$

$$C_{\alpha\beta;1k}^{(1k)} = N_1^{-1/2}\langle\langle \alpha;\beta|\phi_1;\phi_k\rangle\rangle, \quad \alpha\neq\beta,$$

$$C^{(k1)}_{\alpha\beta;k1}=[C^{(1k)}_{\alpha\beta;1k}]^{\text{c.c.}},$$

$$C^{(kk')}_{11;kk'}=\langle\langle 1;1|\phi_k;\phi_{k'}\rangle\rangle, \tag{B11}$$

$$C^{(kk')}_{\alpha\beta;kk'}=\langle\langle \alpha;\beta|\phi_k;\phi_{k'}\rangle\rangle, \quad \alpha\neq\beta.$$

The nonfactorizable elements are associated with the components diagonal in momentum representation, such as kk, i.e.,

$$C^{(0)}_{1l;kk}=\langle\langle 1;l|\tilde{\phi}_k;\tilde{\phi}_k\rangle\rangle+C^{(0)}_{1l;11}\frac{\theta^{(0)}_{11;kk}}{\theta^{(0)}_{11;11}},$$

$$C^{(0)}_{l1;kk}=[C^{(0)}_{1l;kk}]^{\text{c.c.}},$$

$$C^{(1k)}_{kk;1k}=\frac{\lambda V_k}{z_1-\omega_k},$$

$$C^{(1k)}_{ll;1k}=-\frac{\lambda V_k}{\eta^-_d(\omega_k)}\frac{\lambda^2V_l^2}{z_1-\omega_k}\left[\frac{1}{\omega_l-\omega_k+i\epsilon}+\frac{1}{(z-\omega_l)^+_{z_1}}\right], \tag{B12}$$

$$C^{(kk')}_{kk;kk'}=\frac{\lambda V_{k'}}{\eta^-_d(\omega_{k'})}\frac{\lambda V_k}{\omega_{k'}-\omega_k+i\epsilon},$$

$$C^{(kk')}_{ll;kk'}=\frac{\lambda^2V_kV_{k'}}{\eta^-_d(\omega_{k'})\eta^+_d(\omega_k)}\frac{\lambda^2V_l^2}{\omega_k-\omega_{k'}-i\epsilon}\times\left[\frac{1}{\omega_{k'}-\omega_l-i\epsilon}-\frac{1}{\omega_k-\omega_l+i\epsilon}\right],$$

$$C^{(\alpha'\beta')}_{\alpha\alpha;\alpha'\beta'}=[C^{(\beta'\alpha')}_{\alpha\alpha;\beta'\alpha'}]^{\text{c.c.}}.$$

For the destruction operators we present only the matrix elements that are used to determine the normalization constants $N^{(\nu)}_\alpha$ in this paper. These are

$$D^{(0)}_{11;\alpha\beta}=|N_1|^{-1}\langle\langle\tilde{\phi}_1;\tilde{\phi}_1|\alpha;\beta\rangle\rangle,$$

$$D^{(1k)}_{1k;\alpha\beta}=N_1^{-1/2}\langle\langle\tilde{\phi}_1;\tilde{\phi}_k|\alpha;\beta\rangle\rangle, \tag{B13}$$

$$D^{(kk')}_{kk';\alpha\beta}=\langle\langle\tilde{\phi}_k;\tilde{\phi}_{k'}|\alpha;\beta\rangle\rangle, \quad \alpha\neq\beta,$$

which are again factorizable in terms of the complex eigenstates of the Hamiltonian. Similar to Eq. (B12) the elements associated with the kk components are not factorizable. Since these give negligible contributions of order L^{-1} to the normalization constants $N^{(\nu)}_\alpha$, we shall not present their explicit forms. Readers interested in these forms should consult Ref. [30].

The matrix elements of $\theta^{(0)}_C$ are obtained from the relation $\theta^{(0)}_C=\lambda P^{(0)}L_VC^{(0)}$, which leads to

$$\theta^{(0)}_{11;11}=z_1-z_1^{\text{c.c.}}=-2i\gamma, \tag{B14a}$$

$$\theta^{(0)}_{kk;11}=-2i\gamma\lambda^2c_kc_k^{\text{c.c.}}, \tag{B14b}$$

$$\theta^{(0)}_{11;kk}=2i\gamma\frac{\lambda^2V_k^2}{|\eta^+(\omega_k)|^2}, \tag{B14c}$$

$$\theta^{(0)}_{kk;kk}=\lambda^2V_k^2\left(\frac{1}{\eta^+(\omega_k)}-\text{c.c.}\right), \tag{B14d}$$

$$\theta^{(0)}_{k'k';kk}=O(1/L^2), \tag{B14e}$$

where we have neglected L^{-2} order terms.

4. Normalization constants

The normalization constants are given by the relation

$$\langle\langle\tilde{F}^\nu_\alpha|F^\nu_\alpha\rangle\rangle=1. \tag{B15}$$

This leads to

$$[N^{(\nu)}_\alpha]^{-1}=\langle\langle\tilde{v}^{(\nu)}_\alpha|(P^{(\nu)}+D^{(\nu)}C^{(\nu)})|u^{(\nu)}_\alpha\rangle\rangle. \tag{B16}$$

Using the biorthogonality relation of the eigenstates of H [see Eq. (2.23)] we obtain

$$N^{(0)}_1=|N_1|^2=|1+\xi|^{-2},$$

$$N^{(0)}_k=1,$$

$$N^{(1k)}=[N^{(k1)}]^{\text{c.c.}}=N_1=(1+\xi)^{-1}, \tag{B17}$$

$$N^{(kk')}=1.$$

5. Explicit forms of the eigenstates of L_H

Using Eqs. (B9), (B10), (B11), and (B17) and neglecting terms of higher orders in $1/L$ we can write the right eigenstates of L_H as follows:

$$|F^0_1\rangle\rangle=|\phi_1;\phi_1\rangle\rangle, \quad |F^0_k\rangle\rangle=|\tilde{\phi}_k;\tilde{\phi}_k\rangle\rangle, \tag{B18}$$

$$|F^{kk'}\rangle\rangle=|\phi_k;\phi_{k'}\rangle\rangle. \tag{B19}$$

Hence, the eigenstates in the $\nu=0$ and $\nu=kk'$ subspaces are factorizable in terms of the complex eigenstates of the Hamiltonian. In contrast, the eigenstates associated with the $\nu=1k,k1$ subspaces are not factorizable and are given by

$$|F^{1k}\rangle\rangle=|\phi_1;\phi_k\rangle\rangle-\sum_l |l;l\rangle\rangle f(k,l), \tag{B20}$$

$$|F^{k1}\rangle\rangle=[|F^{1k}\rangle\rangle]^\dagger, \tag{B21}$$

where

$$f(k,l)\equiv\langle\langle l;l|\phi_1;\phi_k\rangle\rangle-N_1^{1/2}C^{(1k)}_{ll;1k}. \tag{B22}$$

The appearance of the function $f(k,l)$ is a direct consequence of the nonfactorizability of the creation operator

$C_{ll;1k}^{(1k)}$ [cf. Eq. (B12)], in terms of eigenstates of H.[9] In the stable case it can be verified that $f(k,l)\to 0$ and hence we recover the factorizable eigenstates of L_H.

For the left eigenstates we have

$$\langle\!\langle \tilde{F}_1^0| = \langle\!\langle \tilde{\phi}_1;\tilde{\phi}_1|, \quad \langle\!\langle \tilde{F}_k^0| = \langle\!\langle \tilde{\phi}_k^{\text{c.c.}};\tilde{\phi}_k^{\text{c.c.}}|, \tag{B23}$$

$$\langle\!\langle \tilde{F}^{1k}| = \langle\!\langle \tilde{\phi}_1;\tilde{\phi}_k| - \sum_l \langle\!\langle l;l|\tilde{f}(k,l), \tag{B24}$$

$$\langle\!\langle \tilde{F}^{k1}| = [\langle\!\langle \tilde{F}^{1k}|]^\dagger, \tag{B25}$$

$$\langle\!\langle \tilde{F}^{kk'}| = \langle\!\langle \tilde{\phi}_k;\tilde{\phi}_{k'}|, \tag{B26}$$

where

$$\tilde{f}(k,l) \equiv \langle\!\langle \tilde{\phi}_1;\tilde{\phi}_k|l;l\rangle\!\rangle - N_1^{1/2} D_{1k;ll}^{(1k)}. \tag{B27}$$

We note that

$$\langle\!\langle \tilde{\phi}_1;\tilde{\phi}_k|H\rangle\!\rangle \neq 0, \tag{B28}$$

which is a consequence of $\langle \tilde{\phi}_1|\tilde{\phi}_k\rangle \neq 0$ [see Eq. (2.29)]. In contrast, by a direct calculation and using the explicit form

$$D_{1k;ll}^{(1k)} = \frac{\lambda V_k}{z_1-\omega_k}\left[\delta_{l,k} - \frac{\lambda^2 V_l^2}{\eta^+(\omega_k)} \times\left(\frac{1}{\omega_l-\omega_k-i\epsilon} - \frac{1}{(\omega_l-z)_{z_1}^+}\right)\right], \tag{B29}$$

one can show that our nonfactorizable eigenstate satisfies

$$\langle\!\langle \tilde{F}^{(1k)}|H\rangle\!\rangle = 0. \tag{B30}$$

APPENDIX C: RELATIONS USED IN SECTION VI

In this appendix we derive the relations (6.11b) and (6.12). Using Eq. (6.11a), we obtain Eq. (6.11b),

$$\lambda \sum_k \omega_k c_k^2 = z_1 \sum_k \frac{\lambda V_k^2}{[(z-\omega_k)_{z_1}^+]^2} + \sum_k \frac{\lambda V_k^2}{[(z-\omega_k)_{z_1}^+]^2}(\omega_k - z_1) = z_1\xi - (z_1-\omega_1). \tag{C1}$$

Similarly, for Eq. (6.12a) we have

$$\lambda^2 \sum_k V_k\omega_k c_k = z_1\sum_k \frac{\lambda V_k^2}{(z-\omega_k)_{z_1}^+} + \sum_k \frac{\lambda V_k^2}{(z-\omega_k)_{z_1}^+}(\omega_k - z_1) = z_1(z_1-\omega_1) - \sum_k \lambda^2 V_k^2, \tag{C2}$$

which leads to Eq. (6.12b),

$$\lambda^2\sum_k \omega_k^2 c_k^2 = \sum_k \frac{\lambda V_k^2}{[(z-\omega_k)_{z_1}^+]^2}[(\omega_k - z_1)(\omega_k+z_1)+z_1^2] = -z_1\lambda^2\sum_k V_k c_k - \lambda^2 \sum_k V_k\omega_k c_k + z_1^2\xi = -2z_1(z_1-\omega_1) + z_1^2\xi + \sum_k \lambda^2 V_k^2. \tag{C3}$$

APPENDIX D: HILBERT NORM OF THE DRESSED STATES

In this Appendix we prove Eq. (7.1), i.e., that the Hilbert norm of $|\rho_1^0\rangle\!\rangle$ vanishes. The Hilbert norm is given by

$$\langle\!\langle \rho_1^0|\rho_1^0\rangle\!\rangle = \langle\!\langle 1;1|\chi^\dagger(P+C^\dagger)(P+C)\chi|1;1\rangle\!\rangle, \tag{D1}$$

where we have omitted the (0) superscripts in the creation operator for brevity. Neglecting terms of order L^{-1} we obtain

$$\langle\!\langle \rho_1^0|\rho_1^0\rangle\!\rangle = (\chi^\dagger)_{11;11}(P+C^\dagger C)_{11;11}\chi_{11;11} = |\chi_{11;11}|^2[1+(C^\dagger C)_{11;11}]. \tag{D2}$$

The matrix element $(C^\dagger C)_{11;11}$ can be evaluated as

$$(C^\dagger C)_{11;11} = \sum_k \left[C_{k1;11}C_{k1;11}^{\text{c.c.}} + C_{1k;11}C_{1k;11}^{\text{c.c.}} + \sum_{k'} C_{kk';11}C_{kk';11}^{\text{c.c.}}\right] = \lambda^2\sum_k 2c_k(c_k)^{\text{c.c.}} + \lambda^4\sum_k c_k(c_k)^{\text{c.c.}}\sum_{k'} c_{k'}(c_{k'})^{\text{c.c.}}, \tag{D3}$$

where we used the relations for the matrix elements of C given in Eq. (4.25). Since c_k is a distribution with delayed analytic continuation, the integral of the product $c_k(c_k)^{\text{c.c.}}$ has to be carefully evaluated (see Refs. [2,30]). Taking the continous limit we have

[9]In contrast to $f(k,l)$, the difference $\langle\!\langle l;l|\phi_k;\phi_{k'}\rangle\!\rangle - C_{ll;kk'}^{(kk')}$ vanishes in the sense of distributions. This is the reason why Eq. (B19) is factorizable.

$$\sum_k \lambda^2 c_k(c_k)^{\text{c.c.}} \to \int_{-\infty}^{\infty} dk \frac{\lambda^2 v_k^2}{(\omega_k - z)^+_{z_1}(\omega_k - z)^-_{z_1^{\text{c.c.}}}}$$

$$= \frac{1}{z_1^{\text{c.c.}} - z_1} \int_{-\infty}^{\infty} dk\, \lambda^2 v_k^2 \left[\frac{1}{(\omega_k - z)^+_{z_1}} - \frac{1}{(\omega_k - z)^-_{z_1^{\text{c.c.}}}} \right]$$

$$= -1, \tag{D4}$$

where we have used the relation in Eq. (2.19) to get the last equality. Substituting this value in Eq. (D3), we obtain our desired result [Eq. (7.1)]. This indicates that the state $|\rho_1^0\rangle\rangle$ is not an element in the Hilbert space.

In contrast for $|\rho_k^0\rangle\rangle$ we have

$$\langle\langle \rho_k^0 | \rho_k^0 \rangle\rangle = \langle\langle k;k | \chi^\dagger (P + C^\dagger C) \chi | k;k \rangle\rangle = 1 + O(1/L), \tag{D5}$$

where we have used Eq. (5.15) and $\langle\langle k;k | C^\dagger C | k;k \rangle\rangle \sim 1/L$. This shows that $|\rho_k^0\rangle\rangle$ belongs to the Hilbert space.

APPENDIX E: ORDER ESTIMATION OF $\delta\tilde{\omega}_1$

In this Appendix we evaluate the frequency shift given in Eqs. (7.30) and (7.32). Since the three dimensionality of the atom has been already taken into account in the Hamiltonian discussed by Facchi and Pascazio in Ref. [37], we shall use their Hamiltonian to evaluate the frequency shift, instead of our Hamiltonian [Eq. (2.1)]. In the continous limit one can obtain their Hamiltonian replacing ω_k by ω and the wave-vector integration $\int_{-\infty}^{+\infty} dk$ by the energy integration $\int_0^{+\infty} d\omega$ as

$$H = \omega_1 |1\rangle\langle 1| + \int_0^\infty d\omega\, \omega |\omega\rangle\langle\omega| + \lambda \int_0^\infty d\omega\, v_\omega (|\omega\rangle\langle 1| + |1\rangle\langle\omega|). \tag{E1}$$

We have incorporated the sum over spin and orbital angular momenta indices into the interaction.

Then, we have for the Green function energy in Eq. (7.31),

$$\omega_2^G = \tfrac{1}{2} \int_0^\infty d\omega |v_\omega|^2 \left[\frac{1}{\omega_1 - \omega + i\epsilon} + \text{c.c.} \right] \tag{E2}$$

and

$$\omega_4^G = -\frac{\omega_2^G}{2} \int_0^\infty d\omega |v_\omega|^2 \left[\frac{1}{(\omega_1 - \omega + i\epsilon)^2} + \text{c.c.} \right] + \frac{i\gamma_2}{2} \int_0^\infty d\omega |v_\omega|^2 \left[\frac{1}{(\omega_1 - \omega + i\epsilon)^2} - \text{c.c.} \right]$$

$$= \omega_2^G \frac{\partial \omega_2^G}{\partial \omega_1} - \gamma_2 \frac{\partial \gamma_2}{\partial \omega_1}. \tag{E3}$$

From the expression corresponding to Eq. (7.16) we have

$$\gamma_2 = \pi \omega_1. \tag{E4}$$

From the expression corresponding to Eq. (7.25) we have

$$\delta\tilde{\omega}_1 \approx \frac{3\lambda^4 \gamma_2}{2} \frac{\partial \gamma_2}{\partial \omega_1} = \frac{3\lambda^4}{2} \pi^2 \omega_1. \tag{E5}$$

This leads to the estimation in Eq. (7.30).

Performing the integration in Eq. (E2) with the form factor [Eq. (7.26)], we obtain

$$\omega_2^G \approx \omega_1 \ln\left(\frac{M}{\omega_1}\right) - \frac{5\pi M}{32}, \tag{E6}$$

where we have neglected terms of order $\omega_1/M \ll 1$. Substituting Eq. (E6) into Eq. (E3) we obtain

$$\lambda^4 \omega_4^G \approx -\lambda^4 \frac{5\pi M}{32} \left[\ln\left(\frac{M}{\omega_1}\right) - 1 \right]. \tag{E7}$$

These results give the order estimation presented in Sec. VII.

APPENDIX F: OTHER POSSIBILITIES OF DRESSED UNSTABLE STATE

In this Appendix we comment on other possible choices of the star-unitary transformation Λ. As we shall see, these choices lead to unsatisfactory definitions of the dressed unstable state.

In Ref. [15] we have introduced the star-unitary operator Λ_C defined by

$$\Lambda_C = \sum_\nu \sum_j |u_j^\nu\rangle\rangle\langle\langle \tilde{F}_j^\nu|, \tag{F1a}$$

$$\Lambda_C^{-1} = \sum_\nu \sum_j |F_j^\nu\rangle\rangle\langle\langle \tilde{u}_j^\nu|. \tag{F1b}$$

In contrast to Eq. (6.25), this transformation connects the eigenstates of the Liouvillian to the eigenstates of the collision operator. In other words, we have a similitude relation between L_H and $\theta_C^{(\nu)}$ through this transformation,

$$\Lambda_C L_H \Lambda_C^{-1} = \Theta_C, \tag{F2}$$

where

$$\Theta_C \equiv \sum_\nu \theta_C^{(\nu)}. \tag{F3}$$

For the stable case this transformation is reduced to a unitary transformation. Hence, one could consider the possibility of identifying the dressed states as

$$|r_1^0\rangle\!\rangle \equiv \Lambda_C^{-1}|1;1\rangle\!\rangle = |F_1^0\rangle\!\rangle - \sum_k |F_k^0\rangle\!\rangle \frac{\theta_{kk;11}^{(0)}}{\theta_{11;11}^{(0)}}, \tag{F4}$$

where we have used Eq. (B6) to obtain the last equality. However, this quantity does not satisfy our basic condition (1) in the Introduction. Indeed, using Eqs. (B14a), (B14b), and (B18), one can see that this state reduces in the stable limit $\mathrm{Im}\, z_1 \to 0$ with $\omega_1 < 0$ to

$$|r_1^0\rangle\!\rangle \to |\bar{\phi}_1;\bar{\phi}_1\rangle\!\rangle - \sum_k |\bar{\phi}_k;\bar{\phi}_k\rangle\!\rangle \frac{\lambda^2 V_k^2}{(\omega_1-\omega_k)^2}. \tag{F5}$$

Hence $|r_1^0\rangle\!\rangle$ does not reduce to the dressed stable state $|\bar{\phi}_1;\bar{\phi}_1\rangle\!\rangle$, but reduces to a superposition of degenerate stable eigenstates of L_H with zero eigenvalue. Therefore, $|r_1^0\rangle\!\rangle$ is not a suitable choice of the dressed unstable particle.

Another possibility to construct Λ is that we choose $\chi_{kk;11}^{(0)}$ in such a way that the average energy of the dressed unstable particle is the same as Green's function energy $\tilde{\omega}_1$. This alternative condition leads to a different value of r in Eq. (6.4) as [cf. Eq. (6.17)]

$$r = \frac{1}{q\xi - \mathrm{c.c.}}[q(|1+\xi|-1) - (\tilde{\omega}_1 - \omega)\xi^{\mathrm{c.c.}}], \tag{F6}$$

where

$$q \equiv \tilde{\omega}_1 - \omega_1 - i\gamma(\xi^{\mathrm{c.c.}} - 1). \tag{F7}$$

However, since this leads to an energy fluctuation of the dressed particle that is not of the order of the inverse lifetime, this choice is inadequate to identify the dressed unstable particle.

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