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Cavity-assisted quasiparticle damping in a Bose-Einstein condensate

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We consider an atomic Bose-Einstein condensate held within an optical cavity and interacting with laser fields. We show how the interaction of the cavity mode with the condensate can cause energy due to excitations to be coupled to a lossy cavity mode, which then decays, thus damping the condensate. We show how to choose parameters for damping specific excitations, and how to target a range of different excitations to potentially produce extremely cold condensates.

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The breakthrough success, largely due to improved cooling techniques [1], of producing atomic Bose-Einstein condensates (BEC's) [2], has created an exciting new area of investigative study [3,4]. Within the field of cavity quantum electrodynamics, one is able to precisely control the interaction of the electromagnetic field with an atom [5]; the trapping and cooling of a single atom in an optical cavity has also been successfully experimentally demonstrated [6]. Here we combine [7] the use of a cavity to cool atoms [8] with the ever present goal of producing colder condensates [9], particularly relevant to controlling quasiparticle excitations [10–12] produced in the course of other experimental investigations [13].

We consider a Fabry-Perot optical cavity with decay rate κ, and a pump laser, with a BEC held in a harmonic potential of frequency ω inside the cavity, driven by a separate laser Ω, as shown in Fig. 1. We generally consider only one spatial dimension, considering the radial motion to be frozen out by a tight harmonic potential of frequency ω_r (cigar-shaped configuration [14]). The laser-cavity superposition field is $\hat{E}(\hat{x},t)=\Omega(\hat{x})e^{-i\omega_L t}+g\hat{a}\cos(k\hat{x})$, where $\hat{a}$ is the cavity mode annihilation operator; interacting with this are the BEC atoms, considered to have two levels $|g\rangle$ and $|e\rangle$, with transition frequency ω_0. We initially consider a single particle, and ignore the degrees of freedom due to the free cavity field and the atomic motion. In a rotating frame ($\hat{U}=\exp[-i(\sigma_{ee}+\hat{a}^\dagger\hat{a})\omega_L t]$), the Hamiltonian describing the internal atomic level dynamics is $\hat{H}_{\mathrm{at}}=\hbar[\Delta_a\sigma_{ee}+\tilde{E}(\hat{x})\sigma_{eg}+\tilde{E}^\dagger(\hat{x})\sigma_{ge}]$, where $\sigma_{eg}=|e\rangle\langle g|$, $\Delta_a=\omega_0-\omega_L$, and $\tilde{E}(\hat{x})=\Omega(\hat{x})+g\hat{a}\cos(k\hat{x})$. Assuming $\Delta_a\gg\tilde{E}$, $\Delta_a\gg\gamma$ where γ is the atomic spontaneous emission rate, and $|\Delta|$, where $\Delta=\omega_c-\omega_L$, by far exceeding the frequency scales governing atomic motion and the cavity dynamics, we may adiabatically eliminate $|e\rangle$ to derive a closed time-evolution equation for the $|g\rangle$ wave function component [15], using this to reconstruct an effective Hamiltonian:

$$\hat{H}_{\mathrm{eff}}=-\frac{\hbar\Delta_a}{\gamma^2+\Delta_a^2}\tilde{E}^\dagger(\hat{x})\tilde{E}(\hat{x})+\frac{i\hbar\gamma}{\gamma^2+\Delta_a^2}\tilde{E}^\dagger(\hat{x})\tilde{E}(\hat{x}). \quad (1)$$

Assuming $g\ll\Omega$ and $\langle\hat{a}^\dagger\hat{a}\rangle\leq 1$, we can neglect the terms $\propto g^2$, and as $\gamma\ll\Delta_a$, we ultimately consider only the Hermitian terms. The light field thus provides: a term $\propto|\Omega(\hat{x})|^2$, which, assuming $\Omega(\hat{x})\propto e^{ik_L\hat{x}}$, contributes only a shift; and a contribution linear in $\hat{a}$ and $\hat{a}^\dagger$, which acts like a driving field for the cavity mode. It thereby implements a mechanism to transfer energy to the cavity, which in turn will shed this energy to the environment by means of cavity damping. The effective single-particle Hamiltonian (including the free dynamics of the particle motion and the cavity) is now

$$\hat{H}_{\mathrm{eff}}=\frac{\hat{p}^2}{2m}+\frac{m\omega^2\hat{x}^2}{2}+\hbar(\Delta-i\kappa)\hat{a}^\dagger\hat{a}+\hbar[f(\hat{x})\hat{a}+\mathrm{H.c.}], \quad (2)$$

where $f(\hat{x})=-\Omega(\hat{x})g\cos(k\hat{x})\Delta_a/(\Delta_a^2+\gamma^2)$, and m is the particle mass. We now consider a BEC-cavity system. The one-dimensional atomic interaction potential $=u\delta(x-y)$, where $u=2\hbar\omega_r a_s$ and a_s is the s-wave scattering length [4]. Thus, in second quantized form,

$$\hat{H}_{\mathrm{eff}}=\int dx\hat{\Psi}^\dagger(x)\left\{\hbar[f(x)\hat{a}+f^*(x)\hat{a}^\dagger]-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}+\frac{u}{2}\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\right\}\hat{\Psi}(x)+\hbar[(\Delta-i\kappa)\hat{a}^\dagger\hat{a}-\alpha_p\hat{a}^\dagger-\alpha_p^*\hat{a}], \quad (3)$$

where $\hat{\Psi}(x)$ and $\hat{\Psi}^\dagger(x)$ are atomic field operators, and the α_p terms are provided by the pump laser. We treat $\hat{\Psi}(x)$ and $\hat{a}$ semiclassically, replacing them with the scalar quantities

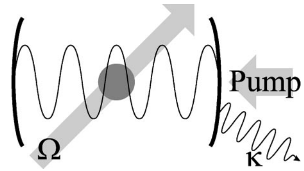

FIG. 1. Proposed configuration: a BEC held in a cavity, driven by a laser Ω and a pump laser, and losing cavity photons with a decay rate κ.

$\varphi(x)$ and α, respectively. It is convenient to rescale to dimensionless harmonic units ($\hbar=m=\omega=1$); this scaling is assumed from now on. The resulting equations of motion are a Gross-Pitaevskii-like (GP) equation

$$i\dot{\varphi}(x)=\{H_{\mathrm{GP}}+[\tilde{f}(x)\tilde{\alpha}+\tilde{f}^*(x)\tilde{\alpha}^*]\}\varphi(x), \tag{4}$$

where $H_{\mathrm{GP}}=-\partial^2/2\partial x^2+x^2/2+\upsilon|\varphi(x)|^2$ is the unperturbed GP ''Hamiltonian,'' with $\upsilon=uN\sqrt{m/\hbar\omega}/\hbar$, $\tilde{f}(x)=\sqrt{N}f(x)/\omega$, and $\tilde{\alpha}=\alpha/\sqrt{N}$, where N is the particle number; and the semiclassical cavity mode equation

$$i\dot{\tilde{\alpha}}=(\Delta-i\kappa)\tilde{\alpha}+\int dx|\varphi(x)|^2\tilde{f}^*(x)-\tilde{\alpha}_p. \tag{5}$$

We set $\tilde{\alpha}_p$ to be $=\int dx|\varphi_0(x)|^2\tilde{f}^*(x)$, so that when $\varphi(x)$ approaches $\varphi_0(x)$ (the ground state of H_{GP}), $\tilde{\alpha}$ will simply decay, without feeding into Eq. (4). The cooling mechanism thus switches off upon reaching the desired steady state $\varphi(x)=\varphi_0(x)$ and $\tilde{\alpha}=0$.

We now consider linearized perturbations $\delta\varphi(x)$, $\delta\alpha$, around the steady state. Linearizing Eq. (4) produces Bogoliubov-like equations [10]. Note in Eq. (5) for $\varphi_0(x)$, no distinction is made between different global phases; in calculations the final phase of $\varphi_0(x)$ is determined by the chosen initial conditions, and is not relevant. We consider only perturbations orthogonal to $\varphi_0(x)$, and can implicitly assume a U(1) gauge transformation [16] such that $\varphi_0(x)$ is always real. Having defined $\delta\varphi(x)$ as orthogonal to $\varphi_0(x)$, i.e., $\delta\varphi(x)=\int dy Q(x,y)\delta\varphi(y)$, where $Q(x,y)=\delta(x-y)-\varphi_0(x)\varphi_0(y)$, we can state that

$$i\begin{pmatrix}\delta\dot{\varphi}(x)\\ \delta\dot{\varphi}^*(x)\end{pmatrix}=\int dy\mathcal{L}(x,y)\begin{pmatrix}\delta\varphi(y)\\ \delta\varphi^*(y)\end{pmatrix}+\int dy\mathcal{Q}(x,y)\times[\tilde{f}(y)\delta\alpha+\tilde{f}^*(y)\delta\alpha^*]\begin{pmatrix}\varphi_0(y)\\ -\varphi_0(y)\end{pmatrix}, \tag{6}$$

where $\mathcal{L}(x,y)=\int\int dz dw\mathcal{Q}(x,z)H_{\mathrm{Bog}}(z,w)\mathcal{Q}(w,y)$ [12], in terms of the usual Bogoliubov Hamiltonian [10]:

$$H_{\mathrm{Bog}}=\begin{pmatrix}H_{\mathrm{GP}}+\upsilon\varphi_0(x)^2-\mu & \upsilon\varphi_0(x)^2\\ -\upsilon\varphi_0(x)^2 & -H_{\mathrm{GP}}-\upsilon\varphi_0(x)^2+\mu\end{pmatrix}, \tag{7}$$

where μ is the ground-state chemical potential; and

$$\mathcal{Q}(x,y)=\begin{pmatrix}Q(x,y) & 0\\ 0 & Q(x,y)\end{pmatrix}. \tag{8}$$

It is convenient to expand $(\delta\varphi(x),\delta\varphi^*(x))$ as

$$\begin{pmatrix}\delta\varphi(x)\\ \delta\varphi^*(x)\end{pmatrix}=\sum_k \zeta_k\begin{pmatrix}u_k(x)\\ \upsilon_k(x)\end{pmatrix}+\zeta_k^*\begin{pmatrix}\upsilon_k(x)\\ u_k(x)\end{pmatrix}, \tag{9}$$

where $\{(u_k(x),\upsilon_k(x)),(\upsilon_k(x),u_k(x))\}$ are eigenstates of $\mathcal{L}(x,y)$, with eigenfrequencies $\pm\omega_k$ [these may be determined numerically by diagonalizing $\mathcal{L}(x,y)$ [17]]. As φ_0 is assumed real, these are also real. We similarly expand

$$\int dy\mathcal{Q}(x,y)\tilde{f}(y)\begin{pmatrix}\varphi_0(y)\\ -\varphi_0(y)\end{pmatrix}=\sum_k \chi_k\left[\begin{pmatrix}u_k(x)\\ \upsilon_k(x)\end{pmatrix}-\begin{pmatrix}\upsilon_k(x)\\ u_k(x)\end{pmatrix}\right], \tag{10}$$

where the coefficients can obviously be defined by

$$\chi_k=\int dx[u_k(x)+\upsilon_k(x)]\tilde{f}(x)\varphi_0(x). \tag{11}$$

Note that treating the evolution of perturbations orthogonal to $\varphi_0(x)$ is equivalent to a number-conserving formalism [11,12]. The $u_k(x)$ and $\upsilon_k(x)$ are orthogonal to $\varphi_0(x)$, and the $\{(u_k(x),\upsilon_k(x)),(\upsilon_k(x),u_k(x))\}$ are used as a convenient time-independent basis. All time dependence in Eq. (9) is thus in the ζ_k,ζ_k^* coefficients, in contrast to Refs. [11,12,16]. We now transform Eq. (6) to an interaction picture. We thus set $(\delta\tilde{\varphi}(x),\delta\tilde{\varphi}^*(x))=e^{i\mathcal{L}t}(\delta\varphi(x),\delta\varphi^*(x))$ and $\delta\tilde{\alpha}=e^{i\Delta t}\delta\alpha$. Substituting Eqs. (9) and (10) into an appropriately transformed Eq. (6), and then making the integration $\int dx u_l(x)\delta\dot{\tilde{\varphi}}(x)-\upsilon_l(x)\delta\dot{\tilde{\varphi}}^*(x)$, we end up with

$$i\dot{\tilde{\zeta}}_l=\delta\tilde{\alpha}\chi_l e^{i(\omega_l-\Delta)t}+\delta\tilde{\alpha}^*\chi_l^* e^{i(\omega_l+\Delta)t}, \tag{12}$$

where $\tilde{\zeta}_l=e^{i\omega_l t}\zeta_l$. From Eqs. (5) and (10), the linearized equation of motion for $\delta\tilde{\alpha}$, after adiabatic elimination (assuming $\kappa\gg|\zeta_k\chi_k|$), produces

$$\delta\tilde{\alpha}=-\frac{i}{\kappa}\sum_k[\tilde{\zeta}_k e^{-i(\omega_k-\Delta)t}+\tilde{\zeta}_k^* e^{i(\omega_k+\Delta)t}]\chi_k^*. \tag{13}$$

We assume that due to our choice of $\tilde{f}(x)$, $|\chi_l|$ dominates $|\chi_{k\neq l}|$, neglecting terms where $k\neq l$, and choose $\Delta=\omega_l$ to match it. Applying the rotating wave approximation (RWA) to Eqs. (12) and (13), we neglect terms involving $\tilde{\zeta}_l^*$ and $\delta\tilde{\alpha}^*$ ($2\omega_l$ should therefore describe the fastest time scale). Combining the resulting expressions, we end up with a simple damping equation:

$$\dot{\tilde{\zeta}}_l=-\frac{|\chi_l|^2}{\kappa}\tilde{\zeta}_l. \tag{14}$$

As $\Omega(x)\propto e^{ik_L x}$, we set $\tilde{f}(x)=g_0[e^{ik_{\mathrm{ex}}x}+e^{i(k_{\mathrm{ex}}\pm 2k)x}]$, where $k_{\mathrm{ex}}=k\pm k_L$, and k is the wave number of the cavity mode. Taking only the $e^{ik_{\mathrm{ex}}x}$ term, we see in Fig. 2(a) that each $|\chi_l|^2$ peaks for some k_{ex} at a point where $|\chi_l|^2>|\chi_{k\neq l}|^2$. Assuming ω_l is chosen to match, for sufficiently large k the $e^{i(k_{\mathrm{ex}}\pm 2k)x}$ term can be ignored, as those modes to which it couples most strongly (i.e., large χ_l) can, by the RWA, be neglected. From now on we consider $\tilde{f}(x)=g_0 e^{ik_{\mathrm{ex}}x}$. As $e^{ik_{\mathrm{ex}}x}\varphi_0(x)$ is equivalent to a momentum-kicked ground state with mean additional energy $=k_{\mathrm{ex}}^2/2$, the inner product of

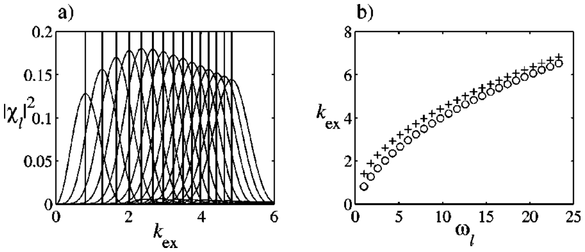

FIG. 2. (a) Plots of $|\chi_l|^2$ for $l=1$–15 as functions of k_{ex}, where χ_l is as defined in Eq. (11), and $\tilde{f}(x)=e^{ik_{ex}x}$. The vertical lines mark the numerically determined optimal values for k_{ex}. (b) Plots of $k_{ex}=\sqrt{2\omega_l}$, i.e., approximately optimal value of k_{ex} (plusses), and numerically determined optimal vales of k_{ex} (circles), against ω_l for $l=1$–25. Units are dimensionless ($\hbar=m=\omega=1$).

$\tilde{f}(x)(\varphi_0(x),-\varphi_0(x))$ with eigenstates of $\mathcal{L}(x,y)$ should be large with those eigenstates of similar energy; one would thus simplistically expect $|\chi_l|^2$ to peak when $k_{ex}^2/2=\omega_l$. In Fig. 2(b) we see that this prediction is a little crude for small ω_L, although it converges to the true values for higher frequencies, as expected. Also correctly predicted is that the optimal k_{ex} values tend to converge for large l. From Fig. 2(a) we see that the $|\chi_{l\pm1}|^2$ (at least) can be significant compared to $|\chi_l|^2$ at the optimal value of k_{ex}, but the fact that these terms are both smaller and rotating means that the lth term will still dominate. For large l, as the peak values of k_{ex} converge and the peaks become less well resolved, Eq. (14) will become less reliable.

Numerically we monitor the damping via the energy $E=\int dx\,\varphi^*(x)[-\partial^2/\partial x^2+x^2+\upsilon|\varphi(x)|^2]\varphi(x)/2$. We define E_0 equivalently, with $\varphi(x)=\varphi_0(x)$. Within the linearized approximation $E=E_0+\Sigma_k\omega_k|\zeta_k|^2$, and thus, if initially only $\zeta_l\neq0$, from Eq. (14) we have

$$\delta\dot{E}=-\frac{2|\chi_l|^2}{\kappa}\,\delta E, \tag{15}$$

where $\delta E=E-E_0$. The energy damping time scale is thus $=2g_0^2A/\kappa$, where $A=|\chi_l|^2$ for $g_0=1$. In Fig. 3 we compare Eq. (15) with numerical integrations of Eqs. (4) and (5). The initial conditions are $\varphi(x)\propto\varphi_0(x)+0.1[u_l(x)+\upsilon_l(x)]$, $\alpha=0$; υ is always $=10$. In Fig. 3(a) we see qualitative agreement of the analytical estimate given by Eq. (15) with the numerical results for different g_0, which, in view of the number of approximations made, is remarkably good. In Fig. 3(b) we see the convergence of the position density $\rho(x)=|\varphi(x)|^2$ to $|\varphi_0(x)|^2$. In Fig. 3(c) we see good qualitative agreement in the damping rates for g_0^2 and $\kappa=1$, 10 with the analytical prediction (which is the same for each). In Fig. 3(d), where $l=1$ rather than 12, but the parameters are otherwise the same, we see heating for the $\kappa=10$ case; $\omega_1=1$ is dominated by κ, and the RWA is therefore not valid. Most significant about Eqs. (14) and (15) is that the procedure for deriving them provides a recipe for determining parameters optimal for the damping of specific excitations.

In the case of a finite temperature BEC, where there is a range of populated quasiparticle (QP) excitations, these can

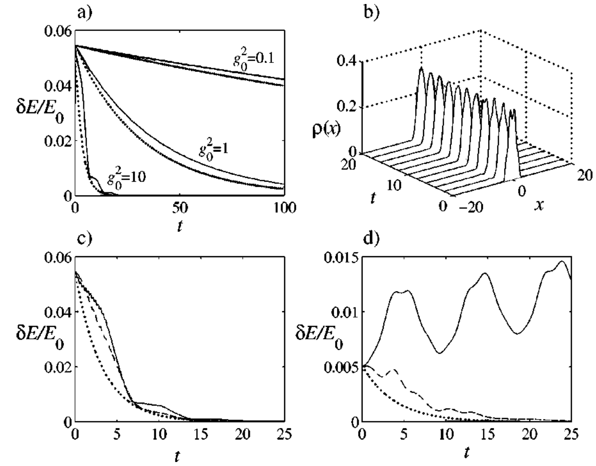

FIG. 3. (a) Plots of $\delta E/E_0$ for different g_0^2 against time, where $l=12$, $\kappa=10$. Solid lines, numerical calculations; dotted lines, prediction of Eq. (15). (b) Time evolution of $\rho(x)=|\varphi(x)|^2$ ($l=12$, κ, $g_0^2=10$). (c) Plots of $\delta E/E_0$ against time, where $l=12$. Solid line, κ, $g_0^2=10$; dashed line, κ, $g_0^2=1$; dotted line, prediction of Eq. (15) for both cases. (d) Plots of $\delta E/E_0$ against time, where $l=1$. Solid line, κ, $g_0^2=10$; dashed line, κ, $g_0^2=1$; dotted line, prediction of Eq. (15). Units are dimensionless ($\hbar=m=\omega=1$).

be individually targeted, resulting in cooling of the BEC. There is a well-known correspondence between linearized perturbations of $\varphi(x)$ and QP excitations [11,12]. Assuming $\hat{a}_0^\dagger\hat{a}_0\approx\hat{N}$, where $\hat{a}_0$ is the annihilation operator for a particle in state $\varphi_0(x)$ and $\hat{N}$ is the total particle number operator, then

$$\hat{\Psi}(x)=\hat{a}_0\left[\varphi_0(x)+\frac{1}{\sqrt{\hat{N}}}\sum_k \hat{b}_ku_k(x)+\hat{b}_k^\dagger\upsilon_k(x)\right], \tag{16}$$

where the $\hat{b}_k^\dagger$, $\hat{b}_k$ are QP creation and annihilation operators, respectively. If the state of the system is assumed thermal, then $\langle\hat{b}_k^\dagger\hat{b}_k\rangle=[\exp(\omega_k/\tau)-1]^{-1}$ and $\langle\hat{b}_k\rangle=\langle\hat{b}_k^\dagger\rangle=0$, where $\tau=k_BT/\hbar\omega$. Semiclassically, one can regard these expectation values as describing the statistics of Gaussian random variables β_k, with mean 0 and variance $\langle\hat{b}_k^\dagger\hat{b}_k\rangle$ [18]. One can make use of the linearized perturbation-QP correspondence by taking a random initial condition:

$$\varphi(x)=\varphi_0(x)+\frac{1}{\sqrt{N}}\sum_k \beta_ku_k(x)+\beta_k^*\upsilon_k(x), \tag{17}$$

with $\alpha=0$, and simulating the cooling process numerically with Eqs. (4) and (5). As we must target a range of excitations, we employ a procedure so that whenever t is a multiple of 4, k_{ex} and Δ are changed (by adjusting the laser Ω), initially targeting $l=25$ (as described above), and then through each of $l=24$–1. We choose g_0^2, $\kappa=1$ (to ensure cooling takes place for small ω_l), $N=1000$, and $\upsilon=10$. In Fig. 4(a) we show damping of states of the form $\varphi(x)\propto\varphi_0(x)+0.1[u_l(x)+\upsilon_l(x)]$, for a range of relevant l. For $l=24$ there are deviations from Eq. (15), as expected; signifi-

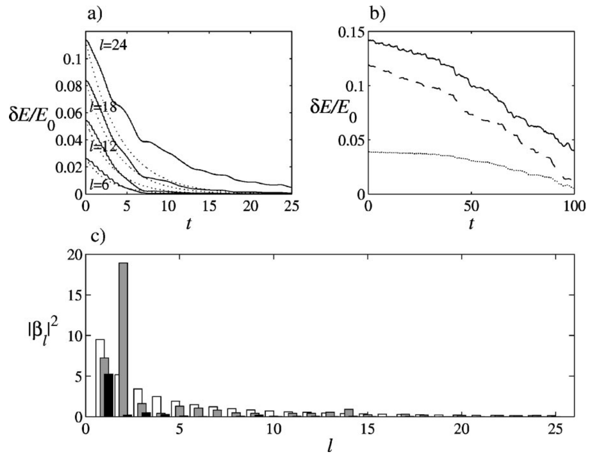

FIG. 4. (a) Damping of states perturbed by a single excitation, for κ, $g_0^2 = 1$. Solid lines, numerical calculation; dotted lines, analytical estimates. (b) Plots of $\delta E/E_0$ against time: solid line, $\tau = 10$; dashed line, same initial condition, but with a cutoff for $l > 25$; dotted line, $\tau = 5$. (c) White, $\langle \hat{b}_l^\dagger \hat{b}_l \rangle$ for $\tau = 10$. Data corresponding to the solid line of (b): gray, $|\beta_l|^2$ for $t = 0$; black, $|\beta_l|^2$ for $t = 100$. Units are dimensionless ($\hbar = m = \omega = 1$).

cant damping nevertheless takes place for each l. In Fig. 4(b) we observe significant damping for initial states where $\tau = 5,10$. Note that the energy separation between the $\tau = 10$ state and the ''cutoff'' state [$k = 1-25$ in Eq. (17), rather than 1–100], remains relatively static, implying that damping is taking place in a targeted manner on the low-lying excitations. Note also from Fig. 4(c) that, although the relative population for $k > 25$ is miniscule, the relative energy contribution is significant. Although the cutoff state was propagated for the sake of comparison, it could conceivably apply to a physical situation, such as occurs in evaporative cooling [1]. In Fig. 4(c) we see clear suppression of the $|\beta_l|^2$, where β_l is determined by $\sqrt{N}\int dx \varphi^*(x) u_l(x) - \varphi(x) v_l(x)$. By our semiclassical analogy, it follows that QP populations will similarly decrease, potentially providing a method to obtain extremely cold condensates. Note that some spontaneous emission is inevitable, and the resulting heating must be slower than the overall cooling rate. The effect of spontaneous emission on BEC temperature is addressed in Ref. [19]; note however, that, within a high Q optical cavity, in the presence of a driving field there can be some quite nontrivial effects on atomic spontaneous emission [20].

We have presented a cavity-laser-BEC configuration whereby, through astute choice of parameters, specific excitations of the BEC can be rapidly damped. We have shown the derivation of a simple equation, which explains all major features of the damping process, and provides a recipe for determining optimal parameters. We have demonstrated how this damping procedure can in principle be used to produce extremely cold condensates.

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