# Parametric time-coherent states for the hydrogen atom

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We obtained coherent states for the hydrogen atom by transforming the problem into four oscillators in the parametric time at a classical level, and quantizing these oscillators by using path integration over their holomorphic coordinates. We showed that for the negative- (positive-) energy coherent states the mean values of the physical position of the electron satisfy the Keplerian ellipses (hyperbolas) and that their dispersions oscillate (increase) in parametric time.

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## I. INTRODUCTION

In 1926 Schrödinger constructed the coherent states for the one-dimensional oscillator [1], and also addressed the problem of constructing localized and nonspreading wave packets for electrons in a Coulomb potential, which travel along the Keplerian ellipses. Recently, ten Wolde *et al.* observed a radially localized electron wave packet [2].

Since there exist coherent states for the harmonic oscillator, it is the natural way to obtain coherent states for a Kepler problem by mapping it onto a system of harmonic oscillators. In 1978, Nieto and Simmons developed a general formalism to construct coherent states for the different potentials [3,4]. In 1986, Bhaumik *et al.* constructed coherent states for the Kepler problem [5], and showed that these states disperse. Even in classical mechanics we cannot express orbits as explicit functions of physical time [6]. For this reason, Gerry later discussed the same problem with the evolution of coherent states in parametric time [7], which was used previously for a path-integral quantization of the hydrogen atom by Duru and Kleinert [8]; this corresponded to an eccentric anomaly in the Kepler problem [6]. Recently, Toyoda and Wakayama also discussed the same problem by using the $SU(2)\times SU(2)$ symmetry of the oscillators [9]. These authors did not discuss the contribution of two double valuedness of the transformation and the dispersions of the wave packets. In these papers the stationary states of the hydrogen atom were mepped onto the four harmonic oscillators by using the Kustaanheimo-Stiefel transformation [10]. The complex form of this transformation was given by Cornish [11].

The aim of this paper is to derive coherent states for a hydrogen atom in parametric time by using path integrals. Path integrals are the most convenient way to discuss state functions and their evolution, and give the normalization of the states. In this approach we first transform the classical Kepler problem onto four oscillators for the negative energies. Since the holomorphic coordinates are classical analogs of the raising and lowering operators of the oscillators, we write the action of the oscillator system in terms of the holomorphic coordinates. Then we derive coherent-state wave functions and their evolution and normalization by using the holomorphic coordinates. We also study coherent states for positive energies, using an analytical continuation of the kernel from negative energies to positive energies.

The classical Kepler problem has six conserved variables: angular momentum and Runge-Lenz vectors. The corresponding dynamical symmetry is SO(4). To represent this dynamical symmetry as the geometric symmetry of a new system, we extend the configuration space of the electron by adding a new free-particle degree of freedom. In Sec. II, to find the appropriate classical system, we first add a free-particle Lagrangian to the Lagrangian of the electron without changing its dynamics. Second, we define a new time parameter. Thus the physical time becomes as a new dynamical degree of freedom for the electron. By the Kustaanheimo-Stiefel transformation the four-dimensional part of the problem becomes one of four harmonic oscillators with the same frequency; then we write the four oscillators in terms of the four holomorphic coordinates. In Sec. III, we quantize this $(4+1)$-dimensional system by using path integrals, and this method gives us normalized eigenfunctions and eigenvalues of lowering operators and their parametric time evolutions.

In Sec. IV, we derive mean values of the space coordinates of the electron, and their dispersions. We show that they oscillate at twice the frequency of the eigenvalues, and that the mean values travel along the Keplerian ellipses. In Sec. V we discuss the analytic continuation of these states into positive-energy states and evaluate the mean values of the physical coordinates and their dispersions. Sec. VI certains conclusions, and in the Appendix we discuss the relation between the kernel evaluated in this paper and the one obtained in Ref. [8], and derive coherent states in spherical coordinates for completeness.

## II. CLASSICAL HYDROGEN ATOM

The action is given as

$$A=\int_a^b dt\left[\vec{p}\cdot\frac{d\vec{x}}{dt}-\left(\frac{1}{2m}\vec{p}^{\,2}-\frac{k}{r}\right)\right], \tag{1}$$

where $t$ is the physical time, $\vec{x}$ and $\vec{p}$ are the canonical conjugate coordinates and the momenta of the electron, $k=e^2$ ($e$ is the electron charge), and $r=|\vec{x}|$. To use the SO(4) dynamical symmetry of the Kepler problem, we transform it to

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a four-dimensional problem by choosing an extra space coordinate $x_4$ and adding a free-particle action to Eq. (1). Then Eq. (1) becomes

$$A=\int_a^b dt\left[p_A\frac{dx_A}{dt}-\left(\frac{1}{2m}p_Ap_A-\frac{k}{r}\right)\right], \tag{2}$$

where $x_A=(\vec{x},x_4)$ and $p_A=(\vec{p},p_4)$. Although the free-particle Lagrangian does not change the dynamics of the electron, it changes the classical trajectories and the transition amplitudes. For this reason we eliminate this degree of freedom after the quantization.

We choose a parametric time λ with the nonholonomic relation,

$$dt=r(\lambda)d\lambda \tag{3}$$

and transform the action in Eq. (2) by the Lagrange multiplier $p_0$:

$$A(x_A(\lambda_b);x_A(\lambda_a))=\int_{\lambda_a}^{\lambda_b}d\lambda\left[p_A\frac{dx_A}{d\lambda}-\left(\frac{1}{2m}p_Ap_Ar-k\right)+(-p_0)\left(\frac{dt}{d\lambda}-r\right)\right]. \tag{4}$$

In Eq. (4), the Lagrange multiplier is choosen in such a way that $t$ is a new coordinate, and $(p_0)$ is the conjugate momentum in the region $-\infty<p_0<+\infty$. Thus Eq. (4) describes a $(4+1)$-dimensional system in non-Cartesian coordinates which moves under the potential $[(-p_0)r-k]$. We introduce complex dimensionless coordinates $\xi_A$ and $\xi_B$ and complex conjugates $\xi_A^*$ and $\xi_B^*$ by using the complex form of the Kustaanheimo-Stifel transformation,

$$\begin{pmatrix}dX\\dY\end{pmatrix}=(m|p_0|)^{-1/2}\begin{pmatrix}\xi_B & \xi_A^*\\ \xi_A & -\xi_B^*\end{pmatrix}\begin{pmatrix}d\xi_A^*\\ d\xi_B\end{pmatrix} \tag{5}$$

where $X$ and $Y$ are

$$X=(x_1+ix_2)/\sqrt{2},$$

$$Y=(x_3+ix_4)/\sqrt{2}.$$

Then $r$ is

$$r=(2m|p_0|)^{-1/2}(\xi_A^*\xi_A+\xi_B^*\xi_B). \tag{6}$$

The momenta are transformed as

$$\begin{pmatrix}p_X\\p_Y\end{pmatrix}=\frac{(m|p_0|)^{1/2}}{(|\xi_A|^2+|\xi_B|^2)}\begin{pmatrix}\xi_B^* & \xi_A\\ \xi_A^* & \xi_B\end{pmatrix}\begin{pmatrix}p_{\zeta_A^*}\\ p_{\zeta_B}\end{pmatrix}, \tag{7}$$

where $p_X$ and $p_Y$ are

$$p_X=\frac{1}{\sqrt{2}}(p_{x_1}-ip_{x_2}),$$

$$p_Y=\frac{1}{\sqrt{2}}(p_{x_3}-ip_{x_4}).$$

This is a double-valued transformation, and the paths in $x_A$ space are mapped into two different classes of paths in the $\xi$ plane: those which go from $x(a)$ to $x(b)$ are mapped into those going from $\xi(a)$ to $\xi(b)$ and $[-\xi(b)]$.

We define the complex spinors $\xi$ and $\xi^\dagger$ as

$$\xi=\begin{pmatrix}\xi_A\\ \xi_B\end{pmatrix} \tag{8}$$

and

$$\xi^\dagger=(\xi_A^*,\xi_B^*). \tag{9}$$

We rewrite the action in Eq. (4) as

$$A(\xi_b^\dagger,t_b;\xi_a,t_a)=\int_{\lambda_a}^{\lambda_b}d\lambda\left[p_{\xi^\dagger}\frac{d\xi^\dagger}{d\lambda}+(-p_0)\frac{dt}{d\lambda}-H\right], \tag{10}$$

where $H$ is the Hamiltonian of the four oscillators in terms of complex spinors $\xi$ and $\xi^\dagger$, and is given by

$$H=\omega[p_\xi p_{\xi^\dagger}+\xi^\dagger\xi]-k, \tag{11}$$

where the frequency of the oscillators is $\omega=\sqrt{(-p_0/2m)}$.

We define the holomorphic coordinates $a$ and $a^\dagger$ as

$$a=\frac{1}{\sqrt{2}}\begin{pmatrix}\xi^\dagger+ip_\xi\\ \xi+ip_{\xi^\dagger}\end{pmatrix}=\begin{pmatrix}a_+\\ b_+\\ a_-\\ b_-\end{pmatrix} \tag{12}$$

and

$$a^\dagger=\frac{1}{\sqrt{2}}[(\xi-ip_{\xi^\dagger}),(\xi^\dagger-ip_\xi)]=(a_+^*\ \ b_+^*\ \ a_-^*\ \ b_-^*). \tag{13}$$

Then Eq. (10) can be written as

$$A(a_b^\dagger,\ t_b;\ a_a,\ t_a)=\int_{\lambda_a}^{\lambda_b}d\lambda\left[\frac{1}{2i}\left(\frac{da^\dagger}{d\lambda}a-a^\dagger\frac{da}{d\lambda}\right)+(-p_0)\frac{dt}{d\lambda}-(\omega a^\dagger a-k)\right], \tag{14}$$

where $H=(\omega a^\dagger a-k)$ is a Hamiltonian or energy functional of the electron in parametric time, which is constant, and the value of this constant is zero. $H=0$ gives the physical energy of the particle as $p_0=-2mk^2(a^\dagger a)^{-2}$. The initial conditions in Eqs. (10) and (14) are different; for this reason their values differ by a constant. To derive the coherent states we use the action in Eq. (14).

### III. QUANTIZATION

Feynman's formulation of path integrals gives us the transition amplitudes or matrix elements of the evolution operator of the system between eigenstates of the position operator $\vec{x}$. They give us the evolution of the position eigenstates. Here we are interested in the evolution of the eigenstates of the lowering operators of the harmonic oscillators, and these correspond to time-dependent coherent states of the system. For this reason we define the kernel of the four-oscillator system in terms of the four-dimensional holomorphic coordinates $a_b^\dagger$ and $a_a$ and $t_b$ and $t_b$. This gives us the matrix elements of the evolution operator between the eigenstates of the lowering operators of the oscillators $|a_a\rangle$ and $|a_b\rangle$. The basic difference between these two kernels is that the eigenvalues of the position operator are real, but the eigenvalues of the lowering operator are complex. In the Appendix we derive the transition amplitudes between $\vec{x}_b, t_b$ and $\vec{x}_a, t_a$ which correspond to Feynman's kernel.

We define the kernel of the hydrogen atom in the holomorphic coordinates as

$$K(a_b^\dagger, t_b; a_a, t_a) = \int_{\lambda_a}^{\infty} d\lambda_b \int \frac{DtD(-p_o)}{[2\pi]} \times e^{i\int_{\lambda_a}^{\lambda_b} d\lambda(-p_0)(dt/d\lambda)} K_\omega(a_b^\dagger, a_a), \quad (15)$$

where $\hbar = 1$, and $K_\omega(a_b^\dagger, a_a)$ is the kernel of the four oscillators in parametric time $\lambda$, defined as

$$K_\omega(a_b^\dagger; a_a) = \int \frac{Da^\dagger Da}{[2\pi i]^4} \exp\left\{ i \int_{\lambda_a}^{\lambda_b} d\lambda \left[ \frac{1}{2i} \left( \frac{da^\dagger}{d\lambda} a - a^\dagger \frac{da}{d\lambda} \right) - \omega(a^\dagger a + 2) + k \right] \right\}. \quad (16)$$

In Eq. (16) the $(2\omega)$ term comes from the ordering of the operators $\hat{a}^\dagger$ and $\hat{a}$. The operators $\hat{a}^\dagger$ and $\hat{a}$ satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (17)$$

We perform the path integration over $t$ and $p_0$ easily. It gives

$$\int \frac{Dt\, D(-p_o)}{[2\pi]} \exp\left[ i \int_{\lambda_a}^{\lambda_b} d\lambda(-p_o) \frac{dt}{d\lambda} \right] = \int_{-\infty}^{+\infty} \frac{d(-p_o)}{2\pi} \exp[-ip_o(t_b - t_a)]. \quad (18)$$

Up to Sec. V we assume that $p_o < 0$, and $\omega$ is real. Then we also perform a path integration over $a^\dagger$ and $a$ in the same way as discussed in Ref. [12]. The result is

$$K_\omega(a_b^\dagger; a_a) = e^{-i(2\omega - k)(\lambda_b - \lambda_a) + a_b^\dagger \exp[-i\omega(\lambda_b - \lambda_a)] a_a}. \quad (19)$$

The Kohn-Sham transformation is double valued for the hydrogen atom problem, and all paths in the $x$ space from $x_a$ to $x_b$ are mapped into two different classes of paths in $u$ space: the paths from $a_a$ to $a_b^\dagger$ and the paths from $a_a$ to $-a_b^\dagger$. Since the hydrogen atom is spinless, the physical transition amplitude is the symmetric sum of the amplitudes $K_\omega(a_b^\dagger; a_a)$ and $K_\omega(-a_b^\dagger; a_a)$:

$$K_\omega^{\rm phys}(a_b^\dagger; a_a) = [K_\omega(a_b^\dagger; a_a) + K_\omega(-a_b^\dagger; a_a)] = e^{-i(2\omega - k)(\lambda_b - \lambda_a)} [e^{a_b^\dagger \exp[-i\omega(\lambda_b - \lambda_a)] a_a} + e^{-a_b^\dagger \exp[-i\omega(\lambda_b - \lambda_a)] a_a}]. \quad (20)$$

Then the kernel in Eq. (15) is

$$K^{\rm phys}(a_b^\dagger, t_b; a_a, t_a) = \int_{\lambda_a}^{\infty} d\lambda_b \int_0^{\infty} \frac{d(-p_0)}{2\pi} \times e^{i[k - 2\omega](\lambda_b - \lambda_a) + i(-p_0)(t_b - t_a)} \times [e^{a_b^\dagger \exp[-i\omega(\lambda_b - \lambda_a)] a_a} + e^{-a_b^\dagger \exp[-i\omega(\lambda_b - \lambda_a)] a_a}]. \quad (21)$$

We parametrize the spinors in Eqs. (12) and (13) as

$$a_a = \begin{pmatrix} a_{+a} \\ b_{+a} \\ a_{-a} \\ b_{-a} \end{pmatrix} \quad (22)$$

and

$$a_b^\dagger = (a_{+b}^*, \; b_{+b}^*, \; a_{-b}^*, \; b_{-b}^*). \quad (23)$$

Then the power-series expansion of the exponential functions in Eq. (21) gives

$$K^{\rm phys}(a_b^\dagger, t_b; a_a, t_a) = \int_0^{\infty} \frac{d(p_0)}{2\pi} e^{-ip_0(t_b - t_a)} \sum_{\substack{n_1, n_2, \\ n_3, n_4 = 0}}^{\infty} \times [1 + (-1)^{n_1 + n_2 + n_3 + n_4}] \times \int_{\lambda_a}^{\infty} d\lambda_b \times e^{i(\lambda_b - \lambda_a)[k - \omega(n_1 + n_2 + n_3 + n_4 + 2)]} \times \frac{(a_{+b}^* a_{+a})^{n_1}}{n_1!} \frac{(b_{+b}^* b_{+a})^{n_2}}{n_2!} \times \frac{(a_{-b}^* a_{-a})^{n_3}}{n_3!} \frac{(b_{-b}^* b_{-a})^{n_4}}{n_4!}. \quad (24)$$

Since the value of the parametric energy is zero, we perform a $\lambda_b$ integration and obtain

$$\omega = \frac{k}{n_1 + n_2 + n_3 + n_4 + 2}, \quad (25)$$

with the condition

$$n_1 + n_2 + n_3 + n_4 = 2n. \quad (26)$$

Thus the physical energy is

$$p_0 = -\frac{m}{2}\frac{k^2}{(n+1)^2}. \tag{27}$$

There are two methods to eliminate the fourth coordinate $x_4$: In the global formulation of the quantum mechanics we integrate over all values of $x_{4b}$ as in Ref. [8], and in the same way we use this method in the Appendix. In the local formulation we choose eigenfunctions that have zero eigenvalues for $p_4(\lambda_b)$ or are independent of $x_{4b}$, as in Refs. [5,7,9], and in this section we use this method by expressing the initial condition $p_4(\lambda_b)$ in terms of $a_b^\dagger$ and $a_b$. The condition $p_4(\lambda_b)=0$ gives

$$|a_{+b}|^2+|b_{+b}|^2-|a_{-b}|^2-|b_{-b}|^2=0. \tag{28}$$

We also express the same condition as

$$n_1+n_2-n_3-n_4=0. \tag{29}$$

Then the physical kernel

$$K^{\rm phys}(a_b^\dagger,t_b;a_a,t_a)=\int_0^\infty \frac{d(-p_0)}{2\pi}e^{i(-p_0)(t_b-t_a)}\times\sum_{n_1',n_2'=0}^{\infty}\sum_{m=-(n_1'+n_2')}^{n_1'+n_2'}\int_{\lambda_a}^{\infty}d\lambda_b e^{i(\lambda_b-\lambda_a)[k-2(n_1'+n_2'+1)\omega]}$$
$$\times\frac{(\rho_b^*\rho_a)^{(n_1'+n_2')}(\sigma_b^*\sigma_a)^{(n_1'-n_2')}(\delta_b^*\delta_a)^{2m}}{\left[\prod_{i=1}^{4}\Gamma(1+n_i)\right]}, \tag{30}$$

where the new quantum numbers $n_1'$, $n_2'$, and $m$ are defined as

$$\begin{aligned} n_1&=n_1'+m,\\ n_2&=n_2'-m,\\ n_3&=n_2'+m,\\ n_4&=n_1'-m. \end{aligned} \tag{31}$$

These are quantum numbers of the hydrogen atom in parabolic coordinates. Three new complex parameters $\rho$, $\sigma$, and $\delta$ are defined as

$$\rho=(a_+b_+a_-b_-)^{1/2},$$
$$\sigma=\left(\frac{a_+b_-}{a_-b_+}\right)^{1/2},$$
$$\delta=\left(\frac{a_+a_-}{b_+b_-}\right)^{1/2}.$$

We can decompose the kernel in Eq. (30) as

$$K^{\rm phys}(a_b^\dagger,t_b;a_a,t_a)=\int_0^\infty\frac{d(-p_0)}{2\pi}\sum_{n_1',n_2'=0}^{\infty}\sum_{m=-(n_1'+n_2')}^{n_1'+n_2'}\int_{\lambda_a}^{\infty}d\lambda_b e^{i(-p_0)(t_b-t_a)}$$
$$\times\langle n_1',n_2',m|a_b\rangle^*\langle n_1',n_2',m|U(\lambda_b-\lambda_a)|a_a\rangle, \tag{32}$$

where $\langle n_1',n_2',m|a_b\rangle$ is the projection of the final coherent states into the energy eigenstates, and $\langle n_1',n_2',m|U(\lambda_b-\lambda_a)|a_a\rangle$ is the projection of the time evolution of the initial coherent state into the energy eigenstates. They are

$$\langle n_1',n_2',m|a_b\rangle=\frac{(\rho_b)^{(n_1'+n_2')}(\sigma_b)^{(n_1'-n_2')}(\delta_b)^{2m}}{\left[\prod_{i=1}^{4}\Gamma(1+n_i)\right]^{1/2}}, \tag{33}$$

$$\langle n_1',n_2',m|U(\lambda_b-\lambda_a)|a_a\rangle=e^{i[k-2(n_1'+n_2'+1)\omega](\lambda_b-\lambda_a)}\frac{(\rho_a)^{(n_1'+n_2')}(\sigma_a)^{(n_1'-n_2')}(\delta_a)^{2m}}{\left[\prod_{i=1}^{4}\Gamma(1+n_i)\right]^{1/2}}. \tag{34}$$

Thus the coherent states are defined as

$$|a(\lambda)\rangle=\sum_{n_1',n_2'=0}^{\infty}\sum_{m=-(n_1'+n_2')}^{n_1'+n_2'}\frac{(\rho)^{(n_1'+n_2')}(\sigma)^{(n_1'-n_2')}(\delta)^{2m}}{\left[\prod_{i=1}^{4}\Gamma(n_i+1)\right]^{\frac{1}{2}}}e^{i\lambda[k-2(n_1'+n_2'+1)\omega]}|n_1n_2n_3n_4\rangle, \tag{35}$$

where $|n_1n_2n_3n_4\rangle$ is the state of four oscillators, and $a$ is the eigenvalue of the lowering operator at $\lambda=0$. The parametric time evolution of the the eigenvalue $a$ is given by

$$a(\lambda)=ae^{-i\omega\lambda}. \tag{36}$$

Thus the coherent state is given by Eq. (35), and there the initial state is described by the three complex parameters $\rho$, $\sigma$, and $\delta$.

## IV. EXPECTATION VALUES OF THE PHYSICAL VARIABLES AND THEIR UNCERTAINTIES

The expectation values of $x_1$, $x_2$, $x_3$, and $r$ can be evaluated by using the transformation in Eqs. (5) and (6). The physical trajectories depend on $\langle p_4\rangle$, and we choose $\langle p_4\rangle=0$ as the initial condition. By using the parametrization in Eq. (22), we write

$$\begin{aligned}\langle 2\hat{\xi}_B\hat{\xi}_A\rangle&=\langle a(\lambda)|[(\hat{a}_-^\dagger+\hat{a}_+)(\hat{a}_+^\dagger+\hat{a}_-)\pm(\hat{b}_-^\dagger+\hat{b}_+)\\&\quad\times(\hat{b}_+^\dagger+\hat{b}_-)]|a(\lambda)\rangle\\&=[(a_-^\dagger+a_+)(a_+^\dagger+a_-)\pm(b_-^*+b_+)(b_-^*+b_+)]\end{aligned} \tag{37}$$

and

$$\begin{aligned}\langle\hat{\xi}_A^*\hat{\xi}_A\pm\hat{\xi}_B^*\hat{\xi}_B\rangle&=\tfrac{1}{2}\langle a(\lambda)|\{[(\hat{a}_-^\dagger+\hat{a}_+)(\hat{a}_+^\dagger+\hat{a}_-)\pm(\hat{a}_-^\dagger+\hat{a}_+)\\&\quad\times(\hat{a}_+^\dagger+\hat{a}_-)]\}|a(\lambda)\rangle=\tfrac{1}{2}\{[|a_+|^2+|a_-|^2\\&\quad+a_-a_++a_-^*a_+^*+1]\pm[|b_-|^2+|b_+|^2\\&\quad+b_-b_++b_-^*b_+^*+1]\}.\end{aligned} \tag{38}$$

In Eqs. (37) and (38) we do not write the explicit parametric time dependence of the eigenvalues $a_\pm$ and $b_\pm$, which is given by Eq. (36). In Eq. (38), the term 1 comes from the ordering of $\hat{a}^\dagger$ and $\hat{a}$. Thus the expectation value of $r$ includes a quantum ordering contribution. We choose $a_\pm$ and $b_\pm$, without losing generality, as

$$\begin{aligned}a_\pm&=|a_\pm|e^{-i\omega\lambda+i\Delta},\\b_\pm&=|b_\pm|e^{-i\omega\lambda+i\Delta},\end{aligned} \tag{39}$$

where $\Delta$ is the argument of the complex numbers $a_\pm$ and $b_\pm$. Then the expectation value of the position vector $\vec{r}$ is

$$\begin{aligned}\langle\vec{r}\rangle&=\frac{\vec{\alpha}}{(2m|p_0|)^{1/2}}[1+e\cos 2(\omega\lambda-\Delta)]\\&\quad+\frac{\vec{\beta}}{(2m|p_0|)^{1/2}}\sin 2(\omega\lambda-\Delta),\end{aligned} \tag{40}$$

where the vectors $\vec{\alpha}$ and $\vec{\beta}$ are given in terms of the unit vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ as

$$\begin{aligned}\vec{\alpha}&=\hat{e}_1(|a_+||b_+|+|a_-||b_-|+|a_-b_+|+|a_+b_-|)\\&\quad+\frac{\hat{e}_3}{2}[(|a_+|+|a_-|)^2-(|b_+|+|b_-|)^2]\\\vec{\beta}&=\hat{e}_2[|a_+b_-|-|a_-b_+|],\end{aligned}$$

and the eccentricity $e$ is

$$e=2(|a_+a_-|)+(|b_+b_-|)/|\vec{\alpha}|. \tag{41}$$

With the initial condition Eq. (28), $\langle\vec{r}\rangle_\lambda$ describes an ellipse with a semimajor axis $\alpha$ and a semiminor axis $\beta$:

$$\alpha=|\vec{\alpha}|=\frac{(2m|p_0|)^{-1/2}}{2}[|a_+|^2+|b_+|^2+|a_-|^2+|b_-|^2], \tag{42}$$

$$\beta=|\vec{\beta}|=\frac{(2m|p_0|)^{-1/2}}{2}[|a_+b_-|^2-|a_-b_+|^2].$$

The expectation values of $p_0^{-1}$ and $r$ are found as

$$\langle p_0^{-1}\rangle=-\frac{2[(|a_+|^2+|b_+|^2+|b_-|^2+|a_-|^2+2)/2]^2}{mk^2}, \tag{43}$$

$$\begin{aligned}\langle r\rangle_\lambda&=\frac{1}{mk}[(|a_+|^2+|b_+|^2+|a_-|^2+|b_-|^2+2)/2]^2\\&\quad\times[1+e\cos 2(\omega\lambda-\delta)],\end{aligned} \tag{44}$$

respectively, and these give the value of $\tilde{\alpha}$ as

$$\tilde{\alpha}=\frac{1}{mk}[(|a_+|^2+|b_+|^2+|a_-|^2+|b_-|^2+2)/2]^2. \tag{45}$$

The difference between $\alpha$ and $\tilde{\alpha}$ in the Eqs. (42) and (45) comes from the quantum ordering contributions. We calculate the dispersions of $x_1$, $x_2$, $x_3$, and $r$ in the same way. The result is

$$(\Delta x_1)^2_\lambda=(\Delta x_2)^2_\lambda=\frac{\tilde{\alpha}^{3/2}}{\sqrt{mk}}[1+e\cos 2(\omega\lambda-\delta)]-\frac{\tilde{\alpha}}{mk}, \tag{46}$$

$$(\Delta x_3)^2_\lambda=(\Delta r)^2_\lambda=\frac{\tilde{\alpha}^{3/2}}{\sqrt{mk}}[1+e\cos 2(\omega\lambda-\delta)]-\frac{\tilde{\alpha}}{2mk}.$$

The ground state is obtained by substituting $|a_+|=|a_-|=|b_+|=|b_-|=0$ in Eqs. (43) and (44). Then there exist only the quantum ordering contributions, and the corresponding orbit becomes a circle with a Bohr radius $a_0$. The dispersions in this state are

$$(\Delta x_1)^2=(\Delta x_2)^2=0$$

and

$$(\Delta x_3)^2=(\Delta r)^2=\frac{a_0^2}{2}.$$

We derive Kepler's third law. The physical period of the orbit $T$ is defined as

$$T=\int_0^{2\pi/2\omega}d\lambda\langle r\rangle_\lambda=\int_0^{2\pi/2\omega}d\lambda\,\tilde{\alpha}[1+e\cos 2(\omega\lambda-\delta)].$$

The result gives the third law of Kepler:

$$T=\frac{a_0\pi}{\omega}=2\pi m\tilde{\alpha}\langle(-2mp_0)^{-1}\rangle^{1/2}=2\pi\tilde{\alpha}^{3/2}\sqrt{\frac{m}{k}}.$$

## V. CONTINUUM STATES

To obtain positive-energy states, we rewrite the kernel as

$$K(a_b^\dagger,t_b,a_a,t_a)=\int_0^\infty\frac{d(-p_0)}{2\pi}e^{i(-p_0)(t_b-t_a)}\int_C dz\,\frac{e^{i\pi z}}{2i\sin\pi z}e^{i(\lambda_b-\lambda_a)[k-2\omega(z+1)]}$$
$$\times\sum_{(n_1'-n_2'),m}\frac{(\rho_b^*\rho_a)^z(\sigma_b^*\sigma_a)^{(n_1'-n_2')}(\delta_b^*\delta_a)^{2m}}{\Pi\varepsilon_1=-1,+1\,\varepsilon_2=-1,+1\Gamma\left(\frac{z+2}{2}+\varepsilon_1(n_1'-n_2')+\varepsilon_2 m\right)}, \tag{47}$$

where the contour $C$ in the complex $z$ plane comes from $+\infty-i\varepsilon$, runs along the axis to the left, passes bound-state poles $n=(n_1'+n_2')=0,1,2,\ldots$, and returns again to $+\infty+i\varepsilon$ above the real axis. When $p_0>0$, $\omega=-i|\omega|$, the function $e^{i\omega z}$ behaves as

$$\lim_{|z|\to\infty}e^{-2|\omega|z}\to 0\quad\text{for}\quad-\pi/2\langle\arg z\langle\pi/2. \tag{48}$$

Then the contour $C$ can be opened up to run along the imaginary axis, and the kernel becomes

$$K(a_b^\dagger,t_b,a_a,t_a)=\int_0^\infty\frac{d(p_0)}{2\pi}e^{ip_0(t_b-t_a)}\int_{-i\infty}^{+i\infty}\frac{dze^{-\pi z}}{i\sinh\pi z}e^{i(\lambda_b-\lambda_a)[k-2|\omega|z]}$$
$$\times\sum_{(n_1'-n_2'),m}\frac{(\rho_b^*\rho_a)^{iz-1}(\sigma_b^*\sigma_a)^{(n_1'-n_2')}(\delta_b^*\delta_a)^{2m}}{\Pi_{\varepsilon_1=-1,+1\varepsilon_2=-1,+1}\Gamma\left(\frac{iz+1}{2}+\varepsilon_1(n_1'-n_2')+\varepsilon_2 m\right)}. \tag{49}$$

For positive energies the coherent states become

$$|a(\lambda)\rangle=\int_0^\infty\frac{d(p_0)}{2\pi}e^{ip_o t}\int_{-i\infty}^{+i\infty}\frac{dze^{-\pi z}}{i\sinh\pi z}e^{i\lambda[k-2|\omega|z]}\sum_{(n_1'-n_2'),m}\frac{\rho^{iz-1}\sigma^{(n_1'-n_2')}\delta^{2m}}{\Pi_{\varepsilon_1=-1,+1\varepsilon_2=-1,+1}\Gamma\left(\frac{iz+1}{2}+\varepsilon_1(n_1'-n_2')+\varepsilon_2 m\right)}. \tag{50}$$

The expectation values of the physical variables and their dispersions show that the trigonometric function in Eqs. (44) and (46) become hyperbolic functions, and the dispersions increase for the positive-energy coherent states of the hydrogen atom.

## VI. CONCLUSION

In the paper we discussed the long-standing problem of quantum mechanics, and derived coherent states for the negative energies of the hydrogen atom. We also showed that the centers of the wave packets travel along the Keplerian ellipse for negative energies. We evaluated the semimajor axis $\alpha$ and the eccentricity $e$ of the ellipse, and showed that $T$, the physical period of the orbit, satisfies Kepler's third law. We expressed classical trajectories in terms of the time parameter $\lambda$, which is proportional to an eccentric anomaly [6]. We showed that mean values of the physical positions of the particles $x_1$, $x_2$, $x_3$, and $r$, and their dispersions, oscillate with the same frequency $2\omega$ in parametric time $\lambda$, and that dispersions have maximum (or minimum) values at the farthest (or nearest) points of the orbit.

We also derived coherent states for the positive energies and showed that in this case the centers of the wave packets travel along the hyperbolas. We showed that dispersions are at a minimum at the nearest point of the orbit, and that they increase as hyperbolic functions of the parametric time. This approach may be applied to other problems which can be transformed to oscillators.

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## APPENDIX: DERIVATION OF THE KERNEL AND THE COHERENT STATES IN CONFIGURATION SPACE

In this paper we derived a kernel between coherent states. The relation between this kernel and the one defined between the points of the complex configuration space is

$$K_\omega^{\rm phys}(\xi_b^\dagger,\xi_b;\xi_a^\dagger,\xi_a)=\int\frac{da_b^\dagger da_b}{(2\pi i)^4}\int\frac{da_a^\dagger da_a}{(2\pi i)^4}\times e^{-a_b^\dagger a_b-a_a^\dagger a_a}\langle\xi_b^\dagger,\xi_b|a_b\rangle\times K_\omega^{\rm phys}(a_b^\dagger;a_a)\langle a_a|\xi_a^\dagger,\xi_a\rangle. \tag{A1}$$

The matrix elements $\langle a|\xi^\dagger,\xi\rangle$ and $\langle\xi^\dagger,\xi|a\rangle$ can be calculated by using the representation of $a^\dagger$ and $a$ in terms of $\xi^\dagger$, $\xi$ and $p_{\xi^\dagger}$, $p_\xi$ in Eqs. (11) and (12). These are

$$\langle\xi^\dagger,\xi|a\rangle=\exp[-(\xi_A^*\xi_A+\xi_B^*\xi_B)+\sqrt{2}(a_+\xi_A+b_+\xi_B+a_-\xi_A^*+b_-\xi_B^*)-\tfrac{1}{2}(a_+a_-+b_+b_-)]$$

and

$$\langle a|\xi^\dagger,\xi\rangle^*=\langle\xi^\dagger,\xi|a\rangle, \tag{A2}$$

where $a_+,b_+,a_-$, and $b_-$ are eigenvalues of the coherent states $|a\rangle$. We substitute $\langle\xi^\dagger,\xi|a\rangle$ into Eq. (A1), and integrate over $a_b^\dagger,a_b$ and $a_a^\dagger,a_a$. The result is

$$K_\omega^{\rm phys}(\xi_b^\dagger,\xi_b,\xi_a^\dagger,\xi_a)=\left(\frac{1}{2i\sin\omega\Lambda}\right)^2\cos\left[\frac{1}{i\sin\omega\Lambda}\times(\xi_b^\dagger\xi_a+\xi_a^\dagger\xi_b)\right]\exp\left[\frac{\cos\omega\Lambda}{i\sin\omega\Lambda}\times(\xi_b^\dagger\xi_b+\xi_a^\dagger\xi_a)\right], \tag{A3}$$

where $\Lambda=\lambda_b-\lambda_a$. We decompose $\xi$ into real and imaginary parts as

$$\xi=\begin{pmatrix}u_1+iu_4\\ u_3+iu_2\end{pmatrix}; \tag{A4}$$

then the substitution of Eq. (A4) into Eq. (A3) gives

$$K_\omega^{\rm phys}(u_b,u_a)=\left(\frac{1}{2i\sin\omega\Lambda}\right)^2\cos\left[\frac{2}{i\sin\omega\Lambda}u_b^\dagger u_a\right]\times\exp\left[\frac{\cos\omega\Lambda}{i\sin\omega\Lambda}(u_b^\dagger u_b+u_a^\dagger u_a)\right]. \tag{A5}$$

This is the expression of the kernel which is given in Ref. [8]. Instead of the Eq. (A5) we can continue from Eq. (A3). By using the transformation in Eq. (7), we write $K_\omega^{\rm phys}$ in terms of $\vec{r}_b$, and $\vec{r}_a$ and obtain the configuration-space expression of the kernel for the hydrogen atom as discussed in Ref. [8]. The integrations over $x_{4b}$ and the time parameter $\lambda_b$ from $\lambda_a$ to $\infty$ give the kernel $K(\vec{r}_b,t_b;\vec{r}_a,t_a)$ as an integral over $p_0$.

To derive coherent states in the spherical coordinates we parametrize, we rewrite $\xi_A$ and $\xi_B$ explicitly as

$$\xi_A=|\xi|\cos\frac{\theta}{2}e^{(i/2)(\varphi-\gamma)},$$

$$\xi_B=|\xi|\sin\frac{\theta}{2}e^{-(i/2)(\varphi+\gamma)}.$$

We substitute $\xi_A$ and $\xi_B$ into Eq. (A2), expand the exponential into power series of $e^{(i/2)\gamma}$ and $e^{-(i/2)\gamma}$, and perform an integration over the angle $\gamma$ from 0 to $2\pi$. The angle $\gamma$ corresponds to the coordinate $x_4$. Then the coherent state becomes

$$\langle|\xi|,\theta,\varphi|a\rangle=4\pi\exp[-|\xi|^2-a^\dagger a]I_0\times[2\sqrt{2a^\dagger a|\xi|^2(1-\cos\Theta)}], \tag{A6}$$

where $\cos\Theta$ and $a^\dagger a$ are

$$\cos\Theta=\hat{a}\cdot\hat{n}\equiv\frac{1}{a^\dagger a}\left(\frac{a_+a_--b_+b_-}{2},-\frac{a_+b_-}{\sqrt{2}},-\frac{a_-b_+}{\sqrt{2}}\right)\cdot\left(\cos\theta,\frac{\sin\theta e^{i\varphi}}{\sqrt{2}},\frac{\sin\theta e^{-i\varphi}}{\sqrt{2}}\right),$$

$$a^{\dagger}a \equiv \frac{a_{+}a_{-}+a_{+}a_{-}}{2}.$$

Then we write $I_0[2\sqrt{2a^{\dagger}a|\xi|^2(1-\cos\Theta)}]$ as infinite series of $L_n^k$ and $P_l(\cos\Theta)$ by writing $I_0[2\sqrt{2a^{\dagger}a|\xi|^2(1-\cos\Theta)}]$ in terms of $I_k[2\sqrt{2a^{\dagger}a|\xi|^2}]\times\cos^k\Theta$:

$$\langle|\xi|,\theta,\varphi|a\rangle = 2\pi\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-a^{\dagger}a)^{n+k}}{\Gamma(n+k+1)}e^{-|\xi|^2}|\xi|^{2k}L_n^k(2|\xi|^2)\sum_{l=0}^{k}\frac{[1+(-1)^{k+l}](2l+1)2^l\Gamma\left[\frac{(l+k)}{2}+1\right]}{\Gamma(l+k+2)\Gamma\left[\frac{(k-l)}{2}+1\right]}P_l(\cos\Theta). \quad \text{(A7)}$$

Equation (A7) is the expression of the coherent states for the hydrogen atom in spherical coordinates.


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