Fundamental solution method applied to time evolution of two-energy-level systems: Exact and adiabatic limit results

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A method of fundamental solutions has been used to investigate transitions in two-energy-level-systems with no level crossing in a real time. Compact formulas for transition probabilities have been found in their exact form as well as in their adiabatic limit. No interference effects resulting from many-level complex crossings as announced by Joye, Mileti, and Pfister [Phys. Rev. A **44**, 4280 (1991)] have been detected in either case. It is argued that these results of this work are incorrect. However, some effects of Berry's phases are confirmed.

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I. INTRODUCTION

Transitions between energy levels in a two-energy-level system evolving in time are of great importance from many points of view. On one side, such systems provide us with the simplest models to investigate transition amplitudes between different energy levels by different approaches [1]. On the other side, these systems play an important role in experimental investigations of basic principles of quantum mechanics [2]. Recently a lot of effort has been devoted to obtain more rigorous results on the adiabatic limit of transition amplitudes for these systems [3–7]. In particular, in a series of recent papers Joye *et al.* have studied this problem by the Hilbert-space methods. Such two-energy-level systems are formally equivalent to a one-half spin system put into time-dependent magnetic field. However good approximate results and more so the exact ones are difficult to obtain for such systems even for simple time evolutions of the effective ''magnetic'' field. Therefore each opportunity of improving this situation is worth trying. A treatment of the problem by a method of fundamental solutions (so fruitful in its application to stationary problems of one-dimensional Schrödinger equation [8–10]) is of first importance, more so that to our knowledge, the method was not used so far to this goal. A possibility of application of the method is related to the fact that a linear system of first-order differential equations describing time evolution of transition amplitudes can always be transformed into a system of decoupled-second-order equations having a form of the stationary-Schrödinger equation, one for each amplitude. This allows us to apply all advantages of the fundamental solution method [10]. The only obstacle related with this approach is a complexity of effective ''potentials'' that appear in the final system of the Schrödinger-type equations.

The paper is organized as follows.

In the next section the problem of transitions in two-energy-level systems is stated and corresponding assumptions about the effective ''magnetic field'' are formulated. A linear system of two differential equations for the transition amplitudes is rewritten in form of two decoupled equations of the Schrödinger type. In Sec. III properties of the fundamental solution method are recalled. In Sec. IV some subtleties of the application of the fundamental solution method to the problems considered in the paper are discussed. The method is first applied to a particular system of the atom-atom scattering within a frame of the Nikitin model [11,12] in Sec. V. In Sec. VI results of Sec. V are generalized to systems with an algebraic time dependence of the effective magnetic field. In Sec. VII another two examples of two-energy-level systems are considered with corresponding magnetic fields depending exponentially on time. These examples, together with the ones of Secs. V and VI, show that a general structure of the transition amplitudes is independent of how the magnetic fields vary in time. This form is not affected either by the number of (complex) energy level crossings on the Stokes lines closest to the real axis of the complex time plane. The latter result confirms the findings of the previous section. Such a dependence resulting with some interference effects has been announced by Joye *et al.* [4]. In Sec. VIII we consider an example of the magnetic field with an explicit contribution of the geometrical (Berry) phase to the transition probability.

We summarize and discuss our results in the last section. In particular, we show there that the results of Joye, Mileti, and Pfister [4] on the effects of interference from many-level crossings are incorrect.

II. ADIABATIC TRANSITIONS IN TWO-ENERGY-LEVEL SYSTEMS

In general, any two-energy-level system is formally equivalent to a one-half spin system put into an external magnetic field $\mathbf{B}(t)$. Therefore, we shall consider just such a system. Its Hamiltonian $H(t)$ is given then by $H(t) = \frac{1}{2}\mu\mathbf{B}(t)\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ are Pauli's matrices so that two energy levels $E_\pm(t)$ of $H(t)$ are given by $E_\pm(t) = \pm(\mu/2)B(t)$ where $B(t)=\sqrt{\mathbf{B}^2(t)}$.

When the adiabatic transitions between the two energy levels $E_\pm(t)$ are considered then the following properties of the field $\mathbf{B}(t)$ are typically assumed.

(1) $\mathbf{B}(t)$ is real being defined for the real t, $-\infty<t<+\infty$.

(2) $\mathbf{B}(t)$ can be continued analytically off the real values

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of t as a meromorphic function defined on some t-Riemann surface $\mathbf{R}_B$. A sheet of $\mathbf{R}_B$ from which $\mathbf{B}(t)$ is originally continued is called physical.

(3) On the physical sheet $\mathbf{B}(t)$ is analytic in an infinite strip $\Sigma=\{t:|\mathrm{Im}\,t|<\delta,\delta>0\}$, without roots in the strip and achieves there finite limits for $\mathrm{Re}\,t=\pm\infty$, i.e., $\mathbf{B}(\mathrm{Re}\,t=\pm\infty)=\mathbf{B}^{\pm}\neq\mathbf{0}$ in the strip;

The field $\mathbf{B}(t)$ depends additionally on a parameter T (>0), i.e., $\mathbf{B}(t)\equiv\mathbf{B}(t,T)$ which introduces a ''natural'' scale of time to the system so that its time evolution is expressed most naturally in units of T. If T is small in comparison with the actual period of the process considered then the latter is ''fast'' or ''sudden.'' If, however, T is large in this comparison then the process is ''slow'' or ''adiabatic.''

In the adiabatic process of the system the following is assumed about $\mathbf{B}(t,T)$.

(4) A dependence of $\mathbf{B}(t,T)$ on T is such that a rescaled field $\mathbf{B}(sT,T)$ has the following asymptotic behavior for $T\to+\infty$:

$$\mathbf{B}(sT,T)\sim\mathbf{B}_0(s)+\frac{1}{T}\mathbf{B}_1(s)+\frac{1}{T^2}\mathbf{B}_2(s)+\cdots, \tag{1}$$

while its s-Riemann surface $\mathbf{R}_B/T$ approaches ''smoothly'' the topological structure of the Riemann surface corresponding to the first term $\mathbf{B}_0(s)$ of the expansion (1).

(5) With respect to its dependence on s, the field $\mathbf{B}_0(s)$ satisfies properties (1)–(3) above with substitutions $t\to s$ and $\mathbf{B}(s)\to\mathbf{B}_0(s)$.

Note that condition (3) excludes periodic fields $\mathbf{B}(t)$.

The time-dependent Schrödinger equation induced by $H(t)$ takes therefore a form

$$\frac{i}{T}\frac{d\Psi(s,T)}{ds}=\frac{1}{2}\mu\mathbf{B}(sT,T)\cdot\boldsymbol{\sigma}\Psi(s,T). \tag{2}$$

The adiabatic regime of evolution of the wave function $\Psi(s,T)$ corresponds now to taking a limit $T\to+\infty$ in Eq. (2).

The main problem of the adiabatic limit in the considered case is to find in this limit the transition amplitude between the two energy levels of the system for $s\to+\infty$ under the assumptions that $\Psi(-\infty,T)$ coincides with one of the two possible eigenstates $\Psi_{\pm}(-\infty,T)$ of $H(-\infty)$ and that there is no level crossing for real t, i.e., $\liminf_{-\infty<t<+\infty}B(t)\geqslant\epsilon>0$. Known approximate solutions of this problem are that of Landau [13] and Zener [14] in a form of the so-called Landau-Zener formula and that of Dykhne [15] who have shown that such an amplitude should be exponentially small in the limit $T\to+\infty$. In the next sections we shall show how to get an exact (i.e., not approximate) result for this amplitude as well as its adiabatic limit with the help of the fundamental solutions.

A typical way of proceeding when the adiabatic limit is investigated is using eigenvectors $\Psi_{\pm}(s,T)$ of $H(sT,T)$ satisfying $(\Psi_{\pm},\dot{\Psi}_{\pm})=0$. Then, such eigenvectors $\Psi_{\pm}(s,T)$ can be chosen as the following ones:

$$\Psi_+(s,T)=\exp\left(-i\int_0^s\dot{\phi}\sin^2\frac{\Theta}{2}d\sigma\right)\begin{bmatrix}\cos\frac{\Theta}{2}\\ \sin\frac{\Theta}{2}e^{i\phi}\end{bmatrix},$$
$$\Psi_-(s,T)=\exp\left(-i\int_0^s\dot{\phi}\cos^2\frac{\Theta}{2}d\sigma\right)\begin{bmatrix}\sin\frac{\Theta}{2}\\ -\cos\frac{\Theta}{2}e^{i\phi}\end{bmatrix}, \tag{3}$$

where Θ and ϕ are polar and azimuthal angles of the vector $\mathbf{B}(t,T)$, respectively, and dots over different quantities mean derivatives with respect to s variable.

The wave function $\Psi(s,T)$ can now be represented as

$$\Psi(s,T)=a_+(s,T)\exp\left(-iT\int_{s'}^{s}E_+(\xi,T)d\xi\right)\Psi_+(s,T)$$
$$+a_-(s,T)\exp\left(-iT\int_{s'}^{s}E_-(\xi,T)d\xi\right)\Psi_-(s,T), \tag{4}$$

where s' takes *any* real but fixed value.

The Schrödinger equation (2) can be rewritten in terms of the coefficients $a_{\pm}(s,T)$ as the following linear system of two equations:

$$\dot{a}_+(s,T)=c(s,T)\exp\left(i\int_{s'}^{s}\omega(\xi,T)d\xi\right)a_-(s,T),$$
$$\dot{a}_-(s,T)=-c^*(s,T)\exp\left(-i\int_{s'}^{s}\omega(\xi,T)d\xi\right)a_+(s,T), \tag{5}$$

where

$$c(s,T)=-\frac{\dot{\Theta}}{2}+\frac{i\dot{\phi}}{2}\sin\Theta$$
$$=-\frac{1}{2}\frac{[\mathbf{B}\times(\mathbf{B}\times\dot{\mathbf{B}})]_z}{B^2\sqrt{B_x^2+B_y^2}}+\frac{i}{2}\frac{(\mathbf{B}\times\dot{\mathbf{B}})_z}{B\sqrt{B_x^2+B_y^2}}, \tag{6}$$

$$\omega(s,T)=T(E_+-E_-)-\dot{\phi}\cos\Theta=\mu TB-\frac{B_z}{B}\frac{(\mathbf{B}\times\dot{\mathbf{B}})_z}{B_x^2+B_y^2}.$$

The system (5) can be rewritten further as the following linear system of second-order equations:

$$\ddot{a}_+-\left(\frac{\dot{c}}{c}+i\omega\right)\dot{a}_++|c|^2a_+=0,$$
$$\ddot{a}_--\left(\frac{\dot{c}^*}{c^*}-i\omega\right)\dot{a}_-+|c|^2a_-=0, \tag{7}$$

where the coefficient functions $a_\pm$ decouple from each other being however still related by Eq. (5).

By the following transformations,

$$a_+(s,T)=\exp\left[\tfrac{1}{2}\int_{s'}^{s}\left(\frac{\dot{c}}{c}+i\omega\right)d\xi\right]b_+(s,T)$$
$$a_-(s,T)=\exp\left[\tfrac{1}{2}\int_{s'}^{s}\left(\frac{\dot{c}^*}{c^*}-i\omega\right)d\xi\right]b_-(s,T), \tag{8}$$

we bring the Eqs. (8) to Schrödinger types

$$\ddot{b}_\pm(s,T)+T^2 q_\pm(s,T)b_\pm(s,T)=0, \tag{9}$$

where

$$q_+(s,T)=\frac{1}{T^2}\left[-\frac{1}{4}\left(\frac{\dot{c}}{c}+i\omega\right)^2+|c|^2\right]+\frac{1}{2T^2}\left(\frac{\dot{c}}{c}+i\omega\right)\cdots, \tag{10}$$

$$q_-(s,T)=\frac{1}{T^2}\left[-\frac{1}{4}\left(\frac{\dot{c}^*}{c^*}-i\omega\right)^2+|c|^2\right]+\frac{1}{2T^2}\left(\frac{\dot{c}^*}{c^*}-i\omega\right)\cdots,$$

so that for real s (and T) we have

$$q_-(s,T)=q_+^*(s,T). \tag{11}$$

The Eqs. (9) are now basic for our further analysis since their form is just of the stationary one-dimensional (1D) Schrödinger equation.

First let us note that the dependence of the ''potential'' function $q_+(s,T)$ on T is given by

$$q_+(s,T)=\frac{1}{4}\mu^2B^2+\frac{i\mu}{2T}\left[\dot{B}-B\left(\frac{\dot{c}}{c}-i\dot{\phi}\cos\Theta\right)\right]+\frac{1}{T^2}\left[-\frac{1}{4}\left(\frac{\dot{c}}{c}-i\dot{\phi}\cos\Theta\right)^2+|c|^2\right]+\frac{1}{2T^2}\left(\frac{\dot{c}}{c}-i\dot{\phi}\cos\Theta\right), \tag{12}$$

where the dependence of B,c,Θ,ϕ on T in Eq. (12) is also anticipated. By Eq. (11) we get a corresponding dependence of $q_-(s,T)$ on T. Taking into account Eqs. (1) and (6) it is easy to check that the last formula provides us with the following type of asymptotic behavior of $q_+(s,T)$ for large T,

$$q_+(s,T)=q_+^{(0)}(s)+\frac{1}{T}q_+^{(1)}(s)+\frac{1}{T^2}q_+^{(2)}(s)+\cdots. \tag{13}$$

Therefore the above form of dependence of $q_\pm(s,T)$ on T permits us to apply to the considered case the method of fundamental solutions. For this reason we shall start the next section with a review of basic principles of the method suitably adapted to the considered case.

III. FUNDAMENTAL SOLUTIONS AND THEIR PROPERTIES

Consider first $q_\pm(s,T)$ as functions of s. They are defined completely by an s dependence of field $\mathbf{B}(Ts,T)$. According to our assumptions, the latter is meromorphic on some Riemann surface $\mathbf{R}_B/T$. However, by Eq. (12), $q_\pm(s,T)$ are algebraic functions of $\mathbf{B}$, $\dot{\mathbf{B}}$, and $\ddot{\mathbf{B}}$ and, therefore, they are also meromorphic functions of s defined again on some other Riemann surfaces $\mathbf{R}_\pm$ determined by these algebraic dependencies. As it follows from Eq. (12) topological structures of $\mathbf{R}_\pm$ can be quite complicated. However, in what follows, we are interested in the adiabatic limit $T\to+\infty$ by which the structure of $\mathbf{R}_\pm$ should be determined for T large enough basically by the first term $q_+^{(0)}(s)$ of the expansion (13). In consequence, by Eq. (12), it should be determined by $\mu\mathbf{B}^{(0)}(s)$, i.e., by the first term of the expansion (1). The structure of $\mathbf{R}_\pm$ can turn out to be much simpler in this limit. Despite this supposed complexity of $q_\pm(s,T)$ and of their Riemann surfaces we shall introduce and discuss the fundamental solutions to the Eqs. (9) without simplifications. We shall do it for the $q_+(s,T)$ case of Eq. (12). An extension of the discussion to the $q_-(s,T)$ case will be obvious.

A standard way of introducing the fundamental solutions is a construction of a Stokes graph [8–10] related to a given $q_+(s,T)$. Such a construction, according to Fröman and Fröman [8] and Fedoriuk [9], can be performed in the following way [10].

Let Z denote a set of all the points of $\mathbf{R}_+$ at which $q_+(s,T)$ has its single or double poles. Let $\delta(x)$ be a meromorphic function on $\mathbf{R}_+$, the unique singularities of which are double poles at the points collected by Z with coefficients at all the poles equal to 1/4 each. (In a case when $\mathbf{R}_+$ is simply a complex plain, the latter function can be constructed in general with the help of the Mittag-Leffler theorem [17]. But for a case of branched $\mathbf{R}_+$ the general procedure is unknown to us.) Consider now a function

$$\tilde{q}_+(s,T)=q_+(s,T)+\frac{1}{T^2}\delta(s). \tag{14}$$

The presence and the role of the δ term in Eq. (14) are explained below. This term contributes to Eq. (14) if and only when the corresponding ''potential'' function $q_+(s,T)$ contains simple or second order poles. (Otherwise the corresponding δ term is put to zero.) It is called the Langer term [10,18].

The Stokes graph corresponding to the function $\tilde{q}_+(s,T)$ consists now of Stokes lines emerging from roots (turning points) of $\tilde{q}_+(s,T)$. Stokes lines satisfy one of the following equations:

$$\mathrm{Im}\int_{s_i}^{s}\sqrt{\tilde{q}_+(\xi,T)}d\xi=0 \tag{15}$$

with s_i being a root of $\tilde{q}_+(s,T)$. We shall assume further a generic situation when all the roots s_i are simple.

Stokes lines that are not closed end at these points of $\mathbf{R}_+$ (i.e., have the latter points as their boundaries) for which the action integral in Eq. (15) becomes infinite. Of course such points are singular for $\tilde{q}_+(s,T)$ and they can be its finite poles or its poles lying at an infinity.

Each such singularity z_k of $\tilde{q}_+(s,T)$ defines a domain called a sector. This is the connected domain of $\mathbf{R}_+$ bounded by Stokes lines and z_k itself. The latter is also a boundary for the Stokes lines being an isolated boundary point of the sector (as it is in the case of the second-order pole).

In each sector the left-hand-side (LHS) in Eq. (15) is only positive or only negative.

Consider now Eq. (9) for $b_+(s,T)$. Following Fröman and Fröman in each sector S_k having a singular point z_k at its boundary one can define a solution of the form

$$b_{+,k}(s,T)=\tilde{q}_+^{-1/4}(s,T)e^{\sigma iTW(s,T)}\chi_{+,k}(s,T), \quad k=1,2,\ldots, \tag{16}$$

where

$$\begin{aligned}\chi_{+,k}(s,T)=1+\sum_{n\geq 1}\left(-\frac{\sigma}{2iT}\right)^n\int_{z_k}^{s}d\xi_1 \\ \times\int_{z_k}^{\xi_1}d\xi_2\cdots\int_{z_k}^{\xi_{n-1}}d\xi_n\Omega(\xi_1)\Omega(\xi_2)\cdots\Omega(\xi_n) \\ \times(1-\exp\{-2\sigma iT[W(s)-W(\xi_1)]\}) \\ \times(1-\exp\{-2\sigma iT[W(\xi_1)-W(\xi_2)]\})\cdots \\ \times(1-\exp\{-2\sigma iT[W(\xi_{n-1})-W(\xi_n)]\})\end{aligned} \tag{17}$$

with

$$\Omega(s,T)=\frac{\delta(s)}{\tilde{q}_+^{1/2}(s,T)}-\frac{1}{4}\frac{\tilde{q}_+''(s,T)}{\tilde{q}_+^{3/2}(s,T)}+\frac{5}{16}\frac{\tilde{q}_+'^2(s,T)}{\tilde{q}_+^{5/2}(s,T)} \tag{18}$$

and

$$W(s,T)=\int_{s_i}^{s}\sqrt{\tilde{q}(\xi,T)}d\xi \tag{19}$$

where s_i is a root of $\tilde{q}(s,T)$ lying at the boundary of S_k.

In Eqs. (16) and (18) a sign of σ $(=\pm 1)$ and an integration path are chosen in such a way to have

$$\sigma\,\mathrm{Im}[W(\xi_j)-W(\xi_{j+1})]\leq 0 \tag{20}$$

for any ordered pair of integration variables (with $\xi_0=s$). Such an integration path is then called canonical. Of course, the condition (20) means that $b_{+,k}(s,T)$ vanishes in its sector when $s\to z_k$ along the canonical path. The Langer δ term appearing in Eqs. (14) and (18) is necessary to ensure all the integrals in Eq. (18) to converge when z_k is a first or a second order pole of $\tilde{q}(s,T)$ or when the solutions (16) are to be continued to such poles. As it follows from Eq. (18) each such pole z_k demands a contribution to $\delta(s)$ of the form $[2(s-z_k)]^{-2}$, what has been already assumed in the corresponding construction of $\delta(s)$.

IV. THE ADIABATIC LIMIT IN THE FUNDAMENTAL SOLUTION APPROACH

Consider now the consequences of taking the large-T limit for the above description. We assume that for a given $\tilde{q}_+(s,T)$ and its Riemann surface $\mathbf{R}_+$ the corresponding Stokes graph $\mathbf{G}_+$ is drawn. It is drawn, of course, on the Riemann surface $\sqrt{\mathbf{R}_+}$ corresponding to $\sqrt{\tilde{q}_+(s,T)}$.

First let us notice that singular points of $\tilde{q}_+(s,T)$ such as its branch points and poles depend in general on T. For both kinds of these singularities this also means a dependence on T of jumps of $\tilde{q}_+(s,T)$ on its cuts as well as the T dependence of coefficients of its poles.

According to the property (4) of the magnetic field $\mathbf{B}$ (see Sec. II) we can expect that the singular structure of $\tilde{q}_+(s,T)$, i.e., positions of its roots and poles, as well as the cut jumps and pole coefficients, change smoothly in this limit to their final positions and values, respectively. This limit structure is defined by the singularity structure of $\tilde{q}_+^{(0)}(s,T)$ [see expansion (13)]. Therefore, both the topology of $\sqrt{\mathbf{R}_+}$ and the associated Stokes graph $\mathbf{G}_+$ change accordingly to coincide eventually with the Riemann surface $\sqrt{\mathbf{R}_+^{(0)}}$ and with the Stokes graph $\mathbf{G}_+^{(0)}$ corresponding to $\sqrt{\tilde{q}_+^{(0)}(s,T)}$. This limit structure can be achieved in the following ways: (a) some branch points and poles of $\tilde{q}_+(s,T)$ escape to infinities of $\mathbf{R}_+$; (b) some branch points and poles of $\tilde{q}_+(s,T)$ approach the respective singularities of $\tilde{q}_+^{(0)}(s,T)$; (c) some branch points and poles of $\tilde{q}_+(s,T)$ disappear because their respective jumps and coefficients vanish in the limit $T\to+\infty$.

Being more specific we expect that for T large enough a set $\mathbf{S}_+$ of all singular points of $\tilde{q}_+(s,T)$ (i.e., containing all its branch points and poles) consists of three well-separated subsets $\mathbf{S}_+^{inf}$, $\mathbf{S}_+^{van}$ and $\mathbf{S}_+^{fin}$. The points of $\mathbf{S}_+^{inf}$ run to infinities of $\mathbf{R}_+$ when $T\to+\infty$. Those of $\mathbf{S}_+^{van}$ disappear in this limit while those of $\mathbf{S}_+^{fin}$ coincide in this limit with the set $\mathbf{S}_+^{(0)}$ of the singular points of $\tilde{q}_+^{(0)}(s,T)$.

Let us remove the points contained in $\mathbf{S}_+^{inf}\cup\mathbf{S}_+^{van}$ from the Riemann surface $\mathbf{R}_+$, i.e., let us consider these points as regular for $\tilde{q}_+(s,T)$. Then $\mathbf{R}_+$ will transform into $\mathbf{R}_+^{fin}$—a Riemann surface whose singular points coincide with those of the set $\mathbf{S}_+^{fin}$.

Together with the previous operation let us remove from $\sqrt{\mathbf{R}_+}$ the Stokes lines generated by the points of $\mathbf{S}_+^{inf}\cup\mathbf{S}_+^{van}$ so that the remaining Stokes lines can uniquely continue to form the Stokes graph $\mathbf{G}_+^{fin}$ generated by the set $\mathbf{S}_+^{fin}$. It is clear that the graph $\mathbf{G}_+^{fin}$ coincides with $\mathbf{G}_+^{(0)}$ in the limit $T\to+\infty$.

The above two operations will be called the adiabatic limit reduction or simply the reduction operation.

As we have mentioned earlier there is a set of sectors and a corresponding set of fundamental solutions defined in them associated with the graph $\mathbf{G}_+$. By the reduction operation, both sets can be reduced, i.e., under this operation some sectors of $\mathbf{G}_+$ transform into corresponding sectors of $\mathbf{G}_+^{fin}$ whereas the others disappear. Obviously, the latter sectors are those that disappear when the limit $T\to+\infty$ is taken.

The following assumption should stabilize the corresponding results obtained with the help of the fundamental solution method.

(6) Among a full set of fundamental solutions associated with the Stokes graph $\mathbf{G}_+$ there is a subset of them associated with graph $\mathbf{G}_+^{fin}$ that allows us to solve the basic problem of the adiabatic transition and that is invariant under the reduction operation.

The dynamical systems described by the Hamiltonian $H(t)$ satisfying assumption (6) will be called the adiabatic limit reducible (ALR) systems.

The above assumption means that to solve the problem of the adiabatic transitions in the ALR system we can first perform the reduction operation and next work with the simplified Stokes graphs $\mathbf{G}_+^{fin}$. A set of fundamental solutions associated with this graph that can be used to solve the problem considered coincide with the corresponding ones of the full graph $\mathbf{G}_+$. The procedure used to construct a solution of the problem with the help of the latter graph is not affected by the reduction operation, i.e., it looks the same when the simplified graph $\mathbf{G}_+^{fin}$ is used instead of $\mathbf{G}_+$. Therefore the aim of the reduction operation is to make easier choosing the proper set of fundamental solution solving the problem. The results obtained in this way can be still exact if the integration paths taken on the graph $\mathbf{G}_+^{fin}$ can be mapped properly on the Stokes graph $\mathbf{G}_+$ restoring in this way the exact condition of the problem. However, if such a map is not known or is difficult to construct (because of the complicated structure of graph $\mathbf{G}_+$) the result obtained in this way can be considered only as an approximation, i.e., valid only in the limit $T\to+\infty$.

According to the above assumptions we can conclude from Eqs. (12) and (13) that there is one-to-one correspondence between the Stokes graphs $\mathbf{G}_+$ and $\mathbf{G}_+^{(0)}$ and the corresponding sets $\mathbf{S}_+^{fin}$ and $\mathbf{S}_+^{(0)}$. Namely, this correspondence is built by aggregations (blobs) of singular points of $\mathbf{S}_+^{fin}$, i.e., the branch points and poles of $\tilde{q}_+(s,T)$, which are transformed into single points of $\mathbf{S}_+^{(0)}$ when the limit $T\to+\infty$ is taken. Also there are sheaves of Stokes lines of $\mathbf{G}_+^{fin}$ emerging from the blobs and transformed into single lines of $\mathbf{G}_+^{(0)}$ in the same limit.

Therefore in the limit $T\to+\infty$ we can eventually consider for potentials (12) Stokes graphs corresponding to first terms $q_\pm^{(0)}(s)$ of the asymptotic expansions for $q_\pm(s,T)$. The first terms of the asymptotic expansions corresponding to $q_\pm^{(0)}(s)$ and $q_\pm(s,T)$ are the same in this limit and equal, according to Eq. (1), to $\frac{1}{4}\mu^2\mathbf{B}_0^2(s)$.

Let us note that properties (1)–(6) above can be satisfied by the field $\mathbf{B}$ for which $\mathbf{B}^2$ is a *meromorphic* function of t. We shall assume just such a dependence of $\mathbf{B}$ on t and of the corresponding rescaled field $\mathbf{B}(sT,T)$ on s. However, for simplicity, instead of continuing our considerations in their most general form we shall investigate first a particular example of the field $\mathbf{B}(t,T)$ that, as it seems to us, will illustrate our method in a satisfactory way.

V. THE NIKITIN MODEL OF THE ATOM-ATOM SCATTERING

The model of Nikitin [12] describes the scattering $A^*+B\to A+B+\Delta\epsilon$ of the exited atom A^* moving with a small velocity v with the impact parameter b' and scattered by the atom B. The interaction between the atoms is of the dipol-dipol type. The latter example was analyzed in the context of the adiabatic limit $v\to 0$ also by Joye *et al.* [4].

The Hamiltonian for this system reads ([11], paragraph 9.3.2 and [12])

$$H(R)=\begin{bmatrix} \frac{\Delta\epsilon}{2} & \frac{C}{R^3} \\ \frac{C}{R^3} & -\frac{\Delta\epsilon}{2} \end{bmatrix}, \tag{21}$$

where $\Delta\epsilon$ and C are constants and $R=\sqrt{b'^2+v^2t^2}$ is the distance between the atoms. Introducing $d=(2C/\Delta\epsilon)^{1/3}$ as a natural distant unit for this case and $T=d/v$ as the corresponding adiabatic parameter and rescaling: $t\to sT$ and $b'\to bd$ we get from Eq. (21)

$$H(s)=\frac{\Delta\epsilon}{2}\begin{bmatrix} 1 & \frac{1}{(b^2+s^2)^{2/3}} \\ \frac{1}{(b^2+s^2)^{2/3}} & -1 \end{bmatrix}. \tag{22}$$

In the ''magnetic field'' language we have of course $\mathbf{B}(sT,T)=((b^2+s^2)^{-3/2},0,1)\Delta\epsilon/\mu$ so that all the assumptions (1)–(6) above are satisfied with $\mathbf{B}^\pm(T)=\mathbf{B}^\pm(\pm\infty,T)=(0,0,1)\Delta\epsilon/\mu$. Since in the considered case $\phi(s)\equiv 0$ then for the corresponding quantities defined by Eqs. (6) and (12) we get,

$$c=\frac{3}{2}\,\frac{s(b^2+s^2)^{1/2}}{1+(b^2+s^2)^3},\qquad \omega=T\Delta\epsilon\left(1+\frac{1}{(b^2+s^2)^3}\right)^{1/2}, \tag{23}$$

$$q_{\pm}(s,T)=\left[\frac{\Delta\epsilon}{2}\left(1+\frac{1}{(b^2+s^2)^3}\right)^{1/2}\pm\frac{i}{2T}\left(\frac{6s(b^2+s^2)^2}{1+(b^2+s^2)^3}-\frac{s}{b^2+s^2}-\frac{1}{s}\right)\right]^2-\frac{3}{2}\frac{i\Delta\epsilon}{T}\frac{s}{[1+(b^2+s^2)^3]^{1/2}(b^2+s^2)^{5/2}}$$
$$-\frac{1}{2T^2}\left[\frac{2s^2+b^2(b^2+s^2)}{s^2(b^2+s^2)}-\frac{3}{2}\frac{4(b^2+s^2)^4(s^2-b^2)-4(b^2+s^2)(b^2+5s^2)+3s^2(b^2+s^2)}{[1+(b^2+s^2)^3]^2}\right].$$

Equations (23) show that in the limit $T\to+\infty$ the Stokes graph for the considered problem is determined by the function

$$q^{(0)}(s,T)=\frac{(\Delta\epsilon)^2}{4}\left(1+\frac{1}{(b^2+s^2)^3}\right). \tag{24}$$

The graph is shown in Fig. 1.

Each $q_{\pm}(s,T)$ has 40 roots, five branch points at $s=\pm ib$ and at $s=s_k=\pm(e^{[(2k+1)\pi i/3]}-b^2)^{1/2}$, $k=1,2,3$, as well as two poles at $s=0$. Therefore only six roots of $q^{(0)}(s,T)$ at $s=s_k$, $k=1,2,3$ and its two poles at $s=\pm ib$ look encouraging. Nevertheless, we shall consider first the case without any approximations.

At first glance the Stokes graphs corresponding to the functions $q_{\pm}(s,T)$ seem to be quite complicated. However it can be handled in the following way.

Functions $q_{\pm}(s,T)$ are determined on two sheeted Riemann surfaces $\mathbf{R}_{\pm}$, respectively, with the branch points at $s=\pm ib$ and at $s=s_k$, $k=1,2,3$ and with 40 roots distributed into halves on each sheet of the surfaces. Therefore the Riemann surfaces $\sqrt{\mathbf{R}_{\pm}}$ corresponding to $\sqrt{q_{\pm}(s,T)}$ (it will turn out that it is not necessary to introduce to the latter functions the corresponding Langer terms) are four sheeted with these 40 roots being square root branch points on them. When $T\to+\infty$ only six of these branch points survive coinciding with the six roots of $q^{(0)}(s,T)$ at $s=\pm s_k$, $k=1,2,3$ whereas $\mathbf{R}_{\pm}$ transforms into the complex s plane since the branch points of $q_{\pm}(s,T)$ at $s=\pm ib$ disappear, being transformed into the second-order poles of $q^{(0)}(s,T)$. It is easy to check, however that for finite but large T these six roots of $q^{(0)}(s,T)$ are each split initially into two. The split is the result of the square root branch points at $s=\pm ib$ to which the recovering of the finite T transforms the poles of $q^{(0)}(s,T)$ at the same points. The two copies of each of these six roots lie of course on different sheets of $\mathbf{R}_{\pm}$. Next, each of these 12 roots is still split into three by the same reason of finiteness of T. In this way, on each of the two sheets of $\mathbf{R}_{\pm}$ there are 36 roots grouped by three around their limit $s=\pm s_k$, $k=1,2,3$ achieved for $T\to+\infty$.

The remaining four roots of $q_{\pm}(s,T)$ are displaced in two pairs, one pair on each sheet of $\mathbf{R}_{\pm}$, close to the points $s=0$ at which the second-order poles of $q_{\pm}(s,T)$ are localized. When $T\to+\infty$ the roots in each pair collapse into $s=0$ multiplying the corresponding second-order poles and thus causing mutual cancellations of the latter and themselves in this limit.

Now we shall focus our attention on the Stokes graph $\mathbf{G}_-$ generated by $q_-(s,T)$ on the first sheet of $\mathbf{R}_-$ as well as on the remaining ones. It looks as in Fig. 2. The Stokes graph $\mathbf{G}_+$ corresponding to $q_+(s,T)$ can be obtained from $\mathbf{G}_-$ by complex conjugation of the latter. On the figure the wavy lines denote the cuts corresponding to the branch points of the fundamental solutions defined on $\mathbf{R}_-$. The sheet on Fig. 2 cut along the wavy lines defines a domain where all the fundamental solutions $b_{-,1}(s,T),\dots,b_{-,\bar{2}}(s,T)$ defined in the corresponding sectors $S_1,\dots,S_{\bar{2}}$ (shown in the figure) are holomorphic.

According to our earlier description of the behavior of the Riemann surface $\sqrt{R_+}$ when $T\to+\infty$ the set $\mathbf{S}_-^{inf}$ correspond-

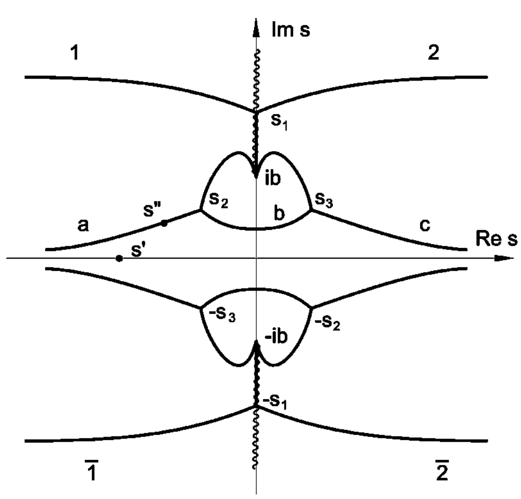

FIG. 1. The Stokes graph corresponding to ''potential'' (24).

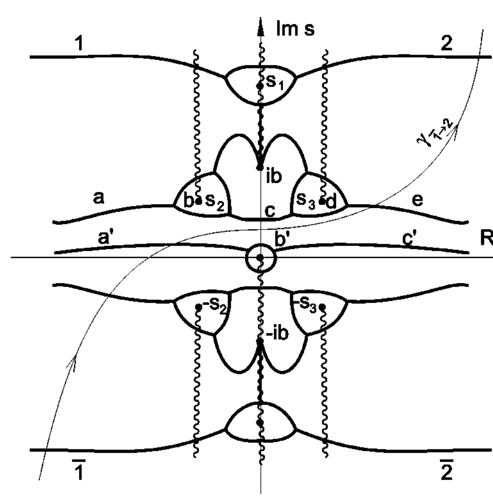

FIG. 2. The Stokes graph corresponding to ''potential'' $q_-(s,T)$ of Eq. (23).

ing to the considered case is empty, $\mathbf{S}_-^{van}$ contains four points at $s=0$ on each of the four sheets of $\sqrt{\mathbf{R}_-}$ [these four points correspond to the second-order poles of $q_-(s,T)$] and the four branch points close to $s=0$, while $\mathbf{S}_-^f$ contains all the remaining singular points of $\sqrt{q_-(s,T)}$. Now, for our case, the solution of the problem stated in this paper is simple. Namely, it can be found in the following steps.

(i) Take a linear combination of the fundamental solutions $b_{-,1}(s,T)$ and $b_{-,\bar{1}}(s,T)$ to construct the amplitude $a_-(s,T)$ with the desired property at $s=-\infty$, i.e., $\lim_{s\to-\infty}|a_-(s,T)|=0$. This amplitude is defined in this way up to a multiplicative constant.

(ii) Use Eq. (5) to construct $a_+(s,T)$ and adjust the constant mentioned earlier so that the limit $\lim_{s\to-\infty}|a_+(s,T)|=1$ can be satisfied.

(iii) Continue canonically $a_-(s,T)$ along the real s axis with the help of the solutions $b_{-,1}(s,T)$ and $b_{-,\bar{1}}(s,T)$ using to this goal the remaining fundamental solutions if necessary.

(iv) Calculate the limit $s\to+\infty$.

(v) Calculate the adiabatic limit $T\to+\infty$.

According to Eq. (9) and to the first of the above steps we have,

$$\begin{aligned}a_-(s,T)=&Aq_-^{-1/4}(s,T)\exp\left[\int_{s'}^{s}\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)(\sigma,T)d\sigma\right.\\&\left.+iT\int_{s_0}^{s}q_-^{1/2}(\sigma,T)d\sigma\right]\chi_{\bar{1}}(s,T)\\&+Bq_-^{-1/4}(s,T)\exp\left[\int_{s'}^{s}\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)(\sigma,T)d\sigma\right.\\&\left.-iT\int_{s_0}^{s}q_-^{1/2}(\sigma,T)d\sigma\right]\chi_1(s,T),\end{aligned}\tag{25}$$

where s' is any point on the real axis that is regular for the integrand while s_0 is the one from the infinite strip bounded by the Stokes line $abcde$ from one side and by $a'b'c'$ from the other (see Fig. 2), being also an arbitrary but regular point for all the integrands. The choice of signatures in Eq. (25) was done due to the fact that $\mathrm{Re}(iT\int_{s_0}^{s}q_-^{1/2}d\sigma)$ is positive (for s sufficiently large) for the sector S_1 and negative for $S_{\bar{1}}$. The latter property follows from the fact that according to Eq. (25) and the Stokes graph on Fig. 2 we have on the first sheet of $\sqrt{\mathbf{R}_-}$: $\mathrm{sgn}(\mathrm{Re}\sqrt{q_-^{1/2}(s,T)})=\mathrm{sgn}(s)$ for $s\to\pm\infty$ along the real axis.

If further we take into account the following asymptotic behavior of the relevant quantities on the real axis,

$$\begin{aligned}&\frac{1}{2}\left[\frac{\dot c}{c}-i\omega\right]+iT\sqrt{q_-}\sim-iT\Delta\epsilon-\frac{4}{s},\quad s\to-\infty,\\&\frac{1}{2}\left[\frac{\dot c}{c}-i\omega\right]-iT\sqrt{q_-}\sim O\left(\frac{1}{s^8}\right),\quad s\to-\infty\end{aligned}\tag{26}$$

then we can conclude that $B=0$ in Eq. (25).

To fix the value of the constant A in Eq. (25) we can use the second of relations (5) and apply the condition mentioned in the second step of the procedure, i.e.,

$$\lim_{s\to-\infty}\left[-\frac{1}{c(s,T)}\cdot\exp\left(i\int_{s'}^{s}\omega d\sigma\right)\dot a_-(s,T)\right]=1$$

to get,

$$\begin{aligned}A=&\frac{1}{T\Delta\epsilon}\sqrt{\frac{\Delta\epsilon}{2}}\exp\left\{-\int_{s'}^{s_0}i\omega ds+\int_{-\infty}^{s_0}\left[-\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)\right.\right.\\&\left.\left.+iT\sqrt{q_-}\right]ds+\ln c(s_0)\right\}.\end{aligned}\tag{27}$$

Therefore, for the amplitude $a_-(s,T)$ we obtain finally

$$\begin{aligned}a_-(s,T)=&\frac{1}{T\Delta\epsilon}\sqrt{\frac{\Delta\epsilon}{2}}q_-^{-1/4}(s,T)\exp\left\{-\int_{s'}^{s_0}i\omega ds\right.\\&+\int_{-\infty}^{s_0}\left[-\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)+iT\sqrt{q_-}\right]ds+\ln c(s_0)\\&\left.+\int_{s_0}^{s}\left[\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)+iT\sqrt{q_-}\right]d\sigma\right\}\chi_{\bar{1}}(s,T).\end{aligned}\tag{28}$$

Now we can take the limit $s\to+\infty$ in the above formula, continuing along the canonical path $\gamma_{\bar{1}\to2}$ shown in Fig. 2, to get,

$$\begin{aligned}a_-(+\infty,T)=&\frac{1}{iT\Delta\epsilon}\exp\left\{-\int_{s'}^{s_0}i\omega ds+\int_{-\infty}^{s_0}\left[-\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)\right.\right.\\&\left.+iT\sqrt{q_-}\right]ds+\ln c(s_0)+\int_{s_0}^{+\infty}\left[\frac{1}{2}\left(\frac{\dot c}{c}-i\omega\right)\right.\\&\left.\left.+iT\sqrt{q_-}\right]d\sigma\right\}\chi_{\bar{1}\to2}(T).\end{aligned}\tag{29}$$

The apparent s_0 dependence in the above formula is illusive. We can use this fact to calculate the integrals in the exponent most accurately. First let us note that we cannot disjoint totally the integrations in the two infinite integrals since the diverging contributions of the three terms in both of these integrals cancel mutually at the corresponding infinities, making the integrals convergent. We can, however, take as the integration paths for these two integrals the Stokes lines abc on Fig. 1 and $abcde$ on Fig. 2. Namely, let the points s_L on line a and s_R on line e be arbitrarily close to the corresponding infinities of the real axis. Let further points s_L' and s_R' be the points on the Stokes lines a and c of Fig. 1, respectively. We choose the latter points to lie on the anti-

Stokes lines of Fig. 1 that pass by the respective points s_L and s_R. Then the integral in the exponential of formula (30) can be rewritten as

$$I\equiv-\int_{s'}^{s_0} i\omega ds+\int_{-\infty}^{s_0}\left[-\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]ds+\ln c(s_0)$$
$$+\int_{s_0}^{+\infty}\left[\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]ds$$
$$=-\int_{s'}^{s''} i\omega ds+\int_{-\infty}^{s_L}\left[-\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]ds$$
$$+\tfrac{1}{2}\int_{s_L}^{s_L'} i\omega ds+\frac{1}{2}\ln c(s_L)+\int_{s_R}^{+\infty}\left[\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]$$
$$+\tfrac{1}{2}\int_{s_R}^{s_R'} i\omega ds+\frac{1}{2}\ln c(s_R)+\int_{s_L}^{s_R} iT\sqrt{q_-}ds+\tfrac{1}{2}\int_{s_L'}^{s''} i\omega ds$$
$$-\int_{s''}^{s_R'} i\omega ds, \tag{30}$$

where the last three integrals run along the respective Stokes lines and, therefore, are purely imaginary. Point s'' in the above formula is an arbitrary point of the Stokes line abc in Fig. 1.

We are interested mainly in the transition probability defined by amplitude $a_-(+\infty,T)$ for which only the real part of the integral $\mathbf{I}$ is important. Formula (30) gives for it,

$$\mathrm{Re}\,\mathbf{I}=-\mathrm{Re}\int_{s'}^{s''} i\omega ds+\mathrm{Re}\int_{-\infty}^{s_L}\left[-\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]ds$$
$$+\tfrac{1}{2}\int_{s_L}^{s_L'} i\omega ds+\frac{1}{2}\mathrm{Re}\ln c(s_L)+\mathrm{Re}\int_{s_R}^{+\infty}\left[\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)\right.$$
$$\left.+iT\sqrt{q_-}\right]ds+\tfrac{1}{2}\int_{s_R}^{s_R'} i\omega ds+\frac{1}{2}\mathrm{Re}\ln c(s_R). \tag{31}$$

We can now calculate $\mathrm{Re}\,\mathbf{I}$ taking in Eq. (32) the limits $s_L\to-\infty$ and $s_R\to+\infty$ along the corresponding Stokes lines. We get in this way,

$$\mathrm{Re}\,\mathbf{I}=-\mathrm{Re}\int_{s'}^{s''} i\omega ds+\frac{1}{2}\lim_{s_L\to-\infty}\left(\int_{s_L}^{s_L'} i\omega ds+\mathrm{Re}\ln c(s_L)\right)$$
$$+\frac{1}{2}\lim_{s_R\to+\infty}\left(\int_{s_R}^{s_R'} i\omega ds+\mathrm{Re}\ln c(s_R)\right)$$
$$=-\mathrm{Re}\int_{s'}^{s''} i\omega ds+\ln\frac{3}{2}. \tag{32}$$

The limits in Eq. (32) can be obtained by estimating the asymptotic behavior of the differences $s_{L,R}'-s_{L,R}$ and the corresponding functions when $|s|\to\infty$ along the Stokes lines, for which direct calculation gives

$$s_{L,R}'-s_{L,R}\sim-\frac{4i}{T\Delta\epsilon}\ln|s|-\frac{5b^2 i}{2T\Delta\epsilon|s|^2}-\frac{i\ln a_{L,R}}{T\Delta\epsilon},$$
$$i\omega\sim iT\Delta\epsilon\left(1+\frac{1}{2s^6}\right), \tag{33}$$
$$\mathrm{Re}\ln c(s)\sim\ln\frac{2}{3}-4\ln|s|-\frac{5b^2}{2|s|^2},$$

where constants $a_{L,R}$ are also independent of T and can be estimated exactly only when the exact equations of the Stokes lines abc of Fig. 1 and $abcde$ of Fig. 2 are known.

The imaginary part of the integral $\mathbf{I}$ can be calculated as the following limit,

$$\gamma(T)\equiv\mathrm{Im}\,\mathbf{I}=\lim_{s_{L,R}\to\mp\infty}\mathrm{Im}\left(\frac{1}{2}\ln c(s_L)+\frac{1}{2}\ln c(s_R)\right.$$
$$+\int_{s_L}^{s_R} iT\sqrt{q_-}ds+\tfrac{1}{2}\int_{s_L'}^{s''} i\omega ds-\tfrac{1}{2}\int_{s''}^{s_R'} i\omega ds$$
$$\left.-\int_{s'}^{s''} i\omega ds\right). \tag{34}$$

Therefore, the final *exact* formula for the transition amplitude is

$$a_-(+\infty,T)=\frac{3a_La_R}{2T\Delta\epsilon}\exp\left(-\int_{s'}^{s''} i\omega(s,T)ds+i\gamma(T)\right)$$
$$\times\chi_{1\to2}^-(T) \tag{35}$$

and the probability P reads,

$$P=\frac{9a_L^2a_R^2}{4T^2(\Delta\epsilon)^2}\exp\left(-2\int_{s'}^{s''} i\omega ds\right)|\chi_{1\to2}^-(T)|^2, \tag{36}$$

where in the last two formulas point s' is an arbitrary point on the real axis while point s'' being the one of line abc of Fig. 1 is taken to lie simultaneously on the anti-Stokes line passing by point s'.

The adiabatic limit of the transition probability is therefore,

$$P=\frac{9a_L^2a_R^2}{4T^2(\Delta\epsilon)^2}\exp\left[-2\,\mathrm{Re}\left(iT\int_{s'}^{s''}\mu B_0(s)ds\right)\right], \tag{37}$$

where s'' is now an arbitrary point of the continuous Stokes line passing by roots of $B_0(s)$ closest to the real axis.

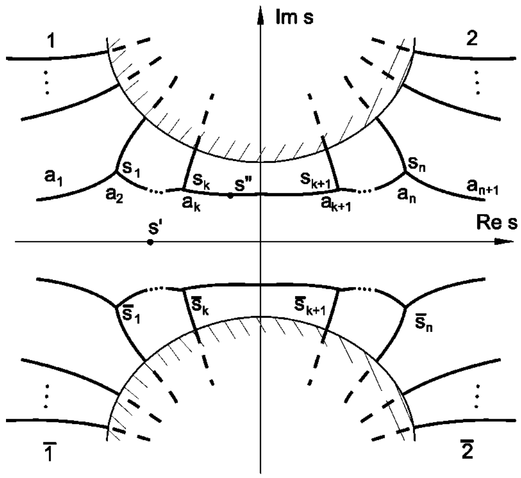

FIG. 3. The Stokes graph corresponding to general $q^{(0)}(s)$ considered in Sec. VI.

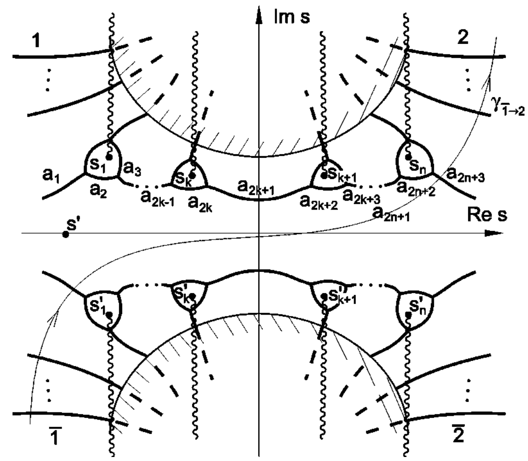

FIG. 4. The Stokes graph corresponding to general $q_-(s,T)$ considered in Sec. VI.

VI. THE GENERAL CASE OF ALGEBRAIC MAGNETIC FIELD

The result given by the formula (35) can be easily generalized. From the way of obtaining formula (30) it follows that the most important is the existence of the continuous Stokes line $abcde$ on Fig. 2 and its $T\to+\infty$-limit, i.e., the Stokes line abc of Fig. 1, which link the respective infinities $\mathrm{Re}\, s=-\infty$ and $\mathrm{Re}=+\infty$ on both Stokes graphs. Another important property was the way field $\mathbf{B}$ approached the limits $\mathbf{B}^\pm$ when $\mathrm{Re}\, t\to\pm\infty$, respectively, in the strip $\mathbf{\Sigma}$ mentioned in the assumption (3). Let us therefore accept the following two additional assumptions.

(7) There are two Stokes lines on each of the Stokes graphs corresponding to $iT\sqrt{q_\pm}$ that can be taken as the boundaries of the strip $\mathbf{\Sigma}$. Each of these two Stokes lines links continuously both infinities of the strip $\mathbf{\Sigma}$, see Fig. 3;

(8) Inside the strip $\mathbf{\Sigma}$ the field $\mathbf{B}$ approaches the infinities of the strip according to the following asymptotic formula:

$$\mathbf{B}(sT,T)\sim\mathbf{B}_0^\pm(T)+\frac{\mathbf{B}_1^\pm(T)}{s^{\alpha_1^\pm}}+\frac{\mathbf{B}_2^\pm(T)}{s^{\alpha_2^\pm}}+\cdots+\frac{\mathbf{B}_k^\pm(T)}{s^{\alpha_k^\pm}}+\cdots,\quad \mathrm{Re}\, s\to\pm\infty,$$
$$\frac{1}{2}<\alpha_1^\pm<\alpha_2^\pm<\cdots<\alpha_k^\pm<\cdots, \tag{38}$$

where $\alpha_1^\pm,\ldots,\alpha_k^\pm$, are rational if $\mathbf{B}^2$ is a meromorphic function of s.

If the Stokes graph corresponding to $iT\sqrt{q_-}$ satisfies the conditions of being a graph of the ALR system described in Sec. IV, then we can claim that there are four sectors $S_1,S_{\bar{1}}^-,S_2,S_{\bar{2}}^-$ of the graph and the corresponding fundamental solutions $\chi_1,\chi_{\bar{1}}^-$ that can be used in exactly the same way as it was done in the case of the Nikitin model to solve the problem stated in Sec. II, see Fig. 4.

Let us choose the xyz axes in the space of vector $\mathbf{B}$ in such a way that one of its limit components $B_{x,0}^\pm$ and $B_{y,0}^\pm$ does not vanish in the corresponding infinities. Let us also assume that vectors $\mathbf{B}_0^\pm(T)$ and $\mathbf{B}_1^\pm(T)$ of expansion (38) are not parallel to each other in the respective infinities [otherwise we should take another pair of vectors appearing in Eq. (38) satisfying the last property and having the smallest sum of the power exponents by which they are accompanied]. Then, if we take into account the following asymptotic that comes out of Eq. (6) and of the above assumptions when $\mathrm{Re}\, s\to\pm\infty$ inside the strip,

$$c\sim\left(-\frac{1}{2}\,\frac{[\mathbf{B}_0^\pm\times(\mathbf{B}_0^\pm\times\mathbf{B}_1^\pm)]_z}{B_0^{\pm 2}\sqrt{B_{x,0}^{\pm 2}+B_{y,0}^{\pm 2}}}+\frac{i}{2}\,\frac{(\mathbf{B}_0^\pm\times\mathbf{B}_1^\pm)_z}{B_0^\pm\sqrt{B_{x,0}^{\pm 2}+B_{y,0}^{\pm 2}}}\right)\frac{1}{s^{\alpha_1^\pm}}\equiv\frac{D^\pm}{s^{\alpha_1^\pm}},$$

$$\omega\sim\mu TB_0^\pm+\left(\mu T\mathbf{B}_0^\pm\cdot\mathbf{B}_1^\pm-\frac{B_{z,0}^\pm}{B_0^\pm}\,\frac{(\mathbf{B}_0^\pm\times\mathbf{B}_1^\pm)_z}{\sqrt{B_{x,0}^{\pm 2}+B_{y,0}^{\pm 2}}}\right)\frac{1}{s^{\alpha_1^\pm}}\equiv\frac{G^\pm}{s^{\alpha_1^\pm}},$$

$$\frac{1}{2}\left(\frac{\dot{c}^*}{c^*}-i\omega\right)+iT\sqrt{q_-}\sim\begin{cases}-i\mu TB_0^-\,\dfrac{\alpha_1^+}{s}\\[2ex] \dfrac{D^-(D^-)^*}{i\mu TB_0^-}\,\dfrac{1}{s^{2\alpha_1^-}},\end{cases} \tag{39}$$

$$\frac{1}{2}\left(\frac{\dot{c}^*}{c^*}-i\omega\right)-iT\sqrt{q_-}\sim\begin{cases}-\dfrac{D^+(D^+)^*}{i\mu TB_0^+}\,\dfrac{1}{s^{2\alpha_1^+}}\\[2ex] -i\mu TB_0^-\,\dfrac{\alpha_1^-}{s},\end{cases}$$

$$\frac{\dot{c}}{c}\sim-\frac{\alpha_1^\pm}{s},$$

then we can repeat the procedure of the previous section to get the analog of formulas (30) and (34). Namely, we have for them,

$$a_-(+\infty,T)=\frac{1}{\mu T\sqrt{B_0^-(T)B_0^+(T)}}\exp\left\{\int_{-\infty}^{s_0}\left[-\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]ds+\ln c(s_0)-\int_{s'}^{s_0}i\omega ds+\int_{s_0}^{+\infty}\left[\frac{1}{2}\left(\frac{\dot{c}}{c}-i\omega\right)+iT\sqrt{q_-}\right]d\sigma\right\}\chi^-_{1\to2}(T) \quad (40)$$

$$=\frac{a_L a_R}{\mu T}\sqrt{\frac{|D^-(T)D^+(T)|}{B_0^-(T)B_0^+(T)}}\times\exp\left(-\int_{s'}^{s''}i\omega(s,T)ds+i\gamma\right)\chi^-_{1\to2}(T),$$

where points s' and s'' have been chosen again on the same anti-Stokes line of the graph corresponding to $i\omega(s,T)$ and

$$P(T)=\frac{a_L^2a_R^2|D^-(T)D^+(T)|}{(\mu T)^2B_0^-(T)B_0^+(T)}\times\exp\left(-2\,\mathrm{Re}\int_{s'}^{s''}i\omega(s,T)ds\right)|\chi^-_{1\to2}(T)|^2, \quad (41)$$

where $D^{\pm}$ are given by

$$D^{\pm}=-\frac{1}{2}\frac{[\mathbf{B}_0^{\pm}\times(\mathbf{B}_0^{\pm}\times\mathbf{B}_1^{\pm})]_z}{B_0^{\pm 2}\sqrt{B_{x,0}^{\pm 2}+B_{y,0}^{\pm 2}}}+\frac{i}{2}\frac{(\mathbf{B}_0^{\pm}\times\mathbf{B}_1^{\pm})_z}{B_0^{\pm}\sqrt{B_{x,0}^{\pm 2}+B_{y,0}^{\pm 2}}} \quad (42)$$

so that

$$|D^{\pm}|=\frac{B_1^{\pm}\sin\phi^{\pm}}{2B_0^{\pm}}, \quad (43)$$

where $\phi^{\pm}(T)$ are the angles between fields $\mathbf{B}_0^{\pm}$ and $\mathbf{B}_1^{\pm}$, respectively.

Again, the exact form of the coefficients $a_{L,R}$ can be found if the exact equations of the Stokes lines corresponding to $\omega(s,T)$ and $q_-(s,T)$ are known.

Therefore, the final forms of the transition probability and its adiabatic limit are

$$P(T)=\frac{a_L^2a_R^2B_1^-(T)B_1^+(T)\sin\phi^-(T)\sin\phi^+(T)}{(2\mu TB_0^-(T)B_0^+(T))^2}\times\exp\left(-2\,\mathrm{Re}\int_{s'}^{s''}i\omega(s,T)ds\right)|\chi^-_{1\to2}(T)|^2 \quad (44)$$

and

$$P^{ad}=\frac{a_L^2a_R^2B_{1,0}^-B_{1,0}^+\sin\phi_0^-\sin\phi_0^+}{(2\mu TB_{0,0}^-B_{0,0}^+)^2}\times\exp\left[-2\,\mathrm{Re}\left(iT\int_{s'}^{s''}\mu B_0(s)ds\right)\right], \quad (45)$$

where to get the last formula, the asymptotic expansion (1) has been applied to fields $\mathbf{B}_0^{\pm}(T)$ and $\mathbf{B}_1^{\pm}(T)$ as well as to ω given by Eq. (6). Point s'' is now an arbitrary point of the continuous Stokes line $a_1a_2\cdots a_na_{n+1}$ passing by the roots of $B_0(s)$ closest to the real axis, as it is shown on Fig. 3. Note that because of our assumption the angles in Eqs. (44) and (45) are different from 0 and π.

VII. OTHER TWO EXAMPLES WITH EXPONENTIALLY DECREASING MAGNETIC FIELDS

We consider here another two examples of magnetic fields depending exponentially on time. The main difference between these cases and those considered in the previous sections lies in the number of level crossings that in the exponential cases is, of course, infinite.

We consider the following two cases of the fields:

$$\text{(a)}\quad \mathbf{B}(t,T)=\mathbf{B}_0+\frac{\mathbf{B}_1}{\cosh\left(\frac{t}{T}\right)},\quad \mathbf{B}_0\cdot\mathbf{B}_1=0,$$

$$B_0=|\mathbf{B}_0|\neq|\mathbf{B}_1|=B_1, \quad (46)$$

$$\text{(b)}\quad \mathbf{B}(t,T)=\mathbf{B}_0+\mathbf{B}_1\tanh\left(\frac{t}{T}\right),\quad \mathbf{B}_0\cdot\mathbf{B}_1=0,\quad |\mathbf{B}_0|=|\mathbf{B}_1|=B_0.$$

Case (a). The relevant quantities for this case have the forms

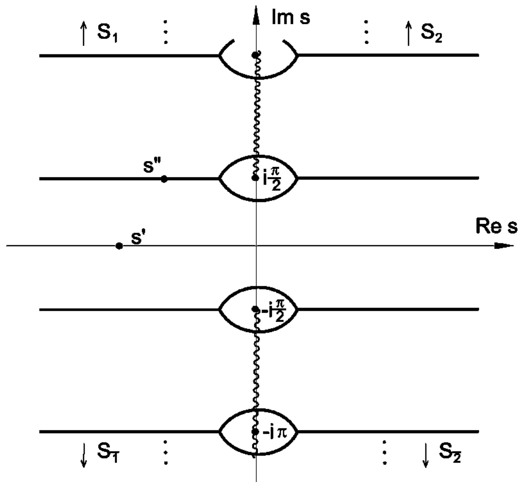

FIG. 5. The Stokes graph corresponding to $q^{(0)}(s)$ of case (a) of Eq. (46).

$$c(s,T)=\frac{1}{2}\frac{B_0B_1\sinh s}{B_1^2+B_0^2\cosh^2 s},$$

$$\omega(s,T)=\mu T\sqrt{B_0^2+\frac{B_1^2}{\cosh^2 s}}, \tag{47}$$

$$\begin{aligned}q_-(s,T)=&-\frac{1}{4T^2}\Bigg(\coth s-\frac{B_0^2\sinh(2s)}{B_1^2+B_0^2\cosh^2 s}\\&-i\mu T\sqrt{B_0^2+\frac{B_1^2}{\cosh^2 s}}\Bigg)^2\\&+\frac{i\mu}{2T}\frac{B_1^2\sinh 2s}{\sinh^2 s(B_1^2+B_0^2\cosh^2 s)^{1/2}}\\&+\frac{1}{4T^2}\frac{B_0^2B_1^2\sinh^2 s}{(B_1^2+B_0^2\cosh^2 s)^2}-\frac{1}{2T^2}\\&\times\Bigg(\frac{1}{\sinh^2 s}+\frac{2B_0^2\cosh 2s}{B_1^2+B_0^2\cosh^2 s}\\&-\frac{B_0^4\sinh^2 2s}{(B_1^2+B_0^2\cosh^2 s)^2}\Bigg)\end{aligned}$$

and the Stokes graphs defined by $\omega(s,T)$ and $q_-(s,T)$ are shown on Figs. 5 and 6, respectively.

The procedure leading us to formula (30) is still valid but the corresponding sectors S_1,S_1^-,S_2,S_2^- are now less exposed. Namely, the first two lie on the left of the imaginary axis, S_1 above and S_1^- below the real axis whereas the next two lie on the right of the imaginary axis and, respectively; above and below the real axis. A peculiarity of this and the next case is that these sectors are cut by the infinite number of the Stokes lines parallel to the real axis and distributed up and down to the imaginary infinities, see Figs. 3 and 4. The fundamental solutions defined in these sectors vanish in their imaginary infinities. Therefore, the corresponding transition amplitude $a_-(s,T)$ from level E_+ to E_- looks as follows:

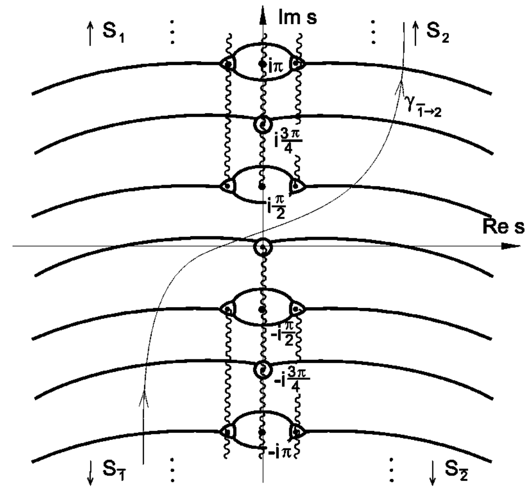

FIG. 6. The Stokes graph corresponding to $q_-(s,T)$ of Eq. (47).

$$a_-(T)=\frac{2iB_1a_La_R}{\mu B_0\sqrt{1+\mu^2T^2B_0^2}}\times\exp\left(-\int_{s'}^{s''}i\omega(s,T)ds+i\gamma(T)\right)\chi_{\bar{1}\to 2}^-(T). \tag{48}$$

To get the above formula we have taken into account the following asymptotic behavior of the quantities determining it.

$$\begin{aligned}&c(x+iy)\sim\begin{cases}+\dfrac{B_1}{B_0}e^{-x-iy}, x\to+\infty\\-\dfrac{B_1}{B_0}e^{+x+iy}, x\to-\infty,\end{cases}\\&\frac{\dot{c}(s)}{c(s)}\sim\begin{cases}-1, \mathrm{Re}\, s\to+\infty\\+1, \mathrm{Re}\, s\to-\infty,\end{cases}\\&\omega(x+iy,T)\sim-\mu TB_0,\quad |x|\to\infty,\\&y'_{L,R}-y_{L,R}\sim\frac{2\ln a_{L,R}}{\mu TB_0}+\frac{|x_{L,R}|}{\mu TB_0},\quad |x_{L,R}|\to\infty,\end{aligned} \tag{49}$$

where $s_{L,R}=x_{L,R}+iy_{L,R}$ and $s'_{L,R}=x'_{L,R}+iy'_{L,R}$ have the same meaning as previously, i.e., lie on the corresponding Stokes lines defined by $q_-(s,T)$ and $\omega(s,T)$, respectively, whilst $a_{L,R}$ measure (together with the terms linear in $x_{L,R}$) the deviations of these lines at the corresponding infinities.

Therefore, for the exact-transition probability and its adiabatic limit, we obtain from Eq. (48),

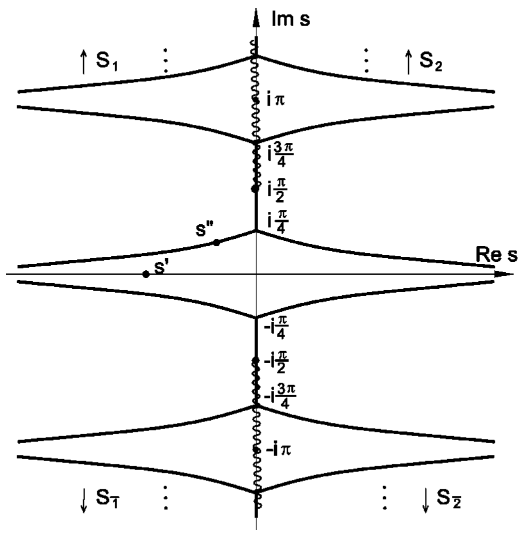

FIG. 7. The Stokes graph corresponding to $q^{(0)}(s)$ of case (b) of Eq. (46).

$$P(T)=\frac{(2B_1a_La_R)^2}{\mu^2B_0^2(1+\mu^2T^2B_0^2)^2}\times\exp\left(-2\,\mathrm{Re}\int_{s'}^{s''}i\omega(s,T)ds\right)|\chi_{\bar{1}\to 2}^{-}(T)|^2 \quad (50)$$

and

$$P^{ad}=\left(\frac{2B_1a_La_R}{\mu^2TB_0^2}\right)^2\times\exp\left(-2\mu T\,\mathrm{Re}\int_{s'}^{s''}i\sqrt{B_0^2+\frac{B_1^2}{\cosh^2 s}}ds\right), \quad (51)$$

respectively.

Case (b). In this case we have,

$$c(s,T)=-\frac{1}{2\cosh(2s)},\quad \frac{\dot{c}(s,T)}{c(s,T)}=-2\tanh(2s),$$

$$\omega(s,T)=\mu TB_0\frac{\sqrt{\cosh(2s)}}{\cosh s},$$

$$q_-(s,T)=-\frac{1}{4T^2}\left(2\tanh(2s)+i\mu TB_0\frac{\sqrt{\cosh(2s)}}{\cosh s}\right)^2-\frac{i\mu B_0}{2T}\frac{\tanh(2s)-\tanh s}{\cosh s}\sqrt{\cosh(2s)}-\frac{7}{4T^2}\frac{1}{\cosh^2(2s)} \quad (52)$$

and the Stokes graphs corresponding to $\omega(s,T)$ and $q_-(s,T)$ are shown in Figs. 7 and 8, respectively.

Again, the transition amplitude can be calculated taking into account the following asymptotic.

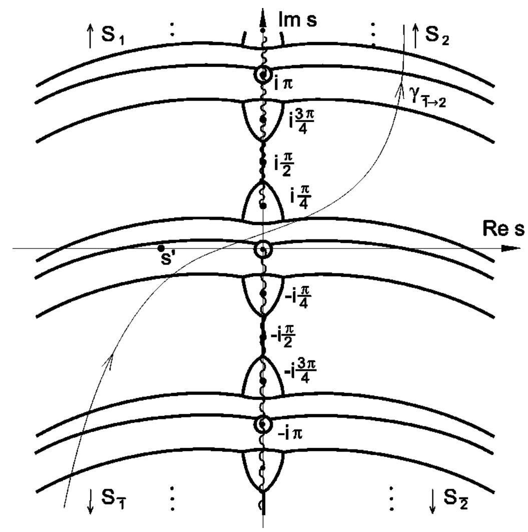

FIG. 8. The Stokes graph corresponding to $q_-(s,T)$ of Eq. (52).

$$c(x+iy)\sim\begin{cases}-e^{-2x-2iy}, x\to+\infty\\ -e^{+2x+2iy}, x\to-\infty,\end{cases}\quad \frac{\dot{c}(s)}{c(s)}\sim\begin{cases}-2, \mathrm{Re}\,s\to+\infty\\ +2, \mathrm{Re}\,s\to-\infty,\end{cases}$$

$$\omega(x+iy,T)\sim-\sqrt{2}\mu TB_0,\quad |x|\to\infty, \quad (53)$$

$$y'_{L,R}-y_{L,R}\sim\frac{2\ln a_{L,R}}{\sqrt{2}\mu TB_0}+\frac{|x_{L,R}|}{\sqrt{2}\mu TB_0},\quad |x_{L,R}|\to\infty,$$

so that we get for it,

$$a_-(T)=\frac{2a_La_R}{\mu\sqrt{4+2\mu^2T^2B_0^2}}\exp\left(-\int_{s'}^{s''}i\omega(s,T)ds+i\gamma(T)\right)\times\chi_{\bar{1}\to 2}^{-}(T). \quad (54)$$

Therefore, for the corresponding transition probabilities we obtain

$$P(T)=\frac{(2a_La_R)^2}{\mu^2(4+2\mu^2T^2B_0^2)}\exp\left(-2\,\mathrm{Re}\int_{s'}^{s''}i\omega(s,T)ds\right)\times|\chi_{\bar{1}\to 2}^{-}(T)|^2 \quad (55)$$

and

$$P^{ad}=\left(\frac{\sqrt{2}a_La_R}{\mu^2TB_0}\right)^2\exp\left(-2\mu TB_0\,\mathrm{Re}\int_{s'}^{s''}i\sqrt{\frac{\cosh(2s)}{\cosh s}}ds\right). \quad (56)$$

VIII. NONVANISHING CONTRIBUTION OF THE BERRY PHASE

The previous sections have provided us with the examples of Hamiltonians in which the corresponding transition probabilities have had no contributions from the term

$$-\frac{B_z}{B}\frac{(\mathbf{B}\times\dot{\mathbf{B}})_z}{B_x^2+B_y^2}$$

of ω [see Eq. (6)] representing (at least) a part of the Berry phase of the transition amplitudes. The Hamiltonian defined by the field

$$\mathbf{B}=\frac{B_0}{1+s^2}[1,\alpha s,s^2], \quad \alpha>\sqrt{2}, \tag{57}$$

provides us with the corresponding positive example of such a contribution. Namely, for this case we have

$$\omega=\mu TB_0\frac{\sqrt{(1+s^2)^2+\alpha^2-2}}{1+s^2} -\frac{1}{\sqrt{(1+s^2)^2+\alpha^2-2}}\frac{s^2}{1+\alpha^2 s^2}. \tag{58}$$

From Eq. (57) it follows easily that for this case the transition probability (44) takes the form

$$P(T)=\frac{a_L^2a_R^2}{(2\mu TB_0)^2}\exp\left(-2\,\mathrm{Re}\int_{s'}^{s''}i\omega(s,T)ds\right)|\chi_{1\to 2}^{-}(T)|^2. \tag{59}$$

It is the second term of Eq. (58) that is responsible for the Berry phase contribution to the transition probability (59). We shall calculate this contribution in the adiabatic limit only and for α close to $\sqrt{2}$. This assumption allows us to calculate the corresponding path integral,

$$\mathbf{I}_\gamma=-i\int_\gamma\frac{1}{\sqrt{(1+s^2)^2+\alpha^2-2}}\frac{s^2}{1+\alpha^2s^2}ds, \tag{60}$$

from point $s=0$ to the closest root $s_0=i(1+i\sqrt{\alpha^2-2})^{1/2}$ of the polynomial $(1+s^2)^2+\alpha^2-2$, lying in the second quadrant of the s plane. For α close to $\sqrt{2}$ we can simplify the integration expanding suitably the square root in the integrand of Eq. (60) and the root s_0 as well. It is easy to check that under the above assumptions the net result of such calculations is,

$$-2\,\mathrm{Re}\,\mathbf{I}_\gamma=\ln\frac{(\alpha-\sqrt{2})^{1/2}}{2^{1/4}(\sqrt{2}-1)^{\sqrt{2}}}+O(\sqrt{\alpha-\sqrt{2}}). \tag{61}$$

Obviously, the above Berry phase contribution to the transition probability (59) modifies its prexponential factor multiplying it by the following additional one,

$$C=\frac{(\alpha-\sqrt{2})^{1/2}}{2^{1/4}(\sqrt{2}-1)^{\sqrt{2}}}. \tag{62}$$

IX. CONCLUSIONS AND DISCUSSION

We have shown in this paper that the fundamental solution method has turned out to be very effective also in its application to the problems of the transition amplitudes in two-energy-level systems. In particular, it has enabled us to obtain compact and exact formulas for these amplitudes and to get easily their adiabatic approximations as well. Due to the clear way of their obtaining and their compact forms, the formulas allow us to claim that there are no particular effects coming out of the many-complex level crossings, i.e., there are no individual contributions of any kind to the transition amplitude from each such crossing leading to any particular interference effects in these amplitudes. Just the opposite, such a contribution is controlled totally by the Stokes line closest to the real axis of the t plane that is however built by these crossing points of the two energy levels. This result is independent of both the number of complex level crossings (i.e., finite or infinite) and of the particular type of the t dependence of the effective magnetic field (i.e., algebraic or exponential). In this way the respective results of Joye, Mileti, and Pfister [4] have not been confirmed by our approach. This last difference seems to be rather dramatic and, as it seems to us, its origin lies in an erroneous calculation of the transition amplitude by the authors mentioned. Namely, it is the formula (6.21) of their paper [4] for the transition matrix $X(z_1)$ that is wrong, particularly if applied further in their *Lemma 6.1* to get the general formula of it. This can be seen if we rewrite our results in terms of the transition matrix.

Namely, let us denote the rhs of the formula (28) by $U_{21}(s,T)$ and the result we obtain calculating $a_+(s,T)$ with the help of the second of Eqs. (5) and of Eq. (28) by $U_{11}(s,T)$.

Reversing the problem we have solved in our paper by assuming that for $s=-\infty$ the vanishing amplitude is rather $a_+(s,T)$ than $a_-(s,T)$ we obtain by exactly the same methods as used in Sec. III and the further ones the remaining elements $U_{12}(s,T)$ and $U_{22}(s,T)$ that construct the transition matrix $\mathbf{U}(s,T)$. For the choice we have done in our paper we have of course,

$$\mathbf{a}(s,T)\equiv\begin{bmatrix}a_+(s,T)\\ a_-(s,T)\end{bmatrix}=\begin{bmatrix}U_{11}(s,T) & U_{12}(s,T)\\ U_{21}(s,T) & U_{22}(s,T)\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}. \tag{63}$$

Matrix $\mathbf{U}(s,T)$ is of course unitary (for real s) and $\mathbf{U}(-\infty,T)=\mathbf{I}$.

It should be clear that the order of $U_{12}(s,T)$ as a function of its arguments is the *same* as that of $U_{21}(s,T)$, the latter element being given by the rhs of Eq. (28) so that the adiabatic limit of $\mathbf{U}(s,T)$ is given by

$$\mathbf{U}^{ad}(s,T)=\begin{bmatrix}1 & U_{12}^{ad}(s,T)\\ U_{21}^{ad}(s,T) & 1\end{bmatrix}. \tag{64}$$

Therefore this matrix is *not* a triangular one in this limit, as it is the case of $X(z_1)$ mentioned earlier, which does *not* contain the *non vanishing* element $U_{12}^{ad}(s,T)$.

Moreover we can not apply matrix $\mathbf{U}(s,T)$ directly to continue the solution (63) along the central strip of the corresponding Stokes graph from point s to another one s'. The proper continuation is of course

$$\mathbf{a}(s',T)=\mathbf{U}(s',T)\mathbf{U}^{-1}(s,T)\mathbf{a}(s,T). \tag{65}$$

In particular, if it is possible to continue the solution $\mathbf{a}(s,T)$ along, say, the upper Stokes line limiting the central strip (i.e., the level crossings $s_1,s_2,\ldots,s_n$ met along this line are not an obstacle to such a continuation) then continuing $\mathbf{a}(s,T)$ in this way to $s=+\infty$ we get

$$\begin{aligned}\mathbf{a}(+\infty,T)=\mathbf{U}(+\infty,T)\mathbf{U}^{-1}(s_n,T)\\ \times\mathbf{U}(s_n,T)\mathbf{U}^{-1}(s_{n-1},T)\cdots,\\ \mathbf{U}(s_2,T)\mathbf{U}^{-1}(s_1,T)\mathbf{U}(s_1,T)\begin{bmatrix}1\\0\end{bmatrix}=\mathbf{U}(+\infty,T)\begin{bmatrix}1\\0\end{bmatrix}.\end{aligned} \tag{66}$$

The above results show that *none* of the contributions from the *individual* level crossings lying on the considered Stokes line survive on the way of continuation.

On the other hand, writing both formula (6.21) and the respective result of *Lemma 6.1* of [4] in terms of the quantities introduced above, we get,

$$X(s_1)=\begin{bmatrix}1 & 0\\ U_{21}^{ad}(s_1,T) & 1\end{bmatrix} \tag{67}$$

and

$$\mathbf{a}^{ad}(+\infty,T)=X(s_n)X(s_{n-1})\cdots X(s_2)X(s_1)\begin{bmatrix}1\\0\end{bmatrix}. \tag{68}$$

Comparing the last two formulas with the respective Eqs. (64) and (66) ones we see that formulas (67) and (68) are wrong. Particularly, it is the incorrect formula (68) that gives rise to the interference effects in the amplitudes of Joye, Mileti, and Pfister [4].

Finally, we would like to mention that, as we have shown this in Sec. VIII there are contributions to the transition probabilities originating from the geometrical (Berry) phase [16] although their geometrical meaning in the context of the transition amplitudes is not clear.

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