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Quantum channel identification problem

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This paper explores an application of quantum entanglement. The problem treated here is the quantum channel identification problem: given a parametric family $\{\Gamma_\theta\}_\theta$ of quantum channels, find the best strategy of estimating the true value of the parameter θ. As a simple example, we study the estimation problem of the isotropic depolarization parameter θ for a two-level quantum system $\mathcal{H}\simeq\mathbf{C}^2$. In the framework of noncommutative statistics, it is shown that the optimal input state on $\mathcal{H}\otimes\mathcal{H}$ to the channel exhibits a transitionlike behavior according to the value of the parameter θ.

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Let $\mathcal{H}$ be a Hilbert space that represents the physical system of interest and let $\mathcal{S}(\mathcal{H})$ be the set of density operators on $\mathcal{H}$. It is well known [1] that a dynamical change $\Gamma:\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})$ of the physical system, called a *quantum channel*, is represented by a trace-preserving completely positive map. But how can we identify the quantum channel that we have in a laboratory? A general scheme may be as follows: input a well-prepared state σ to the quantum channel and estimate the dynamical change $\sigma\mapsto\Gamma(\sigma)$ by performing a certain measurement on the output state $\Gamma(\sigma)$. It is then natural to inquire what is the best strategy of estimating a quantum channel. The purpose of this paper is to study this problem from a noncommutative statistical point of view. For mathematical simplicity, we restrict ourselves to the case in which the quantum channel to be identified lies in a smooth parametric family $\{\Gamma_\theta;\theta=(\theta_1,\ldots,\theta_n)\in\Theta\}$ of quantum channels. When $\mathcal{H}$ is finite-dimensional, this is not an essential restriction [2].

Once an input state σ for the channel is fixed, we have a parametric family $\{\Gamma_\theta(\sigma)\}_{\theta\in\Theta}$ of output states, and as long as the parametrization $\theta\mapsto\Gamma_\theta(\sigma)$ is nondegenerate, the problem of estimating the quantum channel is reduced to a parameter estimation problem for the noncommutative statistical model $\{\Gamma_\theta(\sigma)\}_{\theta\in\Theta}$. As a consequence, the parameter estimation problem for a family $\{\Gamma_\theta\}_{\theta\in\Theta}$ of quantum channels amounts to finding an optimal input state σ for the channel and an optimal estimator for the parametric family $\{\Gamma_\theta(\sigma)\}_\theta$ of output states. One may imagine that this problem does not exceed the realm of conventional quantum estimation theory [3,4]. But, in fact, it opens a new field of research in noncommutative statistics.

Since each channel Γ_θ is completely positive, it can be extended to the composite quantum system $\mathcal{H}\otimes\mathcal{H}$. In view of the statistical parameter estimation, there are two essentially different extensions that have the same parametrization θ as Γ_θ: one is $\Gamma_\theta\otimes I:\mathcal{S}(\mathcal{H}\otimes\mathcal{H})\to\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$, where I denotes the identity channel, and the other is $\Gamma_\theta\otimes\Gamma_\theta:\mathcal{S}(\mathcal{H}\otimes\mathcal{H})\to\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. A question arises naturally: what happens when we use an entangled state as an input to the extended channel? In what follows, we demonstrate a somewhat nontrivial aspect of this problem.

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Let $\mathcal{H}:=\mathbf{C}^2$ and let the channel $\Gamma_\theta:\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})$ be defined by

$$\Gamma_\theta\left(\frac{1}{2}\begin{bmatrix}1+z & x-iy\\ x+iy & 1-z\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1+\theta z & \theta(x-iy)\\ \theta(x+iy) & 1-\theta z\end{bmatrix}.$$

The parameter θ represents the magnitude of isotropic depolarization. The channel can be uniquely extended on the 2×2 matrix algebra $\mathbf{C}^{2\times2}$ as follows:

$$\Gamma_\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1+\theta & 0\\ 0 & 1-\theta\end{bmatrix},$$

$$\Gamma_\theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0 & 2\theta\\ 0 & 0\end{bmatrix},$$

$$\Gamma_\theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0 & 0\\ 2\theta & 0\end{bmatrix},$$

$$\Gamma_\theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1-\theta & 0\\ 0 & 1+\theta\end{bmatrix}.$$

To ensure that Γ_θ is completely positive, the parameter θ must lie in the closed interval $\Theta:=[-\frac{1}{3},1]$. (See [2].) We thus have a one-parameter family $\{\Gamma_\theta;\theta\in\Theta\}$ of quantum channels, and our task is to estimate the true value of θ.

Before proceeding to the parameter estimation for $\{\Gamma_\theta\}_\theta$, we give a brief account of the one-parameter quantum estimation theory for density operators. (Consult [3] or [4] for details.) Given a one-parameter family $\{\rho_\theta\}_\theta$ of density operators, an estimator for the parameter θ is represented by a Hermitian operator T, normally with a requirement that the estimator should be (locally) unbiased: that is, if the system is in the state ρ_θ, then the expectation $E_\theta[T]:=\mathrm{Tr}\rho_\theta T$ of the estimator T should be identical to θ. It is easy to show that every (locally) unbiased estimator T for the parameter θ satisfies the quantum Cramér-Rao inequality $V_\theta[T]\geq(J_\theta)^{-1}$, where $V_\theta[T]:=\mathrm{Tr}\rho_\theta(T-\theta)^2$ is the variance of estimator T, and $J_\theta:=J(\rho_\theta):=\mathrm{Tr}\rho_\theta(L_\theta)^2$ is the quantum Fisher information with L_θ the symmetric logarithmic derivative (SLD), i.e., the Hermitian operator that satisfies the equation

$$\frac{d\rho_\theta}{d\theta}=\tfrac{1}{2}(L_\theta\rho_\theta+\rho_\theta L_\theta).$$

It is important to notice that the lower bound $(J_\theta)^{-1}$ in the quantum Cramér-Rao inequality is achievable (at least locally). In other words, the inverse of the SLD Fisher information gives the ultimate limit of estimation. As a consequence, the larger the SLD Fisher information is, the more accurately we can estimate the parameter θ.

Let us return to the parameter estimation problem for the one-parameter family $\{\Gamma_\theta\}_\theta$ of quantum channels. Taking account of the above-mentioned one-parameter estimation theory for density operators, our task is reduced to finding an optimal input for the channel that maximizes the SLD Fisher information of the corresponding parametric family of output states.

We start with the maximization of the SLD Fisher information of the family $\{\Gamma_\theta(\sigma)\}_\theta$ with respect to the input state $\sigma\in\mathcal{S}(\mathcal{H})$. An important observation is that the maximum is attained by a pure state. To see this, it suffices to prove the convexity of the SLD Fisher information, i.e., if $\rho_\theta=\lambda\sigma_\theta+(1-\lambda)\tau_\theta$ for a constant λ between 0 and 1, then $J(\rho_\theta)\leqslant\lambda J(\sigma_\theta)+(1-\lambda)J(\tau_\theta)$. Let us introduce the states $\tilde{\rho}_\theta:=\lambda\sigma_\theta\oplus(1-\lambda)\tau_\theta$ on $\mathcal{H}\oplus\mathcal{H}$. It is easy to show that $J(\tilde{\rho}_\theta)=\lambda J(\sigma_\theta)+(1-\lambda)J(\tau_\theta)$. By identifying $\mathcal{H}\oplus\mathcal{H}$ with $\mathbf{C}^2\otimes\mathcal{H}$, we consider the partial trace $\mathrm{Tr}_{C^2}:\mathcal{S}(\mathcal{H}\oplus\mathcal{H})\to\mathcal{S}(\mathcal{H})$. Since Tr_{C^2} is a stochastic (i.e., trace-preserving completely positive) map, the monotonicity of the SLD Fisher information with respect to a stochastic map [5] shows that $J(\tilde{\rho}_\theta)\geqslant J(\mathrm{Tr}_{C^2}\tilde{\rho}_\theta)=J(\rho_\theta)$. This completes the proof of the convexity of the SLD Fisher information. Now we are ready to specify an optimal input state. Since our channel Γ_θ is unitarily invariant (i.e., isotropic in the Stokes parameter space), we can take without loss of generality the optimal input to be $\sigma=|\mathbf{e}\rangle\langle\mathbf{e}|$, where $\langle\mathbf{e}|=(1,0)$. The corresponding output state $\rho_\theta:=\Gamma_\theta(\sigma)$ is

$$\rho_\theta=\frac{1}{2}\begin{bmatrix}1+\theta & 0\\ 0 & 1-\theta\end{bmatrix}.$$

Since the state ρ_θ is isomorphic to the classical coin flipping in which "heads" occur with probability $(1+\theta)/2$, the SLD Fisher information becomes

$$J_\theta=\frac{1}{1-\theta^2}.$$

We next study the extended channel $\Gamma_\theta\otimes I:\mathcal{S}(\mathcal{H}\otimes\mathcal{H})\to\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. In this case, we can use a possibly entangled state as the input. For the same reason as above, we can take the input to be a pure state: $\hat{\sigma}=|\psi\rangle\langle\psi|$, where $\psi\in\mathcal{H}\otimes\mathcal{H}$. By the Schmidt decomposition, the vector ψ is represented as

$$|\psi\rangle=\sqrt{x}|\mathbf{e}_1\rangle|\mathbf{f}_1\rangle+\sqrt{1-x}|\mathbf{e}_2\rangle|\mathbf{f}_2\rangle, \tag{1}$$

where x is a real number between 0 and 1, and $\{\mathbf{e}_1,\mathbf{e}_2\}$ and $\{\mathbf{f}_1,\mathbf{f}_2\}$ are orthonormal bases of $\mathcal{H}=\mathbf{C}^2$. Since the channels Γ_θ and I are both unitarily invariant, we can assume without loss of generality that the optimal input takes the form (1) with $\langle\mathbf{e}_1|=\langle\mathbf{f}_1|=(0,1)$ and $\langle\mathbf{e}_2|=\langle\mathbf{f}_2|=(1,0)$. The constant x remains to be determined. The corresponding output state $\hat{\rho}_\theta:=\Gamma_\theta\otimes I(\hat{\sigma})$ becomes

$$\hat{\rho}_\theta=\frac{1}{2}\begin{bmatrix}(1-x)(1+\theta) & 0 & 0 & 2\sqrt{x(1-x)}\,\theta\\ 0 & x(1-\theta) & 0 & 0\\ 0 & 0 & (1-x)(1-\theta) & 0\\ 2\sqrt{x(1-x)}\,\theta & 0 & 0 & x(1+\theta)\end{bmatrix}.$$

The SLD for the family $\{\hat{\rho}_\theta\}_\theta$ is given by

$$\hat{L}_\theta=\begin{bmatrix}\frac{1+2\theta-3\theta^2-8\theta x}{(1-\theta^2)(1+3\theta)} & 0 & 0 & \frac{4\sqrt{x(1-x)}}{(1-\theta)(1+3\theta)}\\ 0 & -\frac{1}{1-\theta} & 0 & 0\\ 0 & 0 & -\frac{1}{1-\theta} & 0\\ \frac{4\sqrt{x(1-x)}}{(1-\theta)(1+3\theta)} & 0 & 0 & \frac{1-6\theta-3\theta^2+8\theta x}{(1-\theta^2)(1+3\theta)}\end{bmatrix},$$

and the SLD Fisher information is

$$\hat{J}_\theta = \frac{1+3\theta+8x(1-x)}{(1-\theta^2)(1+3\theta)}.$$

When $x=0$ or 1, the above SLD Fisher information $\hat{J}_\theta$ is identical to J_θ. This is a matter of course: the input state is disentangled in this case and no information about the parameter θ is available via the independent channel I. When $x\neq 0$ and $\neq 1$, the SLD Fisher information $\hat{J}_\theta$ diverges at $\theta=1$ and $-\frac{1}{3}$. This is because the complete positivity of the channel Γ_θ breaks across these values. Now let us specify the optimal input state. For every θ, the SLD Fisher information $\hat{J}_\theta$ takes the maximum $3/(1-\theta)(1+3\theta)$ at $x=\frac{1}{2}$. Therefore, the optimal input for the channel $\Gamma_\theta\otimes I$ is the maximally entangled state. The implication of this result is profound: although we use the channel Γ_θ only once, extra information about the channel is obtained via entanglement of the input state. In particular, the use of entanglement improves exceedingly the performance of estimation as θ approaches $-\frac{1}{3}$.

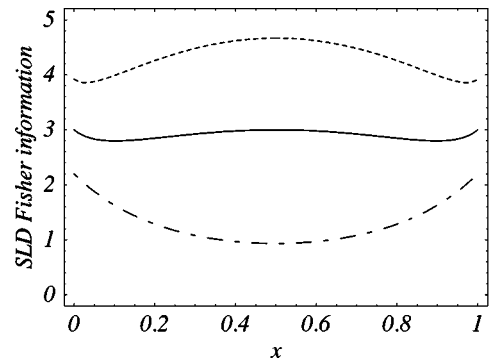

FIG. 1. SLD Fisher information $\check{J}_\theta$ vs x for $\theta=0.7$ (dashed), $1/\sqrt{3}$ (solid), and 0.3 (chained).

Let us proceed to the analysis of the other extended channel $\Gamma_\theta\otimes\Gamma_\theta:\mathcal{S}(\mathcal{H}\otimes\mathcal{H})\to\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. As before, we can take the input to be a pure state $\check{\sigma}=|\psi\rangle\langle\psi|$, where ψ is given byEq. (1) with $\langle\mathbf{e}_1|=\langle\mathbf{f}_1|=(0,1)$ and $\langle\mathbf{e}_2|=\langle\mathbf{f}_2|=(1,0)$. The corresponding output state $\check{\rho}_\theta:=\Gamma_\theta\otimes\Gamma_\theta(\check{\sigma})$ becomes

$$\check{\rho}_\theta=\frac{1}{4}\begin{bmatrix} x(1-\theta)^2+(1-x)(1+\theta)^2 & 0 & 0 & 4\sqrt{x(1-x)}\,\theta^2 \\ 0 & 1-\theta^2 & 0 & 0 \\ 0 & 0 & 1-\theta^2 & 0 \\ 4\sqrt{x(1-x)}\,\theta^2 & 0 & 0 & x(1+\theta)^2+(1-x)(1-\theta)^2 \end{bmatrix}.$$

Since the SLD for the family $\{\check{\rho}_\theta\}_\theta$ is too complicated to write down, we give the SLD Fisher information only,

$$\check{J}_\theta=\frac{4\theta^4+5\theta^2-1}{2\theta^2(1-\theta^4)}+\frac{8\theta^2x(1-x)}{1-\theta^4}+\frac{1-\theta^2}{2\theta^2(1+\theta^2)[1-\theta^2+16\theta^2x(1-x)]}.$$

When $x=0$ or 1, the above SLD Fisher information $\check{J}_\theta$ becomes $2/(1-\theta^2)$, which precisely doubles the J_θ. Again this is a matter of course: the input state is disentangled in this case and the same amount of information about the parameter θ is obtained per independent use of the channel Γ_θ. When $x\neq 0$ and $\neq 1$, the SLD Fisher information $\check{J}_\theta$ diverges at $\theta=1$ but not at $\theta=-\frac{1}{3}$. This is because the requirement of positivity for the channel $\Gamma_\theta\otimes\Gamma_\theta$ is strictly weaker than that for the channel $\Gamma_\theta\otimes I$ (i.e., the complete positivity for Γ_θ). Now we examine a rather unexpected behavior of the optimal input state. For $1/\sqrt{3}\leq\theta<1$, the SLD Fisher information $\check{J}_\theta$ takes the maximum $12\theta^2/(1-\theta^2)(1+3\theta^2)$ at $x=\frac{1}{2}$, while for $-\frac{1}{3}\leq\theta\leq 1/\sqrt{3}$ it takes the maximum $2/(1-\theta^2)$ at $x=0$ and 1. (See Fig. 1.) Namely, the optimal input state "jumps" from the maximally entangled state to a disentangled state at $\theta=1/\sqrt{3}$. It is surprising that the seemingly homogeneous family $\{\Gamma_\theta\}_\theta$ of depolarization channels involves a transitionlike behavior.

Finally, we mention the possibility of extending the channel Γ_θ in the form $\Gamma_\theta\otimes\Gamma'$, where Γ' is a channel that is known to the observer and is independent of θ. Since the channel $\Gamma_\theta\otimes\Gamma'$ is decomposed into $(I\otimes\Gamma')(\Gamma_\theta\otimes I)$, the monotonicity argument for the SLD Fisher information with respect to a stochastic map allows us to deduce that the best choice of the channel Γ' is the identity channel.

To conclude, among those we have considered on the second extension $\mathcal{H}\otimes\mathcal{H}$ of the quantum system, the best strategy of estimating the isotropic depolarization parameter θ is the following. For $1/\sqrt{3}\leq\theta\leq 1$, use $\Gamma_\theta\otimes\Gamma_\theta$ and input a maximally entangled state on $\mathcal{H}\otimes\mathcal{H}$; for $\frac{1}{3}\leq\theta\leq 1/\sqrt{3}$, use Γ_θ twice independently and input any pure state on $\mathcal{H}$ each time; for $-\frac{1}{3}\leq\theta\leq\frac{1}{3}$, use $\Gamma_\theta\otimes I$ and input a maximally entangled state on $\mathcal{H}\otimes\mathcal{H}$.

We have demonstrated a nontrivial aspect of a statistical estimation problem for a quantum channel. Other problems, such as the use of the nth extension $\mathcal{H}^{\otimes n}$ and its asymptotics, or the multiparameter quantum channel estimation, will be presented elsewhere.

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