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Fluidity aspects of Bose-Einstein condensed systems

G. C. Marques
Instituto de Física, Universidade de São Paulo, C.P. 66318, 05389-970 São Paulo, SP, Brazil

V. Bagnato
Instituto de Física de São Carlos, Univesidade de São Paulo, Center for Optics and Photonics Research, C.P. 369, 13560-970 São Carlos, SP, Brazil

D. Spehler
Université Louis Pasteur, IUT, Allée d'Athènes, 67300 Schiltigheim, France

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We analyze some aspects of the fluidity of Bose-Einstein (BE) condensed systems. We show that a two-fluid picture of a condensed system holds true. The condensate is a quantum fluid component of the system. This component of the fluid satisfies the continuity equation and a generalized Bernoulli equation. Under certain conditions, the condensate component becomes a classical superfluid. We suggest that condensed systems might exhibit mechanochalorical and thermomechanical effects, in close analogy with superfluid helium-4. Quantized vortices are also expected to appear in a BE condensed system. We discuss the structure of quantized vortices.

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I. INTRODUCTION

In a recent paper, we presented the hydrodynamic quantization approach in order to understand some properties of a Bose-Einstein condensed system. In Ref. [1], we relied on a variational method in order to find the wave function of the condensed state as well as the wave function of excited states.

In this paper, we analyze the problem of Bose-Einstein condensation from a different standpoint. Here the quantization is achieved by using the functional integral method. Within the functional integral method, physical quantities such as partition function are expressed as sums over all possible field configurations with an appropriate weight. The hydrodynamical approach requires that we sum over all possible densities and phase configurations. For example, by using these hydrodynamical variables, the partition function of a many-body system is obtained as a sum over all possible phases and densities with a weight given by the Lagrangian density $\mathcal{L}$. That is,

$$Z=\int\int\mathcal{D}[\varphi]\mathcal{D}[\rho]e^{i\int\mathcal{L}[\rho,\varphi,\partial_\mu\rho\partial_\mu\varphi]}. \tag{1.1}$$

The free energy $\mathcal{F}$ is defined, as usual, as

$$Z=e^{i\mathcal{F}}. \tag{1.2}$$

The Lagrangian density $\mathcal{L}$ is written, in terms of the hydrodynamical variables, as

$$-\mathcal{L}[\rho,\varphi,\partial_\mu\rho,\partial_\mu\varphi]=\rho\frac{d\varphi}{dt}+\mathcal{H}[\vec{\nabla}\varphi,\rho,\vec{\nabla}\rho], \tag{1.3}$$

where the Hamiltonian can be separated into a kinetic term K (containing derivatives of the fields) and an interaction term depending only on the density. We write

$$\mathcal{H}=K[\vec{\nabla}\varphi,\rho,\vec{\nabla}]+\mathcal{H}^I[\rho]. \tag{1.4}$$

The kinetic term is the one defined in terms of the field ψ as

$$K(x)=\frac{\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x)}{2m}, \tag{1.5}$$

where $\psi(x)=\sqrt{\rho(x)}e^{i\varphi(x)}$.

By using Eq. (1.5), we get for the kinetic term

$$K[\rho(x),\vec{\nabla}\rho(x),\vec{\nabla}\varphi(x)]=\frac{\rho(x)}{2m}[\vec{\nabla}\varphi(x)]^2+\frac{1}{2m}[\vec{\nabla}\sqrt{\rho(x)}]^2. \tag{1.6}$$

As we shall see later, the last term of the kinetic density has a quantum origin. In the case of spin-$\frac{1}{2}$ particles, this term is associated to the zitterbewegung of these particles.

Averages over the ensemble of a physical quantity $\langle\hat{O}(x)\rangle$ are defined as the following sum over densities and phases:

$$\langle\hat{O}(x)\rangle\equiv\frac{1}{Z}\int\int\mathcal{D}\rho\,\mathcal{D}\varphi\,O(x)e^{i\int\mathcal{L}}. \tag{1.7}$$

We shall see, in Sec. II, that the equation of motion is valid in the sense of averages over the ensemble. This reduces the problem to that of summing over all densities and phases with a weight given by the action.

In Sec. III, we compute the contribution of phase eigenstates to the average over the ensemble. This leads to the equation for the condensate wave function. We show explicitly that this wave function is associated to a fully condensed system (all particles are in the condensate).

In Sec. IV, we show that the condensate is a quantum liquid. This quantum liquid becomes superfluid for a quasi-

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uniform condensate. Under this condition, we drop the so-called quantum velocity contribution.

In Sec. V, we show that a two-fluid picture emerges whenever the system is condensed. The velocity and density can be separated in such a way that it becomes obvious that the condensate becomes, in general, a component of the fluid.

In Sec. VI, we consider nonuniform condensate wave functions. Solutions describing quantized vortices are built explicitly.

We end this paper by drawing some conclusions.

II. EHRENFEST THEOREMS

In this section, we shall study averages over the ensemble of time evolution equations. We refer to these averages as Ehrenfest theorems.

Within the field-theoretical approach, we associate to the spin-0 boson a complex field ψ. The equation of motion satisfied by the field $\psi(x)$ is

$$i\frac{\partial\psi(x)}{\partial t}=-\frac{\vec{\nabla}^2}{2m}\psi(x)+\frac{\delta\mathcal{H}^I}{\delta\psi^*}[\psi^*(x),\psi(x)]=\frac{\delta\mathcal{H}}{\delta\psi^*(x)}, \tag{2.1}$$

$$-i\frac{\partial\psi^*(x)}{\partial t}=-\frac{\vec{\nabla}^2}{2m}\psi^*(x)+\frac{\delta\mathcal{H}^I}{\delta\psi}[\psi^*(x),\psi(x)]=\frac{\delta\mathcal{H}}{\delta\psi(x)}. \tag{2.2}$$

Within the hydrodynamical approach [1] we use, instead of ψ, the density $[\rho(x)]$ and phase $[\varphi(x)]$ variables. They are formally defined as

$$\rho(x)=\psi^*(x)\psi(x), \tag{2.3}$$

$$\psi(x)=\sqrt{\rho(x)}\,e^{i\varphi(x)}. \tag{2.4}$$

In terms of these hydrodynamical variables, the equations of motion are written as

$$\frac{\partial\varphi(x)}{\partial t}=-\frac{\delta\mathcal{H}}{\delta\rho(x)}, \tag{2.5}$$

$$\frac{\partial\rho(x)}{\partial t}=+\frac{\delta\mathcal{H}}{\delta\varphi(x)}, \tag{2.6}$$

where $\mathcal{H}$ in Eqs. (2.5) and (2.6) is the Hamiltonian density. From the above equations it follows that $\rho(x)$ and $\varphi(x)$ are canonically conjugated variables.

Quantization stands, as usual, for changing the classical fields into the quantum field operator. The commutation relation satisfied by the field operators and the canonically conjugated field is the basis of the quantization procedure. We write

$$[\hat{\rho}(x),\hat{\varphi}(x')]=i\delta(x-x'), \tag{2.7}$$

We define the local momentum operator as the gradient of the phase operator. That is,

$$\hat{\vec{P}}(x)\equiv\vec{\nabla}\hat{\varphi}(x). \tag{2.8}$$

The momentum equation of motion can be formally written under the form of Newton's law:

$$\frac{\partial\hat{\vec{P}}(x)}{\partial t}=\hat{\vec{F}}(x). \tag{2.9}$$

By taking the gradient of Eq. (2.5), we can identify the force operator $[\hat{\vec{F}}(x)]$ with the operator

$$\hat{\vec{F}}(x)=-\vec{\nabla}\left(\frac{\delta\hat{\mathcal{H}}}{\delta\rho(x)}\right). \tag{2.10}$$

Any of the quantum versions of the equations of motion (2.1) and (2.2) and (2.5) and (2.6), are obviously valid in the sense of expectation values. Defining the trace of an operator as

$$T_r\hat{O}(x)=\sum_{\psi}\langle\psi|\hat{O}(x)|\psi\rangle\frac{1}{\langle\psi|\psi\rangle}, \tag{2.11}$$

we can write the following equations:

$$\frac{\partial}{\partial t}(T_r\hat{\varphi}(x))=-T_r\left(\frac{\delta\hat{\mathcal{H}}}{\delta\rho(x)}\right), \tag{2.12}$$

$$\frac{\partial}{\partial t}(T_r\hat{\rho}(x))=T_r\left(\frac{\delta\hat{\mathcal{H}}}{\delta\varphi(x)}\right), \tag{2.13}$$

$$i\frac{\partial}{\partial t}(T_r\hat{\psi}(x))=T_r\left(\frac{\delta\hat{\mathcal{H}}}{\delta\psi^*(x)}\right). \tag{2.14}$$

Ehrenfest theorems are defined here as averages over the ensemble of quantum equations. In order to define averages over the ensemble, we define the quantum-mechanical partition function of a Bose system.

Remembering that the quantum-mechanical action $\hat{S}$ is defined as the integral of the Lagrangian density,

$$\hat{S}\equiv\int dx\,\hat{\mathcal{L}}[x], \tag{2.15}$$

we define the partition function as

$$Z=T_re^{i\hat{S}}. \tag{2.16}$$

This is just an extension (to the result) of the usual finite-temperature (T) definition

$$Z=T_r(e^{-H/kT}). \tag{2.17}$$

The average over the ensemble of a physical quantity $O(x)$ is, as usual, defined by

$$\langle O(x)\rangle = \frac{1}{Z} T_r\{e^{i\hat{S}}\hat{O}(x)\}. \tag{2.18}$$

From Eqs. (2.18), (2.5), and (2.6), it follows that, in the sense of averages over the ensemble, equation analogous to Eqs. (2.12) and (2.13) are valid:

$$\frac{\partial\langle \varphi(x)\rangle}{\partial t} = -\left\langle \frac{\delta\mathcal{H}}{\delta\rho(x)}\right\rangle, \tag{2.19}$$

$$\frac{\partial\langle \rho(x)\rangle}{\partial t} = +\left\langle \frac{\delta\mathcal{H}}{\delta\varphi(x)}\right\rangle, \tag{2.20}$$

where $\mathcal{H}(x)=\{[\vec{P}(x)\rho(x)\vec{P}(x)]/2m\}+\mathcal{H}^I(\psi^*(x)\psi(x))$.

We assume that the interaction Hamiltonian density has the general form

$$\mathcal{H}^I[\psi^*(x)\psi(x)] = U(x)\psi^*(x)\psi(x) + H^{\text{int}}(\psi^*(x)\psi(x)), \tag{2.21}$$

where $U(x)$ is an external potential. $H^{\text{int}}(\psi^*(x)\psi(x))$ in Eq. (2.21) takes into account the interaction of the boson particles among themselves. The general form of $H^{\text{int}}(\psi^*(x)\psi(x))$ is

$$H^{\text{int}}(\rho(x)) \equiv \rho(x)e(x,\rho), \tag{2.22}$$

where $e(x,\rho)$ is the per-particle internal energy.

Using Eq. (2.21), we can write the following Ehrenfest theorems:

$$-\frac{\partial\langle \varphi(x)\rangle}{\partial t} = \left\langle \frac{\vec{P}^2(x)}{2m}\right\rangle + U(x) + \langle h(x)\rangle, \tag{2.23}$$

$$\frac{\partial\langle \rho(x)\rangle}{\partial t} = -\frac{1}{m}\vec{\nabla}\cdot\langle \rho(x)\,\vec{P}(x)\rangle, \tag{2.24}$$

$$\frac{\partial\langle \vec{P}(x)\rangle}{\partial t} = -\vec{\nabla}\left\langle \frac{\vec{P}^2(x)}{2m}\right\rangle + \vec{F}_{\text{ext}}(x) - \vec{\nabla}\langle h(x)\rangle, \tag{2.25}$$

where

$$\langle h(x)\rangle = \langle e(x,\rho)\rangle + \left\langle \rho(x)\int dy\,\frac{\delta e(y,\rho(y))}{\delta\rho(x)}\right\rangle, \tag{2.26}$$

$$\vec{F}_{\text{ext}}(x) = -\vec{\nabla}U(x). \tag{2.27}$$

We shall refer to these equations as the quantum Bernoulli equation [Eq. (2.23)], the continuity equation [Eq. (2.24)], and the quantum force equation [Eq. (2.25)].

III. QUANTUM COHERENCE AND BOSE-EINSTEIN CONDENSATION

In this section, we consider the equation for the field ψ and we take into account only two-body interaction through a potential $V(x-x')$. Under these circumstances, and using Eq. (2.22), we can write Eq. (2.14) as

$$\frac{i\partial}{\partial t}(T_r\hat{\psi}(x)) = \left(-\frac{\vec{\nabla}^2}{2m}+U(x)\right)T_r\hat{\psi}(x) + T_r \times \int dx'\,\hat{\psi}(x)V(x-x')\hat{\psi}^*(x')\hat{\psi}(x'). \tag{3.1}$$

Since the vacuum is supposed to be $U(1)$-invariant, the contribution of the vacuum state and Fock states to the above equation is zero. For the vacuum state we can write

$$\langle O|\vec{\psi}(x)|O\rangle = 0. \tag{3.2}$$

For any Fock states we expect that it will give no contribution at all to Eq. (3.1). For any Fock state we can write

$$\langle \vec{p}_1\cdots\vec{p}_n|\hat{\psi}(x)|\vec{p}_1\cdots\vec{p}_n\rangle = 0. \tag{3.3}$$

The above result can be understood by taking the decomposition of the field ψ in terms of creation for (ψ^*) and annihilation (for ψ) operators $b^+(\vec{k})$ and $b(\vec{k})$.

Let us consider now the contribution to Eq. (3.1) of states that we have identified in Ref. [1] as phase eigenstates. For these states

$$e^{i\hat{\varphi}(x)}|\psi\rangle = e^{i\varphi(x)}|\psi\rangle, \tag{3.4}$$

where $\varphi(x)$ is a classical phase.

For phase eigenstates, the following factorization property holds true:

$$\frac{1}{\langle\psi|\psi\rangle}\langle\psi|\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)|\psi\rangle = \psi(x_1)\cdots\psi(x_n). \tag{3.5}$$

These states are coherent states of the field for which the number of particles is uncertain. This is clear from the commutation relations between the phase and density operators. States with a well-defined phase have a quite uncertain number of particles.

In Eq. (3.4), we have used the density representation of the algebra of operators in which the density operator is a classical c-number density. That is,

$$\hat{\rho}(x) = \rho(x) \tag{3.6}$$

and the phase operator is represented by the operator

$$\hat{\varphi}(x) = -i\,\frac{\delta}{\delta\rho(x)}. \tag{3.7}$$

The state $\psi[\rho]$ in Eq. (3.4) has the following form, when written as a functional of the density:

$$\psi[\rho] = e^{i\int\rho(x)\varphi(x)dx}. \tag{3.8}$$

The most general states

$$\psi_c[\rho] = e^{\int\rho(x)\ln\,\psi_c(x)dx} \tag{3.9}$$

describe a condensed state since the wave function associated to it is

$$\psi_c(x_1\cdots x_n)=\psi_c(x_1)\cdots\psi_c(x_n)=\prod_i^n \psi_c(x_i). \tag{3.10}$$

$\psi_c(x)$ is then the condensed state wave function.

Together with Eq. (3.5), we can see that the condensate wave function contributes to the average over the ensemble. This contribution leads us to the equation for the condensate wave function $\psi_c(x)$,

$$i\frac{\partial}{\partial t}\psi_c(x)=\left(-\frac{\vec{\nabla}^2}{2m}+U(x)\right)\psi_c(x)+\int dx'\,\psi_c(x)V(x-x')\psi_c^*(x')\psi_c(x'). \tag{3.11}$$

Within the hard-sphere approximation, we write

$$\left(i\frac{\partial}{\partial t}+\frac{\vec{\nabla}^2}{2m}\right)\psi_c(x)=U(x)\psi_c(x)+\lambda|\psi_c^2(x)|\psi_c(x), \tag{3.12}$$

where λ is related to the scattering length *a* defined by

$$\lambda=\frac{4\pi a}{m}. \tag{3.13}$$

Equation (3.12) is the generalized Gross-Pitaevskii equation.

Our conclusion is that coherent states of the field give a nonzero expectation value to the average over the ensemble of the field,

$$\frac{\langle\psi|\hat{\psi}(x)|\psi\rangle}{\langle\psi|\psi\rangle}\equiv\psi_c(x). \tag{3.14}$$

As first advanced by Anderson [2], when BE condensation occurs the system undergoes a spontaneous symmetry breakdown of the $U(1)$ symmetry and $\psi_c(x)$ becomes the (local) order parameter of the system.

The free energy of the condensate is

$$\mathcal{F}_c=\int dx\,\frac{i}{2}[\psi_c^*(x)[\partial_t\psi_c(x)]-[\partial_t\psi_c^*(x)]\psi_c(x)]-\frac{\vec{\nabla}\psi_c^*(x)\cdot\vec{\nabla}\psi_c(x)}{2m}-\mathcal{H}^I[\psi_c^*(x)\psi_c(x)]. \tag{3.15}$$

The above expression holds true when we consider all particles in the condensate. We know that only a fraction of all particles are in the condensate. In order to get more consistent expressions for the thermodynamics of the system, we have to take into account the contribution of other states, such as the excited states.

Taking into account excited states, we would get a precise expression for $\mathcal{F}$. We shall deal with this problem in the future.

The main conclusion is that coherent states of the phase field gives a nonzero contribution to the average of the ensemble of certain equations of motion. These states are states describing all particles in the condensate. The expectation value of the field among these states is the order parameter of the phase transition (uncondensed to condensed BE state). The order parameter is the wave function of the condensate.

IV. BOSE-EINSTEIN CONDENSATION AND SUPERFLUIDITY

We shall show now that the Bose-Einstein condensed system is a quantum fluid that, under certain conditions, becomes superfluid.

First of all, we analyze the conditions under which superfluid flow occurs. We just recall that, as pointed out by Penrose and Onsager [3], the relevant feature of superfluidity is the generalized factorization of the density matrix [2,4],

$$\rho(\vec{x},\vec{x}\,')=\langle\psi^*(x)\psi(x')\rangle=f^*(\vec{x})f(\vec{x}\,'). \tag{4.1}$$

Later on, Beliaev [5] extended the above factorization property to $\vec{x}$- and x^0-dependent Green's functions. For the two-point Green's functions, the factorization property implies

$$\langle\psi^*(\vec{x},x^0)\psi(\vec{x}\,',x'^0)\rangle=f^*(\vec{x},x^0)f(\vec{x}\,',x'^0). \tag{4.2}$$

The factorization properties (4.1) and (4.2) are the basis of the ''off-diagonal long-range order'' (ODLRO) hypothesis [2,4]. Assuming ODLRO, we write

$$\langle\psi(\vec{x},x^0)\rangle=\sqrt{\rho_s(\vec{x},x^0)}e^{i\varphi_s(\vec{x},x^0)} \tag{4.3}$$

$$\begin{aligned}&=\langle\psi(\vec{x}_1,x_1^0)\cdots\psi(\vec{x}_n,x_n^0)\rangle\\&=\sqrt{\rho_s(\vec{x}_1,x_1^0)}e^{i\varphi_s(\vec{x}_1,x_1^0)}\cdots\sqrt{\rho_s(\vec{x}_n,x_n^0)}e^{i\varphi_s(\vec{x}_n,x_n^0)}.\end{aligned} \tag{4.4}$$

The above approximation is also known as the factorization approximation [6]. We have shown that the above properties follow from quantum coherence. This coherence is a result of BE condensation.

Now we shall see that, as far as superfluidity is concerned, the factorization condition is, in fact, necessary but not sufficient.

Considering the Hamiltonian given by Eq. (2.21) and taking into account external fields, the corresponding equations, assuming the factorization, become

$$-\frac{\partial\varphi_c(x)}{\partial t}=\frac{(\vec{\nabla}\varphi_c(x)-e\vec{A})^2}{2m}+\frac{1}{4m}\left[\frac{1}{2}\left(\frac{\vec{\nabla}\rho_c(x)}{\rho_c(x)}\right)^2-\frac{\Delta\rho_c(x)}{\rho_c(x)}\right]+U(x)+h(x) \tag{4.5}$$

and

$$\frac{\partial\rho_c(x)}{\partial t} = -\vec{\nabla}\cdot\vec{J}_c(x), \tag{4.6}$$

where $h(x)$ is the per-particle enthalpy, defined in Eq. (2.26). Using Eq. (2.19),

$$h(x) = e_{\text{int}}(x,\rho_c) + \rho_c(x)\int \frac{\delta e_{\text{int}}(y,\rho(y))}{\delta\rho_c(x)}\,dy. \tag{4.7}$$

We shall see later that

$$h(x) = e_{\text{int}}(x,\rho_c) + m\,\frac{P(x)}{\rho_c(x)}, \tag{4.8}$$

where $P(x)$ is the pressure.

Equations (4.5) and (4.6) describe a condensed system of particles under the action of external electric and magnetic fields whose associated potentials are $U(x)$ and $\vec{A}(x)$.

The momentum of the condensate is

$$\vec{P}_c(x) = +\vec{\nabla}\varphi_c(x) \tag{4.9}$$

whereas the velocity of the condensate is

$$\vec{V}_c(x) = \frac{\vec{P}_c(x) - e\vec{A}(x)}{m} \tag{4.10}$$

and the current is

$$\vec{J}_c(x) = \rho_c(x)\vec{V}_c(x) = \frac{\rho_c(x)}{m}[\vec{P}_c(x) - e\vec{A}(x)]. \tag{4.11}$$

Let us see now under which condition the condensate becomes superfluid.

Equation (4.5) shows that the condensate becomes a quantum fluid. In fact, the quantum fluid associated to the condensate is a generalized Madelung fluid. This Madelung fluid becomes superfluid under certain conditions. This condition implies neglecting the quantum velocity contribution.

We define, for a particle of spin 0, the quantum velocity as the quantity

$$\vec{V}^q(x) = \frac{1}{m}\vec{\nabla}[\ln\sqrt{\rho(x)}] = \frac{\vec{P}^q(x)}{m}. \tag{4.12}$$

For particles of spin s (different from zero), the definition is [7]

$$\vec{V}^q(x) = \frac{1}{m}\vec{\nabla}\times[\ln\rho(x)\vec{s}] \tag{4.13}$$

and the quantum velocity, in this case, has its origin in a quantum effect, namely the zitterbewegung of the particle.

We define the quantum current as

$$\vec{j}^q(x) = \rho_c(x)\vec{V}^q(x). \tag{4.14}$$

The quantum Bernoulli equation with $\vec{A}=0$ becomes, for a particle of spin 0,

$$-\frac{\partial\varphi_c(x)}{\partial t} = \frac{\vec{P}_c^2(x)}{2m} + \frac{(\vec{P}^2)^q(x)}{2m} - \frac{\vec{\nabla}\cdot\vec{j}^q(x)}{2\rho_c(x)} + U(x) + h(x). \tag{4.15}$$

Let us assume that the quantum velocity contribution becomes negligible. This clearly is the case when ρ varies slowly with x,

$$\vec{\nabla}^2\rho_c(x) \ll \rho_c(x) \quad \text{and} \quad [\vec{\nabla}\rho_c(x)]^2 \ll \rho_c^2(x). \tag{4.16}$$

We have called these condensates quasi-incompressible condensates. Under these circumstances, we can write

$$-\frac{\partial\varphi_c(x)}{\partial t} = \frac{\vec{P}_c^2(x)}{2m} + U(x) + h(x) \tag{4.17}$$

whereas for the quantum force defined by $\vec{F}(x) = \partial\vec{P}_c(x)/\partial t$ we get

$$\frac{\partial\vec{P}_c(x)}{\partial t} = -\vec{\nabla}\left[\frac{\vec{P}_c^2(x)}{2m}\right] - \vec{\nabla}h(x) + \vec{F}_{\text{ext}}(x). \tag{4.18}$$

By using Eq. (4.10) for $\vec{A}=0$, we can write Eq. (4.18) under the alternative form

$$\frac{D\vec{P}_c(x)}{Dt} = -\vec{\nabla}h(x) + \vec{F}_{\text{ext}}(x), \tag{4.19}$$

where $\vec{F}_{\text{ext}}(x) = -\vec{\nabla}U(x)$ and D/Dt stands for the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}_c\cdot\vec{\nabla}). \tag{4.20}$$

From Eq. (4.18) it follows that, in the absence of external forces, Eq. (4.18) describes the potential flow of a nonviscous fluid [8]. This means superfluidity.

If the external potential $U(x)$ and the Hamiltonian (2.22) do not depend explicitly on the time, then one can look for a solution of Eq. (4.17), with $\vec{A}=0$, under the form

$$\varphi_c(\vec{x},t) = -\mu t + \phi_c(\vec{x}). \tag{4.21}$$

For $\varphi_c(\vec{x},t)$ given in Eq. (4.21) one gets

$$\mu = \frac{\vec{P}_c^2(x)}{2m} + h(x) + U(x). \tag{4.22}$$

In the case of superfluid ^{4}He, the constant μ in Eq. (4.22) should be identified with the chemical potential. This can be inferred from the paper by Beliaev [5] as well as from the fact that by making the substitution

$$\psi(x) \to e^{i\mu t}\psi(x) \tag{4.23}$$

we get the usual μN term in the thermodynamic potential (1.3).

The context of Bernoulli's equation is simple: along a streamline, the chemical potential is constant. In this sense, Bernoulli's equation is just an extension of the well-known result that the equilibrium condition of a body in an external field is the constant of the chemical potential. In fact by taking $\vec{P}_c(x)=0$ in Eq. (4.22), one gets the equilibrium condition [8]

$$\mu=h(x)+U(x). \tag{4.24}$$

If one writes

$$h(x)=e(x)+m\,\frac{P(x)}{\rho_c(x)}, \tag{4.25}$$

we get Bernoulli's equation. That is,

$$\mu=\frac{\vec{P}_c^2(x)}{2m}+m\,\frac{P(x)}{\rho_c(x)}+e(x)+U(x). \tag{4.26}$$

Assuming $e(x)$ constant and $U(x)=0$, we get the usual expression for Bernoulli's equation,

$$\frac{\vec{V}_c^2(x)}{2}+\frac{P(x)}{\rho_c(x)}=\text{const.} \tag{4.27}$$

The extension of condition (4.17) to finite temperatures will be analyzed later. If the particles interact only through binary forces, then, from Eq. (3.11), we write

$$-\frac{\partial\varphi_c(x)}{\partial t}=\frac{1}{2m}[\vec{\nabla}\varphi_c(x)]^2+\frac{1}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho_c(x)}{\rho_c(x)}\right)^2-\frac{\Delta\rho_c(x)}{\rho_c(x)}\right]+U(x)+\int V(x-x')\rho_c(x')dx'. \tag{4.28}$$

Equation (4.28) assumes the form of the Hamiltonian Jacobi equation for a single particle that is under the action of an external potential $U(x)$ and under the average potential produced by the others,

$$\overline{V}(x)=\int V(x-x')\rho_c(x')dx'. \tag{4.29}$$

For $\overline{V}$ defined in Eq. (4.29), one writes

$$-\frac{\partial\varphi_c(x)}{\partial t}=\frac{1}{2m}[\vec{\nabla}\varphi_c(x)]^2+U(x)+\overline{V}(x). \tag{4.30}$$

Finally, independently of the details of the interparticle potential, one can write for the superfluid component

$$\frac{\partial\rho_c(x)}{\partial t}=-\vec{\nabla}\vec{J}_c(x)=-\vec{\nabla}\cdot\frac{\rho_c(x)\vec{\nabla}\varphi_c(x)}{m}. \tag{4.31}$$

Our conclusion is that the condensate is a quantum fluid obeying a generalized Bernoulli equation and continuity equation. Quasiuniform quantum fluids are superfluids.

V. TWO-FLUID PICTURE

One of the advantages of the functional integral approach is that when performing sums over phases and densities one can always separate the contribution of the condensate from the rest (the particles that are not in the condensate). We can write formally the densities and phases as a sum of two contributions,

$$\rho(x)=\rho_c(x)+\rho_n(x), \tag{5.1}$$

$$\varphi(x)=\varphi_c(x)+\varphi_n(x), \tag{5.2}$$

where ρ_c (φ_c) is the density (phase) of the condensate and ρ_n (φ_n) is the ''normal'' density (phase) of the other component of the fluid.

Taking the gradient of Eq. (5.2), the velocity of the fluid can be written as being due to two contributions:

$$\vec{V}(x)=\vec{V}_c(x)+\vec{V}_n(x), \tag{5.3}$$

where

$$\vec{V}_c(x)=\frac{\vec{\nabla}\varphi_c(x)}{m},\quad \vec{V}_n(x)=\frac{\vec{\nabla}\varphi_n(x)}{m}. \tag{5.4}$$

The conclusion is that it is natural to assume a two-fluid picture when BE condensation occurs.

The number of particles in the condensate is [from Eq. (5.1)]

$$\int\rho_c(x)dx=N_c. \tag{5.5}$$

In this paper, we shall be mainly concerned with some properties of the condensate component. The other component will be denoted, from now on, the normal component.

The free energy of the condensate is, from Eq. (3.15), given by

$$\mathcal{F}_c=\int dx\left[-\rho_c(x)\frac{\partial\varphi_c(x)}{\partial t}-\frac{\rho_c(x)}{2m}[\vec{\nabla}\varphi_c(x)]^2-\frac{1}{2m}[\vec{\nabla}\sqrt{\rho_c(x)}]^2-\mathcal{H}^I[\rho_c(x)]\right]. \tag{5.6}$$

One can think of the free energy as a functional of the density and phase. We define

$$\mathcal{F}[\rho(x),\varphi(x)]=\int dx\left[-\rho(x)\frac{\partial\varphi(x)}{\partial t}-\frac{\rho(x)}{2m}[\vec{\nabla}\varphi(x)]^2-\frac{1}{2m}[\vec{\nabla}\sqrt{\rho(x)}]^2-\mathcal{H}^I\rho(x)\right]. \tag{5.7}$$

In this section, we shall take $U(x)=0$.

Following Anderson [2,4], the free energy can now be thought of as a functional of order parameter. The condensate is associated to the field configurations for which $\mathcal{F}$ is minimum. That is,

$$\left.\frac{\delta\mathcal{F}}{\delta\rho}\right|_{\rho=\rho_c}=0, \tag{5.8}$$

$$\left.\frac{\delta\mathcal{F}}{\delta\varphi}\right|_{\varphi=\varphi_c}=0. \tag{5.9}$$

From the above expression follow Eqs. (4.5) and (4.6), where $\vec{A}=0$. Anderson's [2] supercurrent is defined as

$$\vec{j}_s(x)=\frac{-\delta\mathcal{F}}{\delta(\vec{\nabla}\varphi(x))}. \tag{5.10}$$

From our expression (5.7) it follows that the supercurrent is

$$\vec{j}_s(x)=\left.\frac{-\delta\mathcal{F}}{\delta(\vec{\nabla}\varphi(x))}\right|_{\varphi=\varphi_c\rho=\rho_c}=\rho_c(x)\left(\frac{\vec{\nabla}\varphi_c(x)}{m}\right). \tag{5.11}$$

Anderson's supercurrent is just the current associated to the condensate quantum fluid.

We would like to call attention to the fact that this fairly simple dependence of the free energy is due to the fact that we have considered all particles in the condensate. We know that there is always depletion as a result of quantum effects. A more accurate method will be developed in a future publication.

In order to establish more analogies with Anderson's work, one notes that the average over the ensemble of the energy exhibits a functional dependence with the phase in a way very similar to that proposed by Anderson [2]. We get for a totally condensed system

$$U[\varphi_c]=\int dx\,\frac{\rho_c(x)}{2m}[\vec{\nabla}\varphi_c(x)]^2+\int\frac{1}{2m}\{\vec{\nabla}[\sqrt{\rho_c(x)}]\}^2 +\int dx\,\mathcal{H}_{\rm in}[\rho]. \tag{5.12}$$

The phase dependence is such that

$$\frac{\delta U[\varphi_c]}{\delta[\vec{\nabla}\varphi_c(x)]^2}=\frac{\rho_c(x)}{2m}, \tag{5.13}$$

which differs only by a factor of 2 from Anderson's result [2]. The discussion of some dynamical consequences, such as the Josephson junction, could be done from now on along Anderson's proposal [2].

In the condensed phase, the chemical potential vanishes. We shall see that particles belonging to the condensate have a critical velocity.

We shall analyze the solution (4.21) assuming $\mu=0$ for the nonuniform condensate. For the nonuniform time-independent condensate, we write

$$\varphi_c(x)=\phi_c(\vec{x}). \tag{5.14}$$

Under this condition, Eq. (4.22) becomes

$$0=\frac{\vec{P}_c^2(\vec{x})}{2m}+h(\vec{x}). \tag{5.15}$$

Particles belonging to the nonuniform condensate have a momentum that does not exceed a critical value. There is a critical velocity. One can establish an upper bound for this critical velocity. One gets

$$v_c<v_{\max}=\left(-\frac{\varepsilon}{2m}\right)^{1/2}, \tag{5.16}$$

where ε is the per-particle energy of the ground state. For ^{4}He, $\varepsilon\cong-6\times10^{-4}$ eV/at and $m\cong3.5\times10^9$ eV,

$$v_{\max}\cong90\ \mathrm{m/s}. \tag{5.17}$$

This result is close to the simple estimate made by Landau based on the phonon dispersion relation [9].

VI. QUANTIZED VORTICES

We shall argue in this section that, in close analogy with superfluid ^{4}He, quantized vortices will appear in Bose-Einstein condensed system. This topic has been quite well explored in several recent publications [10] and even observed experimentaly [11]; nevertheless, we offer additional discussion on this topic. In fact, we will show that quantized vortices are just solutions of Eqs. (4.5) and (4.6), where $\vec{A}=\vec{0}$. They provide an explicit example of nonuniform condensates.

By using cylindrical coordinates (r,z,θ) and from the fact that the wave function of the condensate is single-valued, it follows that

$$\psi_c(r,z,\theta+2\pi,t)=\psi_c(r,z,\theta,t). \tag{6.1}$$

$$\psi_c(r,z,\theta+2\pi,t)=\psi_c(r,z,\theta,t). \tag{6.2}$$

$$\psi_c(r,z,\theta+2\pi,t)=\psi_c(r,z,\theta,t). \tag{6.3}$$

The velocity potential $\varphi_c(r,z,\theta)$ is such that

$$\varphi_c(r,z,\theta+2\pi)=\varphi_c(r,z,\theta)+2\pi. \tag{6.4}$$

The generalized equilibrium condition permits us to shed some light on the structure of quantized vortices. Assuming that the fluid rotates around the z axis, one can associate a straight vortex line to a z-independent solution of

$$\left.\frac{\delta\mathcal{F}}{\delta\psi^*(x)}\right|_{\psi^*(x)=\psi_c^*(x)}=0,\quad \left.\frac{\delta\mathcal{F}}{\delta\psi(x)}\right|_{\psi(x)=\psi_c(x)}=0.$$

Let us designate by $\psi_c^{\rm vortex}$ such a solution and write

$$\psi_c^{\rm vortex}(r,\theta,t)=\sqrt{\rho_c^v(r)}\,e^{i(-\mu t+n\theta)}, \tag{6.5}$$

where n is an integer number that ensures condition (6.1).

From Eq. (4.9) it follows that

$$\vec{P}_c(r)=\frac{n}{r}\vec{e}_\theta, \tag{6.6}$$

so that for a closed loop around the z axis the quantization condition follows,

$$\oint \vec{P}_c\cdot d\vec{l}=2\pi n. \tag{6.7}$$

Let us assume further the following asymptotic conditions:

$$\lim_{r\to 0}\rho_c^v(r)=0, \tag{6.8}$$

$$\lim_{r\to\infty}\rho_c^v(r)=\rho_b, \tag{6.9}$$

where ρ_b is the uniform Bogoliubov condensate.

Condition (6.6) implies that close to the origin of the vortex there is only normal fluid. There are two ways of checking this. From Eqs. (5.1) and (6.6) it follows that $\rho_n(0)=\rho(0)$. The other way is just to remember that at the origin $\langle\psi\rangle=0$ and consequently at the origin the fluid can be treated as an ordinary fluid. Far away from the origin, there is a uniform condensate.

Let us turn now to the analysis of the asymptotic value of $\rho_c(r)$. For $\varphi_c(r,\theta,z,t)$ given in Eq. (6.3), we get, from Eq. (4.10), where $\vec{A}=0$,

$$\vec{V}_c(r)=\frac{n}{mr}\vec{e}_\theta. \tag{6.10}$$

It follows from the above expression that

$$\lim_{r\to\infty}\vec{V}_c(r)=0. \tag{6.11}$$

Consequently far away from the vortex, solution (6.3) should tend to the uniform Bogoliubov condensate,

$$\lim_{r\to\infty}\rho_c^v(r)=\rho_c(\infty)=\rho_b. \tag{6.12}$$

From Eqs. (4.11), (6.9), and (6.10) it follows that for vortices

$$\lim_{r\to\infty}\vec{j}_c(r)=0. \tag{6.13}$$

We have pointed out that there is an upper limit for the superfluid velocity. If we take for $\vec{V}_c$ the upper bound (5.16), we get a lower bound for the size of a vortex:

$$r\gtrsim r_c=\left(\frac{2}{m(-\varepsilon)}\right)^{1/2}. \tag{6.14}$$

In the case of ^{4}He, we get

$$r_c\sim 2.5\,\text{Å}. \tag{6.15}$$

The cutoff value obtained here is very close to the so-called coherence length of ^{4}He [6].

The relevant quantity for physical applications, in the case of the vortex solution (6.1), is the energy per unit length of the vortex (‘‘tension’’ of the line). This physical quantity is defined as the difference between the free energy associated to the vortex and the free energy associated to the uniform order parameter (ρ_b) [12]:

$$\Delta\mathcal{T}\equiv\frac{1}{L}\{\mathcal{F}[\psi_c^{\text{vortex}}]-\mathcal{F}[\psi_b]\}, \tag{6.16}$$

where $\psi_b=\sqrt{\rho_b}e^{i\alpha}$, α arbitrary phase.

$\Delta\mathcal{T}$ defined in Eq. (6.16) gives the excess energy (or free energy) that results from the appearance of a single vortex in the superfluid.

Once again field theory provides, through expression (5.7), a well-defined scheme for computing $\Delta\mathcal{T}$. We shall discuss next the results obtained under simple approximations. Let us discuss first the kinetic-energy density.

The kinetic-energy density of a vortex is

$$\begin{aligned}K_c^{\text{vortex}}(r)&=m\rho_c^v(r)\frac{V_c^2(x)}{2}+\frac{1}{2m}\{\vec{\nabla}[\sqrt{\rho_c^v(x)}]\}^2\\&=\frac{\rho_c^v(r)}{2}\frac{n^2}{mr^2}+\frac{\rho_c^v(r)}{8m}(V^{q^2}(x)).\end{aligned} \tag{6.17}$$

Neglecting the quantum velocity, the kinetic-energy contribution to the tension will then be

$$\Delta\mathcal{T}^{(K)}=\frac{\pi}{m}\frac{n^2}{L}\int_0^\infty\frac{dr}{r}\rho_c^v(r). \tag{6.18}$$

If one takes for $\rho_c(r)$ in Eq. (6.16) the asymptotic value (6.7), the vortex tension diverges logarithmically in the long and short distance limits. In both limits we have, however, natural cutoffs. For the long distance we can take the radius of the vessel as the cutoff (R) and for the short distance we take r_c defined in Eq. (6.12). One can then write

$$\Delta\mathcal{T}^{(K)}\cong\frac{\pi n^2}{mL}\rho_b\ln\left(\frac{R}{r_c}\right). \tag{6.19}$$

It is possible to make improvements in the determination of the vortex tension once $\rho_c^v(r)$ is known. In order to find $\rho_c^v(r)$, one has to find the solution of the generalized equilibrium equation,

$$\left.\frac{\delta\mathcal{F}}{\delta\psi^*(x)}\right|_{\psi^*(x)=\psi_c^{*\text{vortex}}(x)}=0\quad\text{and}\quad\left.\frac{\delta\mathcal{F}}{\delta\psi(x)}\right|_{\psi(x)=\psi_c^{\text{vortex}}(x)}=0. \tag{6.20}$$

For $\mathcal{F}$ given in Eq. (5.7) we get, for two-body interactions,

$$-\mu\psi_c^v(r,\theta)-\frac{\vec{\nabla}^2}{2m}\psi_c^v(r,\theta)+U(r,\theta)\psi_c^v(r,\theta)$$
$$+\int d^3\vec{r}\,'V(\vec{r}-\vec{r}\,')\psi_c^{*v}(r',\theta')\psi_c^v(r',\theta')\psi_c^v(r,\theta)=0. \tag{6.21}$$

The above equation is just the one proposed by Gross in his analysis of the structure of vortices. If we consider the hard-sphere approximation and $U=0$, then, in the low-density limit, we get

$$-\frac{1}{2m}\left(\partial_r^2+\frac{1}{2}\partial_r-\frac{n^2}{r^2}\right)(\rho_0^v)^{1/2}(r)+\lambda(\rho_c^v)^{3/2}(r)$$
$$-\mu(\rho_c^v)^{1/2}(r)=0. \tag{6.22}$$

Bogoliubov's condensate is, in this case, given by

$$\rho_b=\frac{\mu}{\lambda}. \tag{6.23}$$

We introduce the variables $\overline{r}$ and $\overline{\rho}$:

$$\overline{r}=r\sqrt{2m\mu},\quad \overline{\rho_c^v}=\left(\frac{\rho_c^v}{\rho_b}\right)^{1/2}; \tag{6.24}$$

then, from Eqs. (6.2) and (6.22), Eq. (6.20) can be written as [13]

$$\left[-\left(\partial_{\overline{r}}^2+\frac{1}{\overline{r}}\partial_{\overline{r}}-\frac{n^2}{\overline{r}^2}\right)+(\overline{\rho_c^v})^2-1\right]\overline{\rho_c^v}=0. \tag{6.25}$$

For small $\overline{r}$ we get

$$\overline{\rho_c^v}(\overline{r})=\rho_bA_mJ_n\overline{r}), \tag{6.26}$$

whereas for large r one gets the series [13]

$$\overline{\rho_c^v}(r)\cong\left[1-\frac{n^2}{2\overline{r}^2}-\left(n^2+\frac{n^4}{8}\right)\frac{1}{\overline{r}^4}\right.$$
$$\left.-\left(8+2n^2+\frac{n^4}{16}\right)\frac{n}{\overline{r}^6}-O\left(\frac{1}{\overline{r}^8}\right)\right]. \tag{6.27}$$

The vortex tension is given by the integral

$$\Delta\mathcal{T}=\frac{\pi\rho_b}{mL}\int_0^\infty d\overline{r}\;\overline{r}\left\{(\partial_{\overline{r}}\overline{\rho_c^v})^2+\frac{n^2}{\overline{r}^2}(\overline{\rho_c^v})^2+\tfrac{1}{2}\,[(\overline{\rho_c^v})^2-1]^2\right\}. \tag{6.28}$$

The dominant term in Eq. (6.26) is the one coming from the kinetic term. One can then write that

$$\Delta\mathcal{T}\cong\frac{\pi\rho_b}{mL}\int_0^\infty d\overline{r}\,\frac{n^2}{\overline{r}}(\overline{\rho_c^v})^2=\frac{\pi n^2}{mL}\int_0^\infty\frac{dr}{r}\rho_c^v. \tag{6.29}$$

VII. CONCLUSIONS

The conclusions that can be drawn from this paper are the following.

The condensate behaves like a quantum fluid. The most important equation governing this fluid is the generalized Bernoulli equation. When the condensate fluid is quasi-incompressible, the condensate becomes superfluid.

Particles belonging to the superfluid have a critical velocity whose upper bound is

$$v_{\max}=\left(-\frac{\varepsilon}{2m}\right)^{1/2},$$

where ε is the per-particle energy of the ground state.

We suggest further that condensed systems will exhibit mechanochaloric as well as thermomechanical effects. We suggest also that London's relation will hold true.

A two-fluid picture is valid in the sense of Landau. That is, the density and the phase variables can be written as a sum over two contributions. We claim that the functional integral approach leads to a well-defined framework for separating out these contributions. As a result, we have written explicitly the free energy in the totally condensed phase.

Quantized vortices are expected to appear in BE quantized systems. At the origin of the vortices, we can find only normal fluid. As we move far away from the vortices, we find the Bogoliubov condensate. We analyze the structure of these quantized vortices.

Quantized vortices are particular solutions of Madelung's equation in which we take into account the quantum velocity term. They are nontrivial solutions of a nonuniform condensate.

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